The Johnson filtration is finitely generated

Andrew Putman

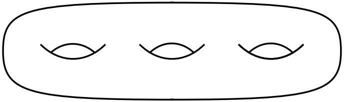
University of Notre Dame

MIT Topology Seminar

Mapping class group

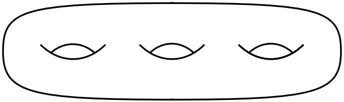
Mapping class group

$\Sigma_g =$ cpt oriented genus g surface



Mapping class group

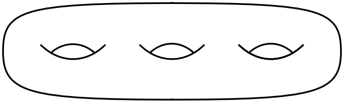
Σ_g = cpt oriented genus g surface



The mapping class group Mod_g is group of homotopy classes of orientation-preserving diffeomorphisms $f\colon \Sigma_g \to \Sigma_g$.

Mapping class group

Σ_g = cpt oriented genus g surface

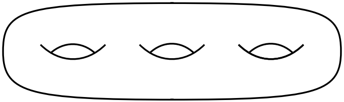


The mapping class group Mod_g is group of homotopy classes of orientation-preserving diffeomorphisms $f: \Sigma_g \to \Sigma_g$.

Basic object in topology:

Mapping class group

Σ_g = cpt oriented genus g surface



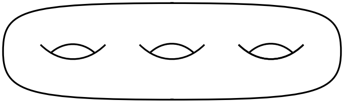
The mapping class group Mod_g is group of homotopy classes of orientation-preserving diffeomorphisms $f\colon \Sigma_g \to \Sigma_g$.

Basic object in topology:

- gluing data for 3-manifolds.

Mapping class group

Σ_g = cpt oriented genus g surface



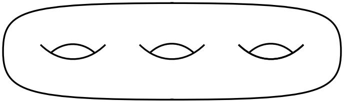
The mapping class group Mod_g is group of homotopy classes of orientation-preserving diffeomorphisms $f : \Sigma_g \to \Sigma_g$.

Basic object in topology:

- gluing data for 3-manifolds.
- monodromies of Σ_g-bundles (e.g. families of alg. curves).

Mapping class group

Σ_g = cpt oriented genus g surface



The mapping class group Mod_g is group of homotopy classes of orientation-preserving diffeomorphisms $f : \Sigma_g \to \Sigma_g$.

Basic object in topology:

- gluing data for 3-manifolds.
- monodromies of Σ_g-bundles (e.g. families of alg. curves).
- π_1 of moduli space of algebraic curves.

Dehn twists

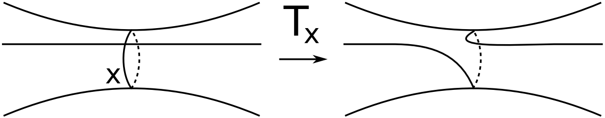
Dehn twists

x = simple closed curve

Dehn twists

x = simple closed curve

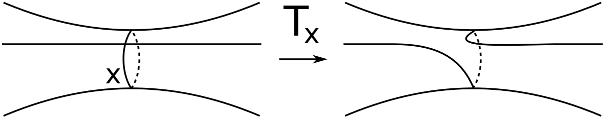
$T_x \in \mathrm{Mod}_g$ = cut, twist, reglue.



Dehn twists

x = simple closed curve

$T_x \in \mathrm{Mod}_g$ = cut, twist, reglue.



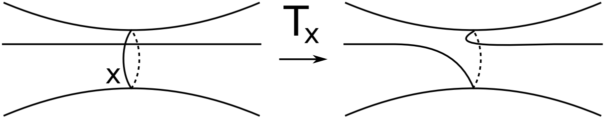
Theorem (Dehn)

Mod_g *is gen by fin many Dehn twists.*

Dehn twists

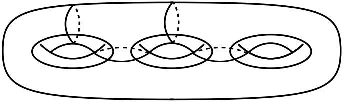
x = simple closed curve

$T_x \in \mathrm{Mod}_g$ = cut, twist, reglue.



Theorem (Dehn)

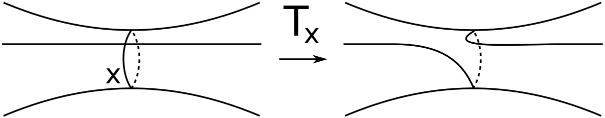
Mod_g *is gen by fin many Dehn twists.*



Dehn twists

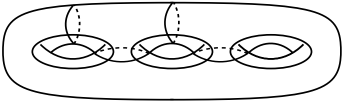
x = simple closed curve

$T_x \in \mathrm{Mod}_g$ = cut, twist, reglue.



Theorem (Dehn)

Mod_g *is gen by fin many Dehn twists.*



Mod_g has many other finiteness properties: finitely presentable (McCool, Hatcher–Thurston), all H_k finitely generated (Harer?), etc.

Torelli group

Torelli group

$\mathrm{Mod}_g \circlearrowright \mathrm{H}_1(\Sigma_g)$, preserves alg. isect. pairing.

Torelli group

$\mathrm{Mod}_g \circlearrowright \mathrm{H}_1(\Sigma_g)$, preserves alg. isect. pairing.

$$\rightsquigarrow \mathrm{Mod}_g \to \mathrm{Sp}_{2g}(\mathbb{Z}).$$

Torelli group

$\mathsf{Mod}_g \circlearrowright \mathsf{H}_1(\Sigma_g)$, preserves alg. isect. pairing.

$$\rightsquigarrow \mathsf{Mod}_g \to \mathsf{Sp}_{2g}(\mathbb{Z}).$$

Torelli group $\mathcal{I}_g$ is kernel: mapping classes acting trivially on $\mathsf{H}_1(\Sigma_g)$.

Torelli group

$\mathrm{Mod}_g \circlearrowright \mathrm{H}_1(\Sigma_g)$, preserves alg. isect. pairing.

$$\rightsquigarrow \mathrm{Mod}_g \to \mathrm{Sp}_{2g}(\mathbb{Z}).$$

Torelli group $\mathcal{I}_g$ is kernel: mapping classes acting trivially on $\mathrm{H}_1(\Sigma_g)$.

$$1 \longrightarrow \mathcal{I}_g \longrightarrow \mathrm{Mod}_g \longrightarrow \mathrm{Sp}_{2g}(\mathbb{Z}) \longrightarrow 1$$

Examples of elts of Torelli

Examples of elts of Torelli

$\mathcal{I}_g = \ker(\mathrm{Mod}_g \circlearrowright \mathrm{H}_1(\Sigma_g))$

Examples of elts of Torelli

$\mathcal{I}_g = \ker(\mathrm{Mod}_g \circlearrowright \mathrm{H}_1(\Sigma_g))$

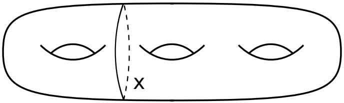
Action of Dehn twist T_x on $\mathrm{H}_1(\Sigma_g)$ determined by $[x] \in \mathrm{H}_1(\Sigma_g)$.

Examples of elts of Torelli

$\mathcal{I}_g = \ker(\mathrm{Mod}_g \circlearrowright \mathrm{H}_1(\Sigma_g))$

Action of Dehn twist T_x on $\mathrm{H}_1(\Sigma_g)$ determined by $[x] \in \mathrm{H}_1(\Sigma_g)$.

Separating twists: T_x w/ $[x] = 0$, i.e. x separating.

