Operads with Homological Stability detect Infinite Loop Spaces

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Figure: Sarah, Kate and Ulrike (WIT II- BIRS 2016)

Outline

- **Introduction**: Operads and infinite loop spaces.
- **Tillmann's surface operad**: Surprise Theorem.
- **Tools**: Bar construction. Group completion theorem.
- **OHS**: Operads with homological stability.
- **Main Theorem**: Group completions of algebras over OHS are infinite loop spaces.
- **Proof sketch.**
- **Examples and applications**

Operads: Useful way to collect multiple input operations and encode their interactions for varying n.

$$\mu_n : A^n \longrightarrow A$$

In particular, useful to encode relations *up to homotopy* between operations.
Example: For a based topological space (X, x_0), concatenation of loops defines operations on

$$\Omega(X) = \text{maps}(([0,1], \partial), (X, x_0)) = \text{ loops space on } (X, x_0)$$

that have inverses and are associative up to homotopy.

Example: For a based topological space (X, x_0), and $n \geq 2$ we obtain operations on

$$\begin{aligned}\Omega^n(X) &= \text{maps}(([0,1]^n, \partial), (X, x_0)) \\ &= \Omega(\Omega(\cdots\Omega(X, x_0)) = \text{ } n\text{-th loop space on } (X, x_0)\end{aligned}$$

that have inverses, are associative and commutative up to homotopy. And, coherent homotopies of homotopies increasing with higher n.

Introduction: Operads

Definition

An **operad** is a collection of spaces

$$\mathcal{O} = \{\mathcal{O}(n)\}_{n\geq 0}$$

with base point $* \in \mathcal{O}(0)$, $1 \in \mathcal{O}(1)$, a right action of the symmetric group Σ_n on $\mathcal{O}(n)$ and structure maps

$$\gamma : \mathcal{O}(k) \times [\mathcal{O}(j_1) \times \ldots \times \mathcal{O}(j_k)] \longrightarrow \mathcal{O}(j_1 + \ldots + j_k)$$

that are required to be associative, unital, and equivariant.
A **map of operads** $\mathcal{O} \longrightarrow \mathcal{V}$ is a a collection of Σ_n equivariant maps $\mathcal{O}(n) \longrightarrow \mathcal{V}(n)$ which commute witht the structure maps and preserve $*$ and 1.

Remark: Note that above we do not insist that $\mathcal{O}(0) = *$.

Introduction: Operads

Definition

An $\mathcal{O}$-**algebra** is a based space $(X,*)$ with equivariant structure maps

$$\mathcal{O}(j)\times X^j \longrightarrow X.$$

For a based space $(X,*)$, the **free $\mathcal{O}$-algebra on** X is

$$\mathbb{O}(X) := \coprod_{n\geq 0} (\mathcal{O}(n)\times_{\Sigma_n} X^n) \Big/ \sim$$

where $\sim$ is a base point relation generated by

$$(\gamma(c;1^i,*,1^{n-i-1});x_1,\dots,x_{n-1}) \sim (c;x_1,\dots x_i,*,x_{i+1},\dots,x_{n-1})).$$

The class of $(1,*)\in\mathcal{O}(1)\times X$ is the base point of $\mathbb{O}(X)$. Note that it coincides with the class of $*\in\mathcal{O}(0)$.

Remark: We will identify $\mathcal{O}(0)$ with $\mathbb{O}(*)$. In the cases of interest it will be a non-trivial $\mathcal{O}$-algebra.

Introduction: Operads

Example: The **little *n*-disks operad** $\mathcal{C}_n$.

$$\mathcal{C}_n(k) \subset Emb(\coprod_k D^n, D^n) \simeq Conf_k(\mathbb{R}^n)$$

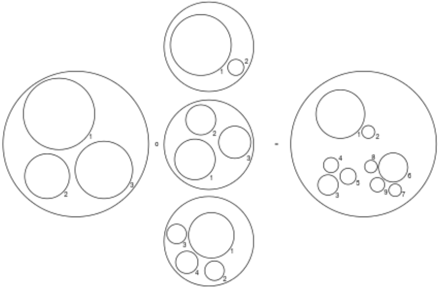

Figure: From Wikipidea

$$\gamma : \mathcal{C}_2(3) \times [\mathcal{C}_2(2) \times \mathcal{C}_2(3) \times \mathcal{C}_2(4)] \longrightarrow \mathcal{C}_2(9)$$

Introduction: Operads

We have maps of operads:

$$\mathcal{C}_1 \longrightarrow \mathcal{C}_2 \longrightarrow \cdots \longrightarrow \mathcal{C}_n \longrightarrow \cdots \longrightarrow \mathcal{C}_\infty$$

Example: $\Omega^n(X)$ is a $\mathcal{C}_n$-algebra.

Recognition Principle (Stasheff, Boardman-Vogt, May, Barrett-Eccless, Milgram ... (1970's):
Connected $\mathcal{C}_n$-algebras are Ω^n. More generally, the group completion of a $\mathcal{C}_n$-algebra is an Ω^n space.

Tillmann's surface operad $\mathcal{M}$

(Motivated by Segal's cobordism category and definition of CFT)

Let $\Gamma_{g,n+1} = \pi_0(Diff^+(F_{g,n+1};\partial))$ the *mapping class group* of an oriented surface of genus g and $n+1$ boundary components.

$$\mathcal{M}(n) \simeq \coprod_{g \geq 0} B\Gamma_{g,n+1}$$

A version of the little 2-disk operad is a sub-operad of $\mathcal{M}$ so that a grouplike $\mathcal{M}$-algebra is in particular a double loop space. But the surprising part is that

Theorem (Tillmann, 2000)

Group like $\mathcal{M}$-algebras are infinite-loop spaces with an infinite loop space action by $\mathcal{M}^+$ (= the group completion of the free $\mathcal{M}$-algebra on a point).

