Homotopical Adjoint Lifting Theorem

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Lifting Quillen equivalences to algebra categories

Joint work with David White.

$$\begin{tikzcd} \mathrm{Alg}(\mathsf{O}) \arrow[r, shift left, "\overline{L}"] \arrow[d, shift left, "U"] & \mathrm{Alg}(\mathsf{P}) \arrow[l, shift left, "R"] \arrow[d, shift left, "U"] \\ \mathsf{M} \arrow[u, shift left, "\mathsf{O}\circ-"] \arrow[r, shift left, "L"] & \mathsf{N} \arrow[l, shift left, "R"] \arrow[u, shift left, "\mathsf{P}\circ-"] \end{tikzcd}$$

$$\begin{tikzcd} \text{operad algebra categories} \arrow[d, shift left, "\text{forget}"] \\ \text{monoidal model categories} \arrow[u, shift left, "\text{free}"] \end{tikzcd}$$

- $L \dashv R$ Quillen equivalence. R lax symmetric monoidal functor.
- O, P are operads on M, N.
- $f : \mathsf{O} \longrightarrow R\mathsf{P}$ operad map with $f : L\mathsf{O} \xrightarrow{\sim} \mathsf{P}$ entrywise weak equivalence.
- $RU = UR$. $\overline{L}(\mathsf{O}\circ -) = (\mathsf{P}\circ -)L$ by Adjoint Lifting Theorem.

Theorem (White-Y.)

With some cofibrancy assumptions, $\overline{L} \dashv R$ is a Quillen equivalence.

Operads and algebras

Definition

An *operad* $\mathsf{O} = \left(\{\mathsf{O}_n\}_{n\geq 0}, \gamma, 1\right)$ on a symmetric monoidal category $(\mathsf{M}, \otimes, \mathbb{1})$:

- Right Σ_n-action on O_n,
- Operadic composition $\mathsf{O}_n \otimes \mathsf{O}_{k_1} \otimes \cdots \otimes \mathsf{O}_{k_n} \xrightarrow{\gamma} \mathsf{O}_{k_1+\cdots+k_n}$,
- Operadic unit $\mathbb{1} \xrightarrow{1} \mathsf{O}_1$

satisfying unity, associativity, and equivariance axioms.

Definition

An O-*algebra* is an object $X \in \mathsf{M}$ with structure maps

$$\mathsf{O}_n \otimes X^{\otimes n} \xrightarrow{\lambda} X ,$$

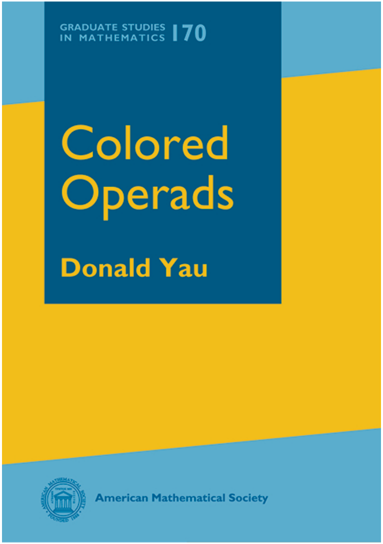
satisfying unity, associativity, and equivariance axioms.
Category of O-algebras = $\mathsf{Alg}(\mathsf{O})$

O_n parametrizes n-ary operations.

Examples

- Endomorphism operad $\mathrm{End}(X)_n = [X^{\otimes n}, X]$
- As-algebras are monoids. $\mathrm{As}_n = \coprod_{\Sigma_n} \mathbb{1}$
- Com-algebras are commutative monoids. $\mathrm{Com}_n = \mathbb{1}$
- Lie-algebras are Lie algebras in dg/simplicial modules.
- Op-algebras are operads.
- $L_\infty \xrightarrow{\sim} \mathrm{Lie}$ cofibrant resolution.
- $A_\infty \xrightarrow{\sim} \mathrm{As}$ cofibrant resolution.
- $E_\infty \xrightarrow{\sim} \mathrm{Com}$ cofibrant resolution.

A word from our sponsor : To learn more about operads ...



Model structure on algebra categories

For maps $X_0 \xrightarrow{f} X_1$ and $Y_0 \xrightarrow{g} Y_1$, the *pushout product* is the map

$$(X_0 \otimes Y_1) \coprod_{X_0 \otimes Y_0} (X_1 \otimes Y_0) \xrightarrow{f \square g} X_1 \otimes Y_1$$

that appears in the pushout product axiom.

Theorem (White-Y.)