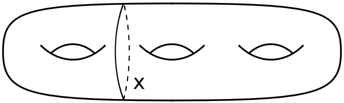


Examples of elts of Torelli

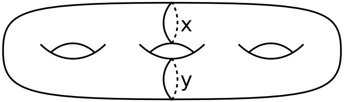
$\mathcal{I}_g = \ker(\mathrm{Mod}_g \circlearrowright \mathrm{H}_1(\Sigma_g))$

Action of Dehn twist T_x on $\mathrm{H}_1(\Sigma_g)$ determined by $[x] \in \mathrm{H}_1(\Sigma_g)$.

Separating twists: T_x w/ $[x] = 0$, i.e. x separating.



Bounding pair: $T_x T_y^{-1}$ w/ $x \cap y = \emptyset$ and $[x] = [y]$, i.e. $x \cup y$ bounds.

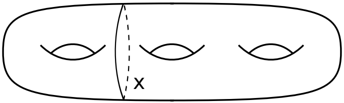


Examples of elts of Torelli

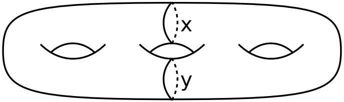
$\mathcal{I}_g = \ker(\mathrm{Mod}_g \circlearrowright \mathrm{H}_1(\Sigma_g))$

Action of Dehn twist T_x on $\mathrm{H}_1(\Sigma_g)$ determined by $[x] \in \mathrm{H}_1(\Sigma_g)$.

Separating twists: T_x w/ $[x] = 0$, i.e. x separating.



Bounding pair: $T_x T_y^{-1}$ w/ $x \cap y = \emptyset$ and $[x] = [y]$, i.e. $x \cup y$ bounds.



Theorem (Birman, Powell)

$\mathcal{I}_g$ is gen. by sep twists and bounding pairs.

Finiteness properties of Torelli

Finiteness properties of Torelli

Theorem (Classical)

$\mathcal{I}_1 = 1$, *so* $\mathsf{Mod}_1 \cong \mathsf{SL}_2(\mathbb{Z})$.

Finiteness properties of Torelli

Theorem (Classical)

$\mathcal{I}_1 = 1$, *so* $\mathrm{Mod}_1 \cong \mathrm{SL}_2(\mathbb{Z})$.

Theorem (Mess)

$\mathcal{I}_2$ *an* ∞*-rank free group.*

Finiteness properties of Torelli

Theorem (Classical)

$\mathcal{I}_1 = 1$, so $\mathrm{Mod}_1 \cong \mathrm{SL}_2(\mathbb{Z})$.

Theorem (Mess)

$\mathcal{I}_2$ an ∞-rank free group.

Theorem (Johnson)

$\mathcal{I}_g$ is fin gen for $g \geq 3$.

Finiteness properties of Torelli

Theorem (Classical)

$\mathcal{I}_1 = 1$, *so* $\mathrm{Mod}_1 \cong \mathrm{SL}_2(\mathbb{Z})$.

Theorem (Mess)

$\mathcal{I}_2$ *an* ∞*-rank free group.*

Theorem (Johnson)

$\mathcal{I}_g$ *is fin gen for* $g \geq 3$.

Open question

Is $\mathcal{I}_g$ fin pres?

Johnson kernel

Johnson kernel

Theorem (Johnson)

For $g \geq 3$, following are same subgroup $\mathcal{K}_g$ of $\mathcal{I}_g$ (Johnson kernel):

Johnson kernel

Theorem (Johnson)

For $g \geq 3$, following are same subgroup $\mathcal{K}_g$ of $\mathcal{I}_g$ (Johnson kernel):

- $\mathcal{K}_g = \ker(\mathcal{I}_g \to \mathrm{H}_1(\mathcal{I}_g)/\textit{torsion})$.

Johnson kernel

Theorem (Johnson)

For $g \geq 3$, following are same subgroup $\mathcal{K}_g$ of $\mathcal{I}_g$ (Johnson kernel):

- $\mathcal{K}_g = \ker(\mathcal{I}_g \to \mathsf{H}_1(\mathcal{I}_g)/\textit{torsion})$.
- $\mathcal{K}_g$ *subgroup of* $\mathcal{I}_g$ *gen. by sep twists.*

Johnson kernel

Theorem (Johnson)

For $g \geq 3$, following are same subgroup $\mathcal{K}_g$ of $\mathcal{I}_g$ (Johnson kernel):

- $\mathcal{K}_g = \ker(\mathcal{I}_g \to \mathrm{H}_1(\mathcal{I}_g)/\mathit{torsion})$.
- $\mathcal{K}_g$ *subgroup of* $\mathcal{I}_g$ *gen. by sep twists.*
- $\mathcal{K}_g$ *is* $f \in \mathcal{I}_g$ *s.t. map. torus* M_f *has same cup products as* $\Sigma_g \times S^1$.

Johnson kernel

Theorem (Johnson)

For $g \geq 3$, following are same subgroup $\mathcal{K}_g$ of $\mathcal{I}_g$ (Johnson kernel):

- $\mathcal{K}_g = \ker(\mathcal{I}_g \to \mathrm{H}_1(\mathcal{I}_g)/torsion)$.
- $\mathcal{K}_g$ *subgroup of* $\mathcal{I}_g$ *gen. by sep twists.*
- $\mathcal{K}_g$ *is* $f \in \mathcal{I}_g$ *s.t. map. torus* M_f *has same cup products as* $\Sigma_g \times S^1$.

Remark

For all $f \in \mathcal{I}_g$, M_f has homology of $\Sigma_g \times S^1$.

Johnson kernel

Theorem (Johnson)

For $g \geq 3$, following are same subgroup $\mathcal{K}_g$ of $\mathcal{I}_g$ (Johnson kernel):

- $\mathcal{K}_g = \ker(\mathcal{I}_g \to \mathrm{H}_1(\mathcal{I}_g)/\mathit{torsion})$.
- *$\mathcal{K}_g$ subgroup of $\mathcal{I}_g$ gen. by sep twists.*
- *$\mathcal{K}_g$ is $f \in \mathcal{I}_g$ s.t. map. torus M_f has same cup products as $\Sigma_g \times S^1$.*

Remark

For all $f \in \mathcal{I}_g$, M_f has homology of $\Sigma_g \times S^1$.

Observation

$\mathcal{K}_g = \ker(\mathcal{I}_g \to \mathrm{H}_1(\mathcal{I}_g)/\text{torsion}) \Rightarrow \mathcal{K}_g$ is commensurable with $[\mathcal{I}_g, \mathcal{I}_g]$.

Johnson kernel

Theorem (Johnson)

For $g \geq 3$, following are same subgroup $\mathcal{K}_g$ of $\mathcal{I}_g$ (Johnson kernel):

- $\mathcal{K}_g = \ker(\mathcal{I}_g \to \mathrm{H}_1(\mathcal{I}_g)/\mathit{torsion})$.
- $\mathcal{K}_g$ *subgroup of* $\mathcal{I}_g$ *gen. by sep twists.*
- $\mathcal{K}_g$ *is* $f \in \mathcal{I}_g$ *s.t. map. torus* M_f *has same cup products as* $\Sigma_g \times S^1$.

Remark

For all $f \in \mathcal{I}_g$, M_f has homology of $\Sigma_g \times S^1$.

Observation

$\mathcal{K}_g = \ker(\mathcal{I}_g \to \mathrm{H}_1(\mathcal{I}_g)/\text{torsion}) \Rightarrow \mathcal{K}_g$ is commensurable with $[\mathcal{I}_g, \mathcal{I}_g]$.
$\Rightarrow \mathcal{K}_g$ has same finiteness properties as $[\mathcal{I}_g, \mathcal{I}_g]$.