Suppose M *is a strongly cofibrantly generated monoidal model category.*

- *For each* $n \geq 1$ *and* $X \in \mathsf{M}^{\Sigma_n^{op}}$, $X \otimes_{\Sigma_n} (-)^{\square n}$ *preserves acyclic cofibrations.*

Then for each operad O, $\mathsf{Alg}(\mathsf{O})$ *admits a projective model structure.*

- strongly : domains of generating (acyclic) cofibrations are small.
- projective : weak equivalences and fibrations are defined in M.
- Examples : $\mathsf{Ch}(\Bbbk)_{(\geq 0)}$, $\mathsf{SSet}_{(*)}$, Sp^{Σ} (positive (flat) stable model)
- There are variations that assume less on M and more on O.

Definition : Weak monoidal Quillen equivalence (Schwede-Shipley)

$\mathsf{M} \underset{R}{\overset{L}{\rightleftarrows}} \mathsf{N}$ Quillen equivalence between monoidal model categories, and

1. R is lax symmetric monoidal.
2. For cofibrant $X, Y \in \mathsf{M}$, $L(X \otimes Y) \xrightarrow[\sim]{L^2} LX \otimes LY$ is a weak equivalence.

   L^2 is adjoint to $X \otimes Y \to RLX \otimes RLY \to R\big(LX \otimes LY\big)$.
3. For some cofibrant replacement $q : Q\mathbb{1}_{\mathsf{M}} \longrightarrow \mathbb{1}_{\mathsf{M}}$, the map

   $$LQ\mathbb{1}_{\mathsf{M}} \xrightarrow{Lq} L\mathbb{1}_{M} \longrightarrow \mathbb{1}_{\mathsf{N}}$$

   is a weak equivalence in N.

Example : $\mathsf{M} = \mathsf{N}$ and $L = R = Id$

Example : Dold-Kan $K : \mathsf{Ch}_{\geq 0} \rightleftarrows (\Bbbk\mathsf{Mod})^{\Delta^{op}} : N$ over a field $\Bbbk$ of char. 0

Example (Castiglioni-Cortiñas) : Monoidal dual Dold-Kan

$$Q : \mathsf{Ch}^{\geq 0} \rightleftarrows (\Bbbk\mathsf{Mod})^{\mathsf{Fin}} : P$$

Definition : Nice Quillen equivalence

1. $\mathsf{M} \underset{R}{\overset{L}{\rightleftarrows}} \mathsf{N}$ weak monoidal Quillen eq., both cofibrantly generated
2. Every generating cofibration in M has cofibrant domain.
3. $U \xrightarrow[\sim]{g} V \in \mathsf{N}^{\Sigma_n^{op}}$, $X \in \mathsf{N}^{\Sigma_n}$, U, V, X cofibrant in N

   $\Rightarrow U \otimes_{\Sigma_n} X \xrightarrow[\sim]{g \otimes_{\Sigma_n} X} V \otimes_{\Sigma_n} X$ is a weak equivalence in N
4. In both M and N: For $W \in \mathsf{M}^{\Sigma_n^{op}}$, $X \in \mathsf{M}^{\Sigma_n}$ cofibrant in M
   - coinvariants $X_{\Sigma_n} \in \mathsf{M}$ is cofibrant
   - $\big[L(W \otimes X)\big]_{\Sigma_n} \xrightarrow[\sim]{(L^2)_{\Sigma_n}} \big[LW \otimes LX\big]_{\Sigma_n}$
   - $W \otimes_{\Sigma_n} (-)^{\square n}$ preserves (acyclic) cofibrations

Examples : $Id : \mathsf{M} \rightleftarrows \mathsf{M} : Id$ (if M is nice), Dold-Kan, monoidal dual Dold-Kan

Main Theorems : Lifting Quillen equivalences

Theorem (Entrywise cofibrant operads)

Suppose $L : \mathsf{M} \rightleftarrows \mathsf{N} : R$ *nice Quillen equivalence.*

- O, P *are entrywise cofibrant operads on* M, N.
- $f : \mathsf{O} \longrightarrow R\mathsf{P}$ *operad map with* $f : L\mathsf{O} \xrightarrow{\sim} \mathsf{P}$ *entrywise weak equivalence.*

Then $L \dashv R$ *lifts to a Quillen equivalence*

$$\mathsf{Alg}(\mathsf{O}) \underset{R}{\overset{\overline{L}}{\underset{\sim}{\rightleftarrows}}} \mathsf{Alg}(\mathsf{P})$$

between algebra categories.

Σ*-cofibrant* means cofibrant in $\prod_{n\geq 0} \mathsf{M}^{\Sigma_n^{op}}$.