Lower central series of Torelli

Lower central series of Torelli

The lower central series of a group G is

$$\gamma_1(G) = G \quad \text{and} \quad \gamma_{k+1}(G) = [\gamma_k(G), G].$$

Lower central series of Torelli

The lower central series of a group G is

$$\gamma_1(G) = G \quad \text{and} \quad \gamma_{k+1}(G) = [\gamma_k(G), G].$$

$$G = \gamma_1(G) \supset \gamma_2(G) \supset \gamma_3(G) \supset \cdots$$

Lower central series of Torelli

The lower central series of a group G is

$$\gamma_1(G) = G \quad \text{and} \quad \gamma_{k+1}(G) = [\gamma_k(G), G].$$

$$G = \gamma_1(G) \supset \gamma_2(G) \supset \gamma_3(G) \supset \cdots$$

$\mathcal{I}_g$ is residually nilpotent:

Lower central series of Torelli

The lower central series of a group G is

$$\gamma_1(G) = G \quad \text{and} \quad \gamma_{k+1}(G) = [\gamma_k(G), G].$$

$$G = \gamma_1(G) \supset \gamma_2(G) \supset \gamma_3(G) \supset \cdots$$

$\mathcal{I}_g$ is residually nilpotent:

$$\bigcap_{k=1}^{\infty} \gamma_k(\mathcal{I}_g) = 1.$$

Lower central series of Torelli

The lower central series of a group G is

$$\gamma_1(G) = G \quad \text{and} \quad \gamma_{k+1}(G) = [\gamma_k(G), G].$$

$$G = \gamma_1(G) \supset \gamma_2(G) \supset \gamma_3(G) \supset \cdots$$

$\mathcal{I}_g$ is residually nilpotent:

$$\bigcap_{k=1}^{\infty} \gamma_k(\mathcal{I}_g) = 1.$$

Each $\gamma_{k+1}(\mathcal{I}_g)$ is infinite-index normal subgroup of $\gamma_k(\mathcal{I}_g)$ with $\gamma_k(\mathcal{I}_g)/\gamma_{k+1}(\mathcal{I}_g)$ fin gen abelian group.

Lower central series of Torelli

The lower central series of a group G is

$$\gamma_1(G) = G \quad \text{and} \quad \gamma_{k+1}(G) = [\gamma_k(G), G].$$

$$G = \gamma_1(G) \supset \gamma_2(G) \supset \gamma_3(G) \supset \cdots$$

$\mathcal{I}_g$ is residually nilpotent:

$$\bigcap_{k=1}^{\infty} \gamma_k(\mathcal{I}_g) = 1.$$

Each $\gamma_{k+1}(\mathcal{I}_g)$ is infinite-index normal subgroup of $\gamma_k(\mathcal{I}_g)$ with $\gamma_k(\mathcal{I}_g)/\gamma_{k+1}(\mathcal{I}_g)$ fin gen abelian group.

$\Rightarrow$ naively, expects finiteness of $\gamma_k(\mathcal{I}_g)$ to get worse as $k \mapsto \infty$.

Deeper finiteness properties

Deeper finiteness properties

Theorem (Dimca–Papadima, 2007)

$H_1([\mathcal{I}_g, \mathcal{I}_g]; \mathbb{Q})$ *is fin. dim. for* $g \geq 4$.

Deeper finiteness properties

Theorem (Dimca–Papadima, 2007)

$H_1([\mathcal{I}_g, \mathcal{I}_g]; \mathbb{Q})$ *is fin. dim. for* $g \geq 4$.

Theorem (Ershov–He, 2017)

$[\mathcal{I}_g, \mathcal{I}_g]$ *is fin. gen. for* $g \geq 12$.

Deeper finiteness properties

Theorem (Dimca–Papadima, 2007)

$H_1([\mathcal{I}_g, \mathcal{I}_g]; \mathbb{Q})$ *is fin. dim. for* $g \geq 4$.

Theorem (Ershov–He, 2017)

$[\mathcal{I}_g, \mathcal{I}_g]$ *is fin. gen. for* $g \geq 12$.

Theorem (Church–Ershov–P, 2017)

$\gamma_k(\mathcal{I}_g)$ *is fin. gen. for* $g \geq \max(2k-1, 4)$

Deeper finiteness properties

Theorem (Dimca–Papadima, 2007)
$H_1([\mathcal{I}_g, \mathcal{I}_g]; \mathbb{Q})$ *is fin. dim. for* $g \geq 4$.

Theorem (Ershov–He, 2017)
$[\mathcal{I}_g, \mathcal{I}_g]$ *is fin. gen. for* $g \geq 12$.

Theorem (Church–Ershov–P, 2017)
$\gamma_k(\mathcal{I}_g)$ *is fin. gen. for* $g \geq \max(2k-1, 4)$

Goal for rest of talk
Prove that $[\mathcal{I}_g, \mathcal{I}_g]$ (and hence Johnson kernel) is fin gen. for $g \geq 4$.

Bieri–Neumann-Strebel (BNS) invariants

Bieri–Neumann-Strebel (BNS) invariants

G grp w/ fin genset S.

Bieri–Neumann-Strebel (BNS) invariants

G grp w/ fin genset S.

Cayley graph: $\mathrm{Cay}(G, S) =$ vertices G, edges g–gs for $g \in G$, $s \in S$.

Bieri–Neumann-Strebel (BNS) invariants

G grp w/ fin genset S.

Cayley graph: $\mathrm{Cay}(G, S)$ = vertices G, edges g–gs for $g \in G$, $s \in S$.

Characters: $G^* = \mathrm{Hom}(G, \mathbb{R})$.

Bieri–Neumann-Strebel (BNS) invariants

G grp w/ fin genset S.

Cayley graph: $\mathrm{Cay}(G, S) =$ vertices G, edges g–gs for $g \in G$, $s \in S$.

Characters: $G^* = \mathrm{Hom}(G, \mathbb{R})$.

Definition

The BNS invariant $\Sigma(G) \subset G^*$ is set of all $f \in G^*$ s.t. $\{g \in G \mid f(g) \geq 0\}$ is connected subgraph of $\mathrm{Cay}(G, S)$.

Bieri–Neumann-Strebel (BNS) invariants

G grp w/ fin genset S.

Cayley graph: $\mathrm{Cay}(G, S)$ = vertices G, edges g–gs for $g \in G$, $s \in S$.

Characters: $G^* = \mathrm{Hom}(G, \mathbb{R})$.

Definition

The BNS invariant $\Sigma(G) \subset G^*$ is set of all $f \in G^*$ s.t. $\{g \in G \mid f(g) \geq 0\}$ is connected subgraph of $\mathrm{Cay}(G, S)$.

Nonobvious fact: independent of genset S.

Definition

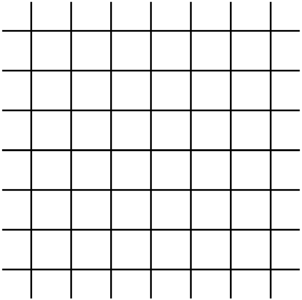
The BNS invariant $\Sigma(G) \subset G^* = \text{Hom}(G, \mathbb{R})$ is set of all $f \in G^*$ s.t. $\{g \in G \mid f(g) \geq 0\}$ is connected subgraph of $\text{Cay}(G, S)$.

Example: $\Sigma(\mathbb{Z}^n) = (\mathbb{Z}^n)^*$

Definition

The BNS invariant $\Sigma(G) \subset G^* = \mathrm{Hom}(G, \mathbb{R})$ is set of all $f \in G^*$ s.t. $\{g \in G \mid f(g) \geq 0\}$ is connected subgraph of $\mathrm{Cay}(G, S)$.

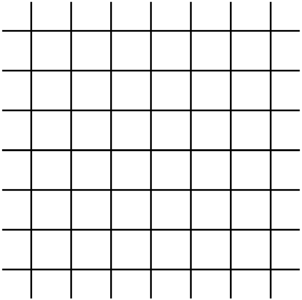
Example: $\Sigma(\mathbb{Z}^n) = (\mathbb{Z}^n)^*$



Definition

The BNS invariant $\Sigma(G) \subset G^* = \mathrm{Hom}(G, \mathbb{R})$ is set of all $f \in G^*$ s.t. $\{g \in G \mid f(g) \geq 0\}$ is connected subgraph of $\mathrm{Cay}(G, S)$.

Example: $\Sigma(\mathbb{Z}^n) = (\mathbb{Z}^n)^*$

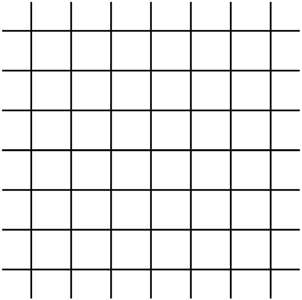


Consider nonzero $f \in (\mathbb{Z}^n)^*$.

Definition

The BNS invariant $\Sigma(G) \subset G^* = \text{Hom}(G, \mathbb{R})$ is set of all $f \in G^*$ s.t. $\{g \in G \mid f(g) \geq 0\}$ is connected subgraph of $\text{Cay}(G, S)$.

Example: $\Sigma(\mathbb{Z}^n) = (\mathbb{Z}^n)^*$

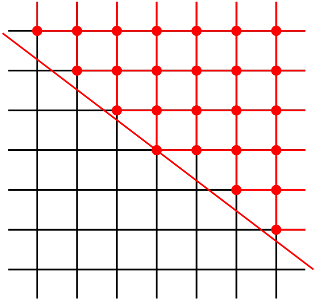


Consider nonzero $f \in (\mathbb{Z}^n)^*$.
$f(x) \geq 0$ is a halfspace.

Definition

The BNS invariant $\Sigma(G) \subset G^* = \mathrm{Hom}(G, \mathbb{R})$ is set of all $f \in G^*$ s.t. $\{g \in G \mid f(g) \geq 0\}$ is connected subgraph of $\mathrm{Cay}(G, S)$.

Example: $\Sigma(\mathbb{Z}^n) = (\mathbb{Z}^n)^*$



Consider nonzero $f \in (\mathbb{Z}^n)^*$.
$f(x) \geq 0$ is a halfspace.

Definition

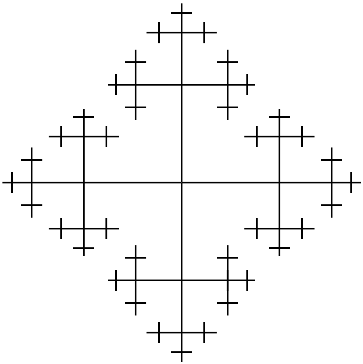
The BNS invariant $\Sigma(G) \subset G^* = \mathrm{Hom}(G, \mathbb{R})$ is set of all $f \in G^*$ s.t. $\{g \in G \mid f(g) \geq 0\}$ is connected subgraph of $\mathrm{Cay}(G, S)$.

Example: $\Sigma(F_2) = 0$

Definition

The BNS invariant $\Sigma(G) \subset G^* = \mathrm{Hom}(G, \mathbb{R})$ is set of all $f \in G^*$ s.t. $\{g \in G \mid f(g) \geq 0\}$ is connected subgraph of $\mathrm{Cay}(G, S)$.

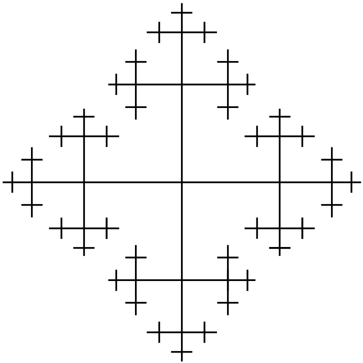
Example: $\Sigma(F_2) = 0$



Definition

The BNS invariant $\Sigma(G) \subset G^* = \mathrm{Hom}(G, \mathbb{R})$ is set of all $f \in G^*$ s.t. $\{g \in G \mid f(g) \geq 0\}$ is connected subgraph of $\mathrm{Cay}(G, S)$.

Example: $\Sigma(F_2) = 0$

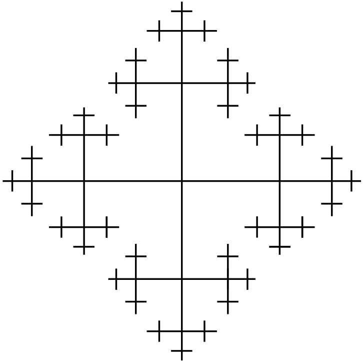


$F_2 = \langle a, b \rangle$

Definition

The BNS invariant $\Sigma(G) \subset G^* = \mathrm{Hom}(G, \mathbb{R})$ is set of all $f \in G^*$ s.t. $\{g \in G \mid f(g) \geq 0\}$ is connected subgraph of $\mathrm{Cay}(G, S)$.

Example: $\Sigma(F_2) = 0$

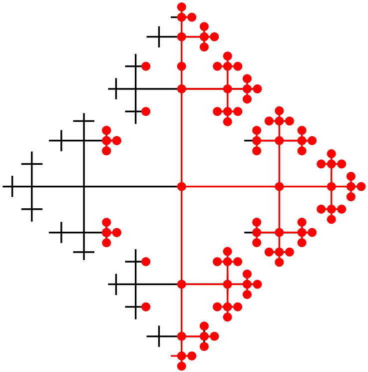


$F_2 = \langle a, b \rangle$
Consider $f\colon F_2 \to \mathbb{R}$, $f(a) = 1$ and $f(b) = 0$.

Definition

The BNS invariant $\Sigma(G) \subset G^* = \mathrm{Hom}(G, \mathbb{R})$ is set of all $f \in G^*$ s.t. $\{g \in G \mid f(g) \geq 0\}$ is connected subgraph of $\mathrm{Cay}(G, S)$.

Example: $\Sigma(F_2) = 0$

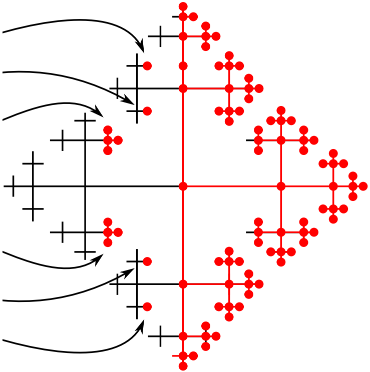


$F_2 = \langle a, b \rangle$
Consider $f: F_2 \to \mathbb{R}$, $f(a) = 1$ and $f(b) = 0$.

Definition

The BNS invariant $\Sigma(G) \subset G^* = \mathrm{Hom}(G, \mathbb{R})$ is set of all $f \in G^*$ s.t. $\{g \in G \mid f(g) \geq 0\}$ is connected subgraph of $\mathrm{Cay}(G, S)$.

Example: $\Sigma(F_2) = 0$



$F_2 = \langle a, b \rangle$

Consider $f\colon F_2 \to \mathbb{R}$, $f(a) = 1$ and $f(b) = 0$.