Theorem (Σ-cofibrant operads)

If O, P *are* Σ*-cofibrant, then we can replace nice Quillen eq with:*

- $L \dashv R$ *is a weak monoidal Quillen equivalence.*
- *Every generating cofibration in* M *has cofibrant domain.*

Proof outline

For cofibrant $A \in \mathrm{Alg}(\mathrm{O})$ and fibrant $B \in \mathrm{Alg}(\mathrm{P})$, need to show:

$$(\S) \qquad \overline{L}A \xrightarrow[\sim]{\varphi} B \in \mathrm{Alg}(\mathrm{P}) \quad \Longleftrightarrow \quad A \xrightarrow[\sim]{\varphi^{\#}} RB \in \mathrm{Alg}(\mathrm{O})$$

$$\begin{tikzcd}
\mathrm{Alg}(\mathrm{O}) \arrow[r, shift left, "\overline{L}"] \arrow[d, shift right, "U"'] & \mathrm{Alg}(\mathrm{P}) \arrow[l, shift left, "R"] \arrow[d, shift right, "U"'] \\
\mathrm{M} \arrow[u, shift right, "\mathrm{O}\circ-"'] \arrow[r, shift left, "L"] & \mathrm{N} \arrow[u, shift right] \arrow[l, shift left]
\end{tikzcd}
\qquad
\begin{tikzcd}
LUA \arrow[rr, "\chi_A", "\text{comparison map}"'] \arrow[dr, bend right, "(U\varphi)\chi_A"'] & & U\overline{L}A \arrow[dl, bend left, "U\varphi"] \\
& UB \in \mathrm{N} &
\end{tikzcd}$$

Key Lemma : For cofibrant $A \in \mathrm{Alg}(\mathrm{O})$, χ_A is a weak equivalence.

- Up to retract, $\varnothing = A^0 \longrightarrow A^1 \longrightarrow A^2 \longrightarrow \cdots \longrightarrow A$, where for $t \geq 1$

$$\begin{tikzcd}
\mathrm{O} \circ X \arrow[r] \arrow[d, "\mathrm{O}\circ i_t"'] \arrow[dr, phantom, "\text{pushout}"] & A^{t-1} \arrow[d] \\
\mathrm{O} \circ Y \arrow[r] & A^t
\end{tikzcd}$$

  for some generating cofibration $i_t : X_t \longrightarrow Y_t$ in M.
- Use this filtration to successively approximate the comparison map χ_A.

Special cases of the Main Theorems

$$\begin{tikzcd}
\mathsf{Alg}(\mathsf{O}) \arrow[r, shift left, "\overline{L}"] \arrow[d, shift left, "U"] & \mathsf{Alg}(\mathsf{P}) \arrow[l, shift left, "R"] \arrow[d, shift left, "U"] \\
\mathsf{M} \arrow[u, shift left, "\mathsf{O}\circ-"] \arrow[r, shift left, "L"] & \mathsf{N} \arrow[l, shift left, "R"] \arrow[u, shift left, "\mathsf{P}\circ-"]
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
\text{operad algebra categories} \arrow[d, shift left, "\text{forget}"] \\
\text{monoidal model categories} \arrow[u, shift left, "\text{free}"]
\end{tikzcd}$$

1. (Rectification) $Id : \mathsf{M} \rightleftarrows \mathsf{N} : Id$ nice, $\mathsf{O} \xrightarrow{\sim} \mathsf{P}$ of entrywise cofibrant operads

   $$\mathsf{Alg}(\mathsf{O}) \underset{\sim}{\rightleftarrows} \mathsf{Alg}(\mathsf{P})$$

   Similar rectification : Berger-Moerdijk, Elmendorf-Mandell, Harper, Muro, Pavlov-Scholbach

2. Fixed operad $\mathsf{O} = \mathsf{P}$:

   $$\begin{tikzcd}
   \mathsf{Alg}_{\mathsf{M}}(\mathsf{O}) \arrow[r, shift left] \arrow[d, shift left] & \mathsf{Alg}_{\mathsf{N}}(\mathsf{O}) \arrow[l, shift left, "\sim"] \arrow[d, shift left] \\
   \mathsf{M} \arrow[u, shift left] \arrow[r, shift left] & \mathsf{N} \arrow[l, shift left, "\sim"] \arrow[u, shift left]
   \end{tikzcd}$$

   For example, $\mathsf{O} = \mathsf{P} =$ As (Schwede-Shipley), Com (Richter-Shipley), Op (Berger-Moerdijk), $\mathsf{Op}^{\mathrm{non}-\Sigma}$ (Muro), Σ-cofibrant operads (Fresse)