BNS Properties

BNS Properties

Basic facts:

BNS Properties

Basic facts:

- Cone on open subset of sphere.

BNS Properties

Basic facts:

- Cone on open subset of sphere.
- For 3-manifold group, is cone on interiors of fibered faces.

BNS Properties

Basic facts:

- Cone on open subset of sphere.
- For 3-manifold group, is cone on interiors of fibered faces.

Fundamental Theorem (Bieri–Neumann-Strebel)

G fin gen grp, $H < G$ w/ $[G, G] \subset H \subset G$.

BNS Properties

Basic facts:

- Cone on open subset of sphere.
- For 3-manifold group, is cone on interiors of fibered faces.

Fundamental Theorem (Bieri–Neumann-Strebel)

G fin gen grp, $H < G$ w/ $[G, G] \subset H \subset G$.

$$H \text{ is fin gen} \quad \Longleftrightarrow \quad \{f \in G^* \mid f|_H = 0\} \subset \Sigma(G).$$

BNS Properties

Basic facts:

- Cone on open subset of sphere.
- For 3-manifold group, is cone on interiors of fibered faces.

Fundamental Theorem (Bieri–Neumann-Strebel)

G fin gen grp, $H < G$ w/ $[G, G] \subset H \subset G$.

$$H \text{ is fin gen} \iff \{f \in G^* \mid f|_H = 0\} \subset \Sigma(G).$$

Previous examples reflect that all $0 \subset H \subset \mathbb{Z}^n$ are fin gen, but no $[F_n, F_n] \subset H \subset F_n$ are fin gen except H finite-index in F_n.

BNS Properties

Basic facts:

- Cone on open subset of sphere.
- For 3-manifold group, is cone on interiors of fibered faces.

Fundamental Theorem (Bieri–Neumann-Strebel)

G fin gen grp, $H < G$ w/ $[G, G] \subset H \subset G$.

$$H \text{ is fin gen} \iff \{f \in G^* \mid f|_H = 0\} \subset \Sigma(G).$$

Previous examples reflect that all $0 \subset H \subset \mathbb{Z}^n$ are fin gen, but no $[F_n, F_n] \subset H \subset F_n$ are fin gen except H finite-index in F_n.

Special Case

$[G, G]$ is fin gen iff $\Sigma(G) = G^*$.

Main goal

Main goal

Goal

$[\mathcal{I}_g, \mathcal{I}_g]$ is fin gen for $g \geq 4$

Main goal

Goal

$[\mathcal{I}_g, \mathcal{I}_g]$ is fin gen for $g \geq 4$, i.e. $\Sigma(\mathcal{I}_g) = (\mathcal{I}_g)^*$.

Main goal

Goal

$[\mathcal{I}_g, \mathcal{I}_g]$ is fin gen for $g \geq 4$, i.e. $\Sigma(\mathcal{I}_g) = (\mathcal{I}_g)^*$.

Step 1: Find large piece of BNS invariant

Exists fin genset $S \subset \mathcal{I}_g$ of genus 1 bounding pairs s.t.

$$\{f \in (\mathcal{I}_g)^* \mid f(s) \neq 0 \text{ for all } s \in S\} \subset \Sigma(\mathcal{I}_g).$$

Step 1: Find large piece of BNS invariant

Exists fin genset $S \subset \mathcal{I}_g$ of genus 1 bounding pairs s.t.

$$\{f \in (\mathcal{I}_g)^* \mid f(s) \neq 0 \text{ for all } s \in S\} \subset \Sigma(\mathcal{I}_g).$$

Step 1: Find large piece of BNS invariant

Exists fin genset $S \subset \mathcal{I}_g$ of genus 1 bounding pairs s.t.

$$\{f \in (\mathcal{I}_g)^* \mid f(s) \neq 0 \text{ for all } s \in S\} \subset \Sigma(\mathcal{I}_g).$$

Lemma (Folklore)

G grp w/ fin genset S. Assume graph w/ vertices S and edge between $s, s' \in S$ when $[s, s'] = 1$ connected. Then

$$\{f \in G^* \mid f(s) \neq 0 \textit{ for all } s \in S\} \subset \Sigma(G).$$

Step 1: Find large piece of BNS invariant

Exists fin genset $S \subset \mathcal{I}_g$ of genus 1 bounding pairs s.t.

$$\{f \in (\mathcal{I}_g)^* \mid f(s) \neq 0 \text{ for all } s \in S\} \subset \Sigma(\mathcal{I}_g).$$

Lemma (Folklore)

G grp w/ fin genset S. Assume graph w/ vertices S and edge between $s, s' \in S$ when $[s, s'] = 1$ connected. Then

$$\{f \in G^* \mid f(s) \neq 0 \textit{ for all } s \in S\} \subset \Sigma(G).$$

Lemma

For $g \geq 4$, graph w/ vertices genus 1 bounding pairs and edges between disjoint bounding pairs is connected.

Step 1: Find large piece of BNS invariant

Exists fin genset $S \subset \mathcal{I}_g$ of genus 1 bounding pairs s.t.

$$\{f \in (\mathcal{I}_g)^* \mid f(s) \neq 0 \text{ for all } s \in S\} \subset \Sigma(\mathcal{I}_g).$$

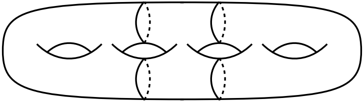
Lemma (Folklore)

G grp w/ fin genset S. Assume graph w/ vertices S and edge between $s, s' \in S$ when $[s, s'] = 1$ connected. Then

$$\{f \in G^* \mid f(s) \neq 0 \textit{ for all } s \in S\} \subset \Sigma(G).$$

Lemma

For $g \geq 4$, graph w/ vertices genus 1 bounding pairs and edges between disjoint bounding pairs is connected.



Step 1: Find large piece of BNS invariant

Exists fin genset $S \subset \mathcal{I}_g$ of genus 1 bounding pairs s.t.

$$\{f \in (\mathcal{I}_g)^* \mid f(s) \neq 0 \text{ for all } s \in S\} \subset \Sigma(\mathcal{I}_g).$$

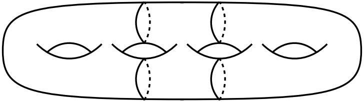
Lemma (Folklore)

G grp w/ fin genset S. Assume graph w/ vertices S and edge between $s, s' \in S$ when $[s, s'] = 1$ connected. Then

$$\{f \in G^* \mid f(s) \neq 0 \text{ for all } s \in S\} \subset \Sigma(G).$$

Lemma

For $g \geq 4$, graph w/ vertices genus 1 bounding pairs and edges between disjoint bounding pairs is connected.



Take S finite subgraph containing genset for $\mathcal{I}_g$.

Main goal

Goal

$[\mathcal{I}_g, \mathcal{I}_g]$ is fin gen for $g \geq 4$, i.e. $\Sigma(\mathcal{I}_g) = (\mathcal{I}_g)^*$.

Step 1: Find large piece of BNS invariant

Exists fin genset $S \subset \mathcal{I}_g$ of genus 1 bounding pairs s.t.

$$\{f \in (\mathcal{I}_g)^* \mid f(s) \neq 0 \text{ for all } s \in S\} \subset \Sigma(\mathcal{I}_g).$$

Main goal

Goal

$[\mathcal{I}_g, \mathcal{I}_g]$ is fin gen for $g \geq 4$, i.e. $\Sigma(\mathcal{I}_g) = (\mathcal{I}_g)^*$.

Step 1: Find large piece of BNS invariant

Exists fin genset $S \subset \mathcal{I}_g$ of genus 1 bounding pairs s.t.

$$\{f \in (\mathcal{I}_g)^* \mid f(s) \neq 0 \text{ for all } s \in S\} \subset \Sigma(\mathcal{I}_g).$$

$\mathrm{Mod}_g \circlearrowright \mathcal{I}_g$ by conjugation

Main goal

Goal

$[\mathcal{I}_g, \mathcal{I}_g]$ is fin gen for $g \geq 4$, i.e. $\Sigma(\mathcal{I}_g) = (\mathcal{I}_g)^*$.

Step 1: Find large piece of BNS invariant

Exists fin genset $S \subset \mathcal{I}_g$ of genus 1 bounding pairs s.t.

$$\{f \in (\mathcal{I}_g)^* \mid f(s) \neq 0 \text{ for all } s \in S\} \subset \Sigma(\mathcal{I}_g).$$

$\mathrm{Mod}_g \circlearrowright \mathcal{I}_g$ by conjugation $\rightsquigarrow$ $\mathrm{Mod}_g \circlearrowright (\mathcal{I}_g)^*$ preserving $\Sigma(\mathcal{I}_g)$.

Main goal

Goal

$[\mathcal{I}_g, \mathcal{I}_g]$ is fin gen for $g \geq 4$, i.e. $\Sigma(\mathcal{I}_g) = (\mathcal{I}_g)^*$.

Step 1: Find large piece of BNS invariant

Exists fin genset $S \subset \mathcal{I}_g$ of genus 1 bounding pairs s.t.

$$\{f \in (\mathcal{I}_g)^* \mid f(s) \neq 0 \text{ for all } s \in S\} \subset \Sigma(\mathcal{I}_g).$$

$\text{Mod}_g \circlearrowright \mathcal{I}_g$ by conjugation $\rightsquigarrow$ $\text{Mod}_g \circlearrowright (\mathcal{I}_g)^*$ preserving $\Sigma(\mathcal{I}_g)$.

Step 2: Push everything into that piece of BNS

For nonzero $f \in (\mathcal{I}_g)^*$, exists $\phi \in \text{Mod}_g$ s.t. $(\phi \cdot f)(s) \neq 0$ for all $s \in S$.

Main goal

Goal

$[\mathcal{I}_g, \mathcal{I}_g]$ is fin gen for $g \geq 4$, i.e. $\Sigma(\mathcal{I}_g) = (\mathcal{I}_g)^*$.

Step 1: Find large piece of BNS invariant

Exists fin genset $S \subset \mathcal{I}_g$ of genus 1 bounding pairs s.t.

$$\{f \in (\mathcal{I}_g)^* \mid f(s) \neq 0 \text{ for all } s \in S\} \subset \Sigma(\mathcal{I}_g).$$

$\text{Mod}_g \circlearrowright \mathcal{I}_g$ by conjugation $\rightsquigarrow$ $\text{Mod}_g \circlearrowright (\mathcal{I}_g)^*$ preserving $\Sigma(\mathcal{I}_g)$.

Step 2: Push everything into that piece of BNS

For nonzero $f \in (\mathcal{I}_g)^*$, exists $\phi \in \text{Mod}_g$ s.t. $(\phi \cdot f)(s) \neq 0$ for all $s \in S$.

$$\Longrightarrow \phi \cdot f \in \Sigma(\mathcal{I}_g)$$

Main goal

Goal

$[\mathcal{I}_g, \mathcal{I}_g]$ is fin gen for $g \geq 4$, i.e. $\Sigma(\mathcal{I}_g) = (\mathcal{I}_g)^*$.

Step 1: Find large piece of BNS invariant

Exists fin genset $S \subset \mathcal{I}_g$ of genus 1 bounding pairs s.t.

$$\{f \in (\mathcal{I}_g)^* \mid f(s) \neq 0 \text{ for all } s \in S\} \subset \Sigma(\mathcal{I}_g).$$

$\mathrm{Mod}_g \circlearrowright \mathcal{I}_g$ by conjugation $\quad\rightsquigarrow\quad$ $\mathrm{Mod}_g \circlearrowright (\mathcal{I}_g)^*$ preserving $\Sigma(\mathcal{I}_g)$.

Step 2: Push everything into that piece of BNS

For nonzero $f \in (\mathcal{I}_g)^*$, exists $\phi \in \mathrm{Mod}_g$ s.t. $(\phi \cdot f)(s) \neq 0$ for all $s \in S$.

$$\implies \phi \cdot f \in \Sigma(\mathcal{I}_g) \implies f \in \Sigma(\mathcal{I}_g)$$

Step 2: Push everything into that piece of BNS

For nonzero $f \in (\mathcal{I}_g)^*$, exists $\phi \in \mathrm{Mod}_g$ s.t. $(\phi \cdot f)(s) \neq 0$ for all $s \in S$.

Step 2: Push everything into that piece of BNS

For nonzero $f \in (\mathcal{I}_g)^*$, exists $\phi \in \mathrm{Mod}_g$ s.t. $(\phi \cdot f)(s) \neq 0$ for all $s \in S$.

Give Mod_g pullback of Zariski topology under

$$\mathrm{Mod}_g \to \mathrm{Aut}\left((\mathcal{I}_g)^*\right) \cong \mathrm{GL}_n(\mathbb{R}).$$

Step 2: Push everything into that piece of BNS

For nonzero $f \in (\mathcal{I}_g)^*$, exists $\phi \in \mathrm{Mod}_g$ s.t. $(\phi \cdot f)(s) \neq 0$ for all $s \in S$.

Give Mod_g pullback of Zariski topology under

$$\mathrm{Mod}_g \to \mathrm{Aut}\left((\mathcal{I}_g)^*\right) \cong \mathrm{GL}_n(\mathbb{R}).$$

Claim 1 (proved next slide)

Mod_g is irreducible space (not finite union of proper closed subspaces).

Step 2: Push everything into that piece of BNS

For nonzero $f \in (\mathcal{I}_g)^*$, exists $\phi \in \text{Mod}_g$ s.t. $(\phi \cdot f)(s) \neq 0$ for all $s \in S$.

Give Mod_g pullback of Zariski topology under

$$\text{Mod}_g \to \text{Aut}\left((\mathcal{I}_g)^*\right) \cong \text{GL}_n(\mathbb{R}).$$

Claim 1 (proved next slide)

Mod_g is irreducible space (not finite union of proper closed subspaces).

For $s \in S$, set $Z_s = \{\phi \in \text{Mod}_g \mid (\phi \cdot f)(s) = 0\}$ (a closed set!).

Step 2: Push everything into that piece of BNS

For nonzero $f \in (\mathcal{I}_g)^*$, exists $\phi \in \mathrm{Mod}_g$ s.t. $(\phi \cdot f)(s) \neq 0$ for all $s \in S$.

Give Mod_g pullback of Zariski topology under

$$\mathrm{Mod}_g \to \mathrm{Aut}\left((\mathcal{I}_g)^*\right) \cong \mathrm{GL}_n(\mathbb{R}).$$

Claim 1 (proved next slide)

Mod_g is irreducible space (not finite union of proper closed subspaces).

For $s \in S$, set $Z_s = \{\phi \in \mathrm{Mod}_g \mid (\phi \cdot f)(s) = 0\}$ (a closed set!).

Claim 2 (proved slide after that)

Z_s proper subspace of Mod_g.

Step 2: Push everything into that piece of BNS

For nonzero $f \in (\mathcal{I}_g)^*$, exists $\phi \in \mathrm{Mod}_g$ s.t. $(\phi \cdot f)(s) \neq 0$ for all $s \in S$.