$\mathsf{M} = \mathsf{N} = \mathsf{Ch}_{\geq 0}(\Bbbk)$, $\Bbbk$ a field of characteristic 0

$$\begin{tikzcd}
\mathsf{Alg}(\mathsf{O}) \arrow[r, shift left, "\overline{L}"] \arrow[d, shift left, "\text{forget}"] & \mathsf{Alg}(\mathsf{P}) \arrow[l, shift left, "R", "\sim"'] \arrow[d, shift left, "\text{forget}"] \\
\mathsf{Ch}(\Bbbk)_{\geq 0} \arrow[u, shift left, "\text{free}"] \arrow[r, equal] & \mathsf{Ch}(\Bbbk)_{\geq 0} \arrow[u, shift left, "\text{free}"]
\end{tikzcd}$$

cofibrant replacement $\mathsf{O} \xrightarrow{\sim} \mathsf{P}$	Quillen equivalence $\mathsf{Alg}(\mathsf{O}) \rightleftarrows_{\sim} \mathsf{Alg}(\mathsf{P})$
$A_\infty \xrightarrow{\sim} \mathsf{As}$	A_∞-algebras $\rightleftarrows_{\sim}$ DGA
$E_\infty \xrightarrow{\sim} \mathsf{Com}$	E_∞-algebras $\rightleftarrows_{\sim}$ CDGA
$L_\infty \xrightarrow{\sim} \mathsf{Lie}$	L_∞-algebras $\rightleftarrows_{\sim}$ Lie algebras

Commutative monoids / algebras and operads

Corollary

Suppose $L : \mathsf{M} \rightleftarrows \mathsf{N} : R$ *is a nice Quillen equivalence.*

1. *If the monoidal units in* M, N *are cofibrant, then:*

commutative monoids	$\mathsf{CMonoid}(\mathsf{M}) \underset{\sim}{\rightleftarrows} \mathsf{CMonoid}(\mathsf{N})$
operads	$\mathsf{Operad}(\mathsf{M}) \underset{\sim}{\rightleftarrows} \mathsf{Operad}(\mathsf{N})$
enriched categories, objects $= \mathfrak{C}$	$\mathsf{Cat}^{\mathfrak{C}}(\mathsf{M}) \underset{\sim}{\rightleftarrows} \mathsf{Cat}^{\mathfrak{C}}(\mathsf{N})$

2. *If L is lax symmetric monoidal, and* $T \in \mathsf{CMonoid}(\mathsf{M})$ *cofibrant in* M*, then:*

$$\mathsf{CAlg}(T) \underset{\sim}{\rightleftarrows} \mathsf{CAlg}(LT)$$

$\mathsf{CAlg}(T)$ = *commutative* T*-algebras = commutative monoids in* $\mathsf{Mod}(T)$

Recover Quillen's Quillen equivalence in *Rational homotopy theory*:

- start with Dold-Kan between reduced dg/simplicial $\Bbbk$-modules
- apply Main Theorem to O = P = Lie operad
- $(\cdots)^{\mathrm{r}}$ = reduced

$$\begin{tikzcd} (\mathrm{DGLie})^{\mathrm{r}} \arrow[r, shift left, "\mathrm{Quillen}"] \arrow[d, shift left, "\mathrm{forget}"] & (\mathrm{SLie})^{\mathrm{r}} \arrow[l, shift left, "\sim"] \arrow[d, shift left, "\mathrm{forget}"] \\ \mathrm{Ch}^{\mathrm{r}}_{\geq 0}(\Bbbk) \arrow[u, shift left, "\mathrm{free}"] \arrow[r, shift left, "\mathrm{Dold\text{-}Kan}"] & \mathrm{SMod}^{\mathrm{r}}(\Bbbk) \arrow[l, shift left, "\sim"] \arrow[u, shift left, "\mathrm{free}"] \end{tikzcd}$$

Quillen's original proof is a calculation of the normalization

$$\mathrm{DGLie} \xleftarrow{N} \mathrm{SLie}\ .$$

Shipley : HR-algebra spectra $\simeq$ DGA

Over each commutative unital ring R, there are Quillen equivalences:

$$\mathrm{Mod}(HR) \underset{U}{\overset{Z}{\underset{\sim}{\rightleftarrows}}} \mathrm{Sp}^{\Sigma}(\mathrm{SMod}) \underset{N}{\overset{L}{\underset{\sim}{\leftrightarrows}}} \mathrm{Sp}^{\Sigma}(\mathrm{Ch}_{\geq 0}) \underset{C}{\overset{D}{\underset{\sim}{\rightleftarrows}}} \mathrm{Ch} \qquad (\dagger)$$