Give Mod_g pullback of Zariski topology under

$$\mathrm{Mod}_g \to \mathrm{Aut}\left((\mathcal{I}_g)^*\right) \cong \mathrm{GL}_n(\mathbb{R}).$$

Claim 1 (proved next slide)

Mod_g is irreducible space (not finite union of proper closed subspaces).

For $s \in S$, set $Z_s = \{\phi \in \mathrm{Mod}_g \mid (\phi \cdot f)(s) = 0\}$ (a closed set!).

Claim 2 (proved slide after that)

Z_s proper subspace of Mod_g.

$$\Longrightarrow \bigcup_{s \in S} Z_s \subsetneq \mathrm{Mod}_g$$

Step 2: Push everything into that piece of BNS

For nonzero $f \in (\mathcal{I}_g)^*$, exists $\phi \in \text{Mod}_g$ s.t. $(\phi \cdot f)(s) \neq 0$ for all $s \in S$.

Give Mod_g pullback of Zariski topology under

$$\text{Mod}_g \to \text{Aut}\left((\mathcal{I}_g)^*\right) \cong \text{GL}_n(\mathbb{R}).$$

Claim 1 (proved next slide)

Mod_g is irreducible space (not finite union of proper closed subspaces).

For $s \in S$, set $Z_s = \{\phi \in \text{Mod}_g \mid (\phi \cdot f)(s) = 0\}$ (a closed set!).

Claim 2 (proved slide after that)

Z_s proper subspace of Mod_g.

$$\Longrightarrow \bigcup_{s \in S} Z_s \subsetneq \text{Mod}_g$$

Desired ϕ is any elt of Mod_g not in this union.

Mod_g has pullback of Zariski topology under

$$\mathrm{Mod}_g \to \mathrm{Aut}\left((\mathcal{I}_g)^*\right) \cong \mathrm{GL}_n(\mathbb{R}).$$

Claim 1

Mod_g irreducible space (not finite union of proper closed subspaces).

Mod_g has pullback of Zariski topology under

$$\mathrm{Mod}_g \to \mathrm{Aut}\left((\mathcal{I}_g)^*\right) \cong \mathrm{GL}_n(\mathbb{R}).$$

Claim 1

Mod_g irreducible space (not finite union of proper closed subspaces).

Theorem (Johnson)

For $g \geq 3$, $\mathrm{H}_1(\mathcal{I}_g; \mathbb{R}) \cong (\wedge^3 H)/H$ w/ $H = \mathrm{H}_1(\Sigma_g; \mathbb{R})$.

Mod_g has pullback of Zariski topology under

$$\mathrm{Mod}_g \to \mathrm{Aut}\left((\mathcal{I}_g)^*\right) \cong \mathrm{GL}_n(\mathbb{R}).$$

Claim 1

Mod_g irreducible space (not finite union of proper closed subspaces).

Theorem (Johnson)

For $g \geq 3$, $\mathrm{H}_1(\mathcal{I}_g; \mathbb{R}) \cong (\wedge^3 H)/H$ *w/* $H = \mathrm{H}_1(\Sigma_g; \mathbb{R})$.

$\Rightarrow$ Have factorization

$$\mathrm{Mod}_g \to \mathrm{Sp}_{2g}(\mathbb{R}) \hookrightarrow \mathrm{Aut}\left((\mathcal{I}_g)^*\right).$$

Mod_g has pullback of Zariski topology under

$$\mathsf{Mod}_g \to \mathsf{Aut}\left((\mathcal{I}_g)^*\right) \cong \mathsf{GL}_n(\mathbb{R}).$$

Claim 1

Mod_g irreducible space (not finite union of proper closed subspaces).

Theorem (Johnson)

For $g \geq 3$, $\mathsf{H}_1(\mathcal{I}_g; \mathbb{R}) \cong (\wedge^3 H)/H$ w/ $H = \mathsf{H}_1(\Sigma_g; \mathbb{R})$.

$\Rightarrow$ Have factorization

$$\mathsf{Mod}_g \twoheadrightarrow \mathsf{Sp}_{2g}(\mathbb{Z}) \hookrightarrow \mathsf{Sp}_{2g}(\mathbb{R}) \hookrightarrow \mathsf{Aut}\left((\mathcal{I}_g)^*\right).$$

Mod_g has pullback of Zariski topology under

$$\mathrm{Mod}_g \to \mathrm{Aut}\left((\mathcal{I}_g)^*\right) \cong \mathrm{GL}_n(\mathbb{R}).$$

Claim 1

Mod_g irreducible space (not finite union of proper closed subspaces).

Theorem (Johnson)

For $g \geq 3$, $\mathrm{H}_1(\mathcal{I}_g; \mathbb{R}) \cong (\wedge^3 H)/H$ w/ $H = \mathrm{H}_1(\Sigma_g; \mathbb{R})$.

$\Rightarrow$ Have factorization

$$\mathrm{Mod}_g \twoheadrightarrow \mathrm{Sp}_{2g}(\mathbb{Z}) \hookrightarrow \mathrm{Sp}_{2g}(\mathbb{R}) \hookrightarrow \mathrm{Aut}\left((\mathcal{I}_g)^*\right).$$

Induced topology on $\mathrm{Sp}_{2g}(\mathbb{R})$ is usual Zariski topology.

Mod_g has pullback of Zariski topology under

$$\mathrm{Mod}_g \to \mathrm{Aut}\left((\mathcal{I}_g)^*\right) \cong \mathrm{GL}_n(\mathbb{R}).$$

Claim 1

Mod_g irreducible space (not finite union of proper closed subspaces).

Theorem (Johnson)

For $g \geq 3$, $\mathrm{H}_1(\mathcal{I}_g; \mathbb{R}) \cong (\wedge^3 H)/H$ w/ $H = \mathrm{H}_1(\Sigma_g; \mathbb{R})$.

$\Rightarrow$ Have factorization

$$\mathrm{Mod}_g \twoheadrightarrow \mathrm{Sp}_{2g}(\mathbb{Z}) \hookrightarrow \mathrm{Sp}_{2g}(\mathbb{R}) \hookrightarrow \mathrm{Aut}\left((\mathcal{I}_g)^*\right).$$

Induced topology on $\mathrm{Sp}_{2g}(\mathbb{R})$ is usual Zariski topology.

$\mathrm{Sp}_{2g}(\mathbb{R})$ connected alg. group, so $\mathrm{Sp}_{2g}(\mathbb{R})$ irreducible.

Mod_g has pullback of Zariski topology under

$$\mathrm{Mod}_g \to \mathrm{Aut}\left((\mathcal{I}_g)^*\right) \cong \mathrm{GL}_n(\mathbb{R}).$$

Claim 1

Mod_g irreducible space (not finite union of proper closed subspaces).

Theorem (Johnson)

For $g \geq 3$, $\mathrm{H}_1(\mathcal{I}_g;\mathbb{R}) \cong (\wedge^3 H)/H$ w/ $H = \mathrm{H}_1(\Sigma_g;\mathbb{R})$.

$\Rightarrow$ Have factorization

$$\mathrm{Mod}_g \twoheadrightarrow \mathrm{Sp}_{2g}(\mathbb{Z}) \hookrightarrow \mathrm{Sp}_{2g}(\mathbb{R}) \hookrightarrow \mathrm{Aut}\left((\mathcal{I}_g)^*\right).$$

Induced topology on $\mathrm{Sp}_{2g}(\mathbb{R})$ is usual Zariski topology.