- HR : Eilenberg-Mac Lane spectrum of R
- $\mathrm{Sp}^{\Sigma}(\cdots)$: symmetric spectra in $(\cdots)$
- U : induced by forgetful $\mathrm{SMod} \longrightarrow \mathrm{SSet}_*$
- N : induced by normalization $\mathrm{SMod} \longrightarrow \mathrm{Ch}_{\geq 0}$
- $C = \left\{ C_0(? \otimes R[m]) \right\}_{m \geq 0}$, C_0 = connective cover

Apply the Main Theorem to $(\dagger)$ and the associative operad As:

Theorem (Shipley)

Each Quillen equivalence in $(\dagger)$ lifts to monoids, so

$$\mathrm{Alg}(HR) \underset{\sim}{\rightleftarrows} \mathrm{Monoid}(\cdots) \underset{\sim}{\leftrightarrows} \mathrm{Monoid}(\cdots) \underset{\sim}{\rightleftarrows} \mathrm{DGA}\,.$$

Commutative HR-algebra spectra $\simeq$ CDGA

Applying the Main Theorem to Shipley's Quillen equivalences

$$\mathrm{Mod}(HR) \underset{U}{\overset{Z}{\underset{\sim}{\rightleftarrows}}} \mathrm{Sp}^{\Sigma}(\mathrm{SMod}) \underset{N}{\overset{L}{\underset{\sim}{\leftrightarrows}}} \mathrm{Sp}^{\Sigma}(\mathrm{Ch}_{\geq 0}) \underset{C}{\overset{D}{\underset{\sim}{\rightleftarrows}}} \mathrm{Ch}$$

and the commutative operad Com:

Corollary (White-Y.)

If char$(R) = 0$, then there is a zig-zag of three Quillen equivalences:

$$\mathrm{CAlg}(HR) \underset{\sim}{\rightleftarrows} \mathrm{CMonoid}(\cdots) \underset{\sim}{\leftrightarrows} \mathrm{CMonoid}(\cdots) \underset{\sim}{\rightleftarrows} \mathrm{CDGA}$$

- char$(R) = 0$ is needed to make sure CDGA inherits a model structure.
- Shipley suggested that this might be true, but did not prove it.
- This is an improvement of a recent result of Richter-Shipley.

Commutative HR-algebra spectra $\simeq$ CDGA

Theorem (Richter-Shipley)

If char$(R) = 0$, then there is a zig-zag of six Quillen equivalences:

$$
\begin{tikzcd}
\mathrm{CAlg}(HR) \arrow[r, shift left] & C\big(\mathrm{Sp}^\Sigma(\mathrm{SMod})\big) \arrow[l, shift left, "\sim"] \arrow[r, shift right, "\sim"'] & C\big(\mathrm{Sp}^\Sigma(\mathrm{Ch}_{\geq 0})\big) \arrow[l, shift right] \arrow[r, shift left] & C\big(\mathrm{Sp}^\Sigma(\mathrm{Ch})\big) \arrow[l, shift left, "\sim"] \arrow[d, shift left] \\
\mathrm{CDGA} \arrow[r, shift right, "\sim"'] & E_\infty \mathrm{Ch} \arrow[l, shift right] \arrow[r, shift right, "\sim"'] & E_\infty\big(\mathrm{Sp}^\Sigma(\mathrm{Ch})\big) \arrow[l, shift right] & E_\infty\big(\mathrm{Sp}^\Sigma(\mathrm{Ch})\big) \arrow[u, shift left, "\sim"]
\end{tikzcd}
$$

All six Quillen equivalences are consequences of the Main Theorem:

- $\longrightarrow$ uses Com. $\longrightarrow$ uses E_∞-operad. $\longrightarrow$ uses $E_\infty \xrightarrow{\sim}$ Com

However, our version

Corollary (White-Y.)

If char$(R) = 0$, then there is a zig-zag of three Quillen equivalences:

$$
\mathrm{CAlg}(HR) \underset{\sim}{\rightleftarrows} C\big(\mathrm{Sp}^\Sigma(\mathrm{SMod})\big) \underset{\sim}{\leftrightarrows} C\big(\mathrm{Sp}^\Sigma(\mathrm{Ch}_{\geq 0})\big) \underset{\sim}{\rightleftarrows} \mathrm{CDGA}
$$

- involves three instead of six Quillen equivalences
- does not go through E_∞-algebras
- every Quillen equivalence is an application of the same theorem