$\mathrm{Sp}_{2g}(\mathbb{R})$ connected alg. group, so $\mathrm{Sp}_{2g}(\mathbb{R})$ irreducible.

$\mathrm{Sp}_{2g}(\mathbb{Z})$ Zariski dense in $\mathrm{Sp}_{2g}(\mathbb{R})$, so $\mathrm{Sp}_{2g}(\mathbb{Z})$ irreducible.

Mod_g has pullback of Zariski topology under

$$\mathrm{Mod}_g \to \mathrm{Aut}\left((\mathcal{I}_g)^*\right) \cong \mathrm{GL}_n(\mathbb{R}).$$

Claim 1

Mod_g irreducible space (not finite union of proper closed subspaces).

Theorem (Johnson)

For $g \geq 3$, $\mathrm{H}_1(\mathcal{I}_g; \mathbb{R}) \cong (\wedge^3 H)/H$ w/ $H = \mathrm{H}_1(\Sigma_g; \mathbb{R})$.

$\Rightarrow$ Have factorization

$$\mathrm{Mod}_g \twoheadrightarrow \mathrm{Sp}_{2g}(\mathbb{Z}) \hookrightarrow \mathrm{Sp}_{2g}(\mathbb{R}) \hookrightarrow \mathrm{Aut}\left((\mathcal{I}_g)^*\right).$$

Induced topology on $\mathrm{Sp}_{2g}(\mathbb{R})$ is usual Zariski topology.

$\mathrm{Sp}_{2g}(\mathbb{R})$ connected alg. group, so $\mathrm{Sp}_{2g}(\mathbb{R})$ irreducible.

$\mathrm{Sp}_{2g}(\mathbb{Z})$ Zariski dense in $\mathrm{Sp}_{2g}(\mathbb{R})$, so $\mathrm{Sp}_{2g}(\mathbb{Z})$ irreducible.

$\mathrm{Mod}_g \twoheadrightarrow \mathrm{Sp}_{2g}(\mathbb{Z})$ surjective, so Mod_g irreducible.

For $s \in S$, set $Z_s = \{\phi \in \mathsf{Mod}_g \mid (\phi \cdot f)(s) = 0\}$.

Claim 2: Push everything into that piece of BNS

Z_s proper subspace of Mod_g.

For $s \in S$, set $Z_s = \{\phi \in \mathrm{Mod}_g \mid (\phi \cdot f)(s) = 0\}$.

Claim 2: Push everything into that piece of BNS

Z_s proper subspace of Mod_g.

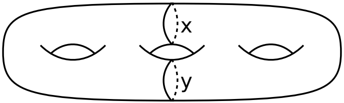
Assume $Z_s = \mathrm{Mod}_g$. Write $s = T_x T_y^{-1}$.

For $s \in S$, set $Z_s = \{\phi \in \mathrm{Mod}_g \mid (\phi \cdot f)(s) = 0\}$.

Claim 2: Push everything into that piece of BNS

Z_s proper subspace of Mod_g.

Assume $Z_s = \mathrm{Mod}_g$. Write $s = T_x T_y^{-1}$.

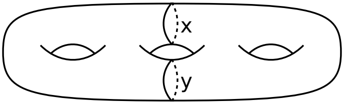


For $s \in S$, set $Z_s = \{\phi \in \mathrm{Mod}_g \mid (\phi \cdot f)(s) = 0\}$.

Claim 2: Push everything into that piece of BNS

Z_s proper subspace of Mod_g.

Assume $Z_s = \mathrm{Mod}_g$. Write $s = T_x T_y^{-1}$.



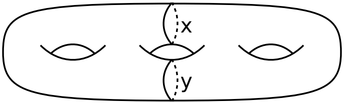
For $\phi \in \mathrm{Mod}_g$,

For $s \in S$, set $Z_s = \{\phi \in \mathrm{Mod}_g \mid (\phi \cdot f)(s) = 0\}$.

Claim 2: Push everything into that piece of BNS

Z_s proper subspace of Mod_g.

Assume $Z_s = \mathrm{Mod}_g$. Write $s = T_x T_y^{-1}$.



For $\phi \in \mathrm{Mod}_g$,

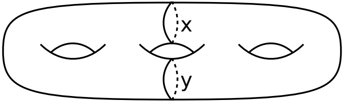
$$0 = (\phi \cdot f)(T_x T_y^{-1})$$

For $s \in S$, set $Z_s = \{\phi \in \mathrm{Mod}_g \mid (\phi \cdot f)(s) = 0\}$.

Claim 2: Push everything into that piece of BNS

Z_s proper subspace of Mod_g.

Assume $Z_s = \mathrm{Mod}_g$. Write $s = T_x T_y^{-1}$.



For $\phi \in \mathrm{Mod}_g$,

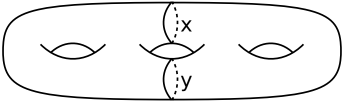
$$0 = (\phi \cdot f)(T_x T_y^{-1}) = f(\phi T_x T_y^{-1} \phi^{-1})$$

For $s \in S$, set $Z_s = \{\phi \in \mathrm{Mod}_g \mid (\phi \cdot f)(s) = 0\}$.

Claim 2: Push everything into that piece of BNS

Z_s proper subspace of Mod_g.

Assume $Z_s = \mathrm{Mod}_g$. Write $s = T_x T_y^{-1}$.



For $\phi \in \mathrm{Mod}_g$,

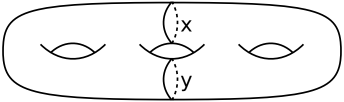
$$0 = (\phi \cdot f)(T_x T_y^{-1}) = f(\phi T_x T_y^{-1} \phi^{-1}) = f(T_{\phi(x)} T_{\phi(y)}^{-1}).$$

For $s \in S$, set $Z_s = \{\phi \in \mathrm{Mod}_g \mid (\phi \cdot f)(s) = 0\}$.

Claim 2: Push everything into that piece of BNS

Z_s proper subspace of Mod_g.

Assume $Z_s = \mathrm{Mod}_g$. Write $s = T_x T_y^{-1}$.



For $\phi \in \mathrm{Mod}_g$,

$$0 = (\phi \cdot f)(T_x T_y^{-1}) = f(\phi T_x T_y^{-1} \phi^{-1}) = f(T_{\phi(x)} T_{\phi(y)}^{-1}).$$

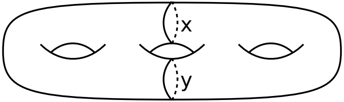
All genus 1 bounding pairs of form $T_{\phi(x)} T_{\phi(y)}^{-1}$ for some $\phi \in \mathrm{Mod}_g$.

For $s \in S$, set $Z_s = \{\phi \in \mathrm{Mod}_g \mid (\phi \cdot f)(s) = 0\}$.

Claim 2: Push everything into that piece of BNS

Z_s proper subspace of Mod_g.

Assume $Z_s = \mathrm{Mod}_g$. Write $s = T_x T_y^{-1}$.



For $\phi \in \mathrm{Mod}_g$,

$$0 = (\phi \cdot f)(T_x T_y^{-1}) = f(\phi T_x T_y^{-1} \phi^{-1}) = f(T_{\phi(x)} T_{\phi(y)}^{-1}).$$

All genus 1 bounding pairs of form $T_{\phi(x)} T_{\phi(y)}^{-1}$ for some $\phi \in \mathrm{Mod}_g$.

These generate $\mathcal{I}_g$, so $f = 0$, contradiction.