Tillmann's surface operad theorem

Main ingredient on the proof:
Harer's homology stability theorem: $H_* B\Gamma_{g,n+1}$ is independent of g and n for g large enough.

Inconvenient feature of the proof:
Requires strict multiplication: some surfaces had to be identified and diffeomorphisms are replaced by mapping class groups.

Tools: Bar construction

Lemma (May (GILS))

For a monad $\mathbb{T}$, a $\mathbb{T}$-functor F and a $\mathbb{T}$-algebra X, define

$$B_\bullet(F, \mathbb{T}, X) := \{q \mapsto F(\mathbb{T}^q X)\}$$

1. *For any functor G, $|B_\bullet(GF, \mathbb{T}, X)| \cong |GB_\bullet(F, \mathbb{T}, X)|$.*
2. $|B_\bullet(\mathbb{T}, \mathbb{T}, X)| \simeq X$.
3. $|B_\bullet(F, \mathbb{T}, \mathbb{T}(X))| \simeq F(X)$.
4. *If $\delta : \mathbb{T} \longrightarrow \mathbb{T}'$ is a natural transformation of monads, then $\mathbb{T}'$ is an $\mathbb{T}$-functor and $B_\bullet(\mathbb{T}', \mathbb{T}, X)$ is a simplicial $\mathbb{T}'$-algebra.*

Tools: Monoid rectification

Corollary

Let $\mathcal{A}$ be an A_∞-operad and let $\delta\colon \mathbb{A} \longrightarrow \mathbb{A}s$ be the map of monads associated to the augmentation of operads $\mathcal{A} \longrightarrow \mathcal{A}s$. For an $\mathcal{A}$-algebra X, there is a topological monoid $\mathbf{M}_{\mathcal{A}}(X) := |B_\bullet(\mathbb{A}s, \mathbb{A}, X)|$ and a strong deformation retract

$$\rho\colon X \longrightarrow \mathbf{M}_{\mathcal{A}}(X)$$

that is natural in X and induces an isomorphism of homology Pontryagin rings.

Group Completion

Algebraic monoids: $M \longrightarrow \mathcal{G}M$ the *Grothendieck group* of M.
Topological monoids: $M \longrightarrow \mathcal{G}M = \Omega BM$ where $BM = |N_\bullet M|$
A_∞ algebras: $X \longrightarrow \mathcal{G}X = \Omega B\mathbf{M}_{\mathcal{A}}(X)$ the composite

$$X \longrightarrow \mathbf{M}_{\mathcal{A}}(X) \longrightarrow \Omega B\mathbf{M}_{\mathcal{A}}(X)$$

Theorem (Quillen, McDuff-Segal)

Let $M = \coprod_{n\geq 0} M_n$ be a topological monoid such that the multiplication on $H_*(M)$ is commutative. Then

$$H_*(\Omega BM) = \mathbb{Z} \times \lim_{n\longrightarrow\infty} H_*(M_n) = \mathbb{Z} \times H_*(M_\infty).$$

OHS: Preliminary definitions

Definition

Let I be a commutative, finitely generated monoid. An I-**grading** on an operad $\mathcal{O}$ is a decomposition

$$\mathcal{O}(n) = \coprod_{g \in I} \mathcal{O}_g(n)$$

for each n so that:

1. the basepoint $*$ lies in $\mathcal{O}_0(0)$;
2. the Σ_n action on $\mathcal{O}(n)$ restricts to an action on each $\mathcal{O}_g(n)$;
3. the structure maps restrict to maps

$$\gamma : \mathcal{O}_g(k) \times \left[\mathcal{O}_{g_1}(j_1) \times \ldots \times \mathcal{O}_{g_k}(j_k)\right] \longrightarrow \mathcal{O}_{g+g_1+\ldots+g_k}(j_1 + \ldots + j_k).$$

OHS: Notation

For an I-graded operad $\mathcal{O}$ let s be the product of a set of generators for I, and choose a **propagator** $\tilde{s} \in \mathcal{O}_s(1)$. Let $D = \gamma(-; *, \cdots, *)$ and $\tilde{s} := \gamma(\tilde{s}, -)$. The diagram

$$\begin{array}{ccc} \mathcal{O}_g(n) & \xrightarrow{\tilde{s}} & \mathcal{O}_{g+s}(n) \\ \downarrow D & & \downarrow D \\ \mathcal{O}_g(0) & \xrightarrow{\tilde{s}} & \mathcal{O}_{g+s}(0) \end{array}$$

commutes and defines a map $D_\infty : \mathcal{O}_\infty(n) \longrightarrow \mathcal{O}_\infty(0)$ where

$$\mathcal{O}_\infty(n) =: \operatorname*{hocolim}_{\tilde{s}} \mathcal{O}_g(n)$$

Operads with homological stability

Definition

An operad $\mathcal{O}$ is an **operad with homological stability (OHS)** if

1. it is I-graded;
2. there is an A_∞-operad $\mathcal{A}$ and a map of graded operads

$$\mu\colon \mathcal{A} \longrightarrow \mathcal{O} \qquad \text{(multiplication map)}$$

   with $\mu(\mathcal{A}(2)) \subset \mathcal{O}_0(2)$ path connected; and
3. the maps

$$D_\infty : \mathcal{O}_\infty(n) \longrightarrow \mathcal{O}_\infty(0)$$

   induce homology isomorphisms.

Operads with homological stability

Examples:

1. $\mathcal{C}_\infty$ is and OHS concentrated in degree zero and multiplication $\mu : \mathcal{C}_1 \longrightarrow \mathcal{C}_\infty$. Since $\mathcal{C}_\infty(n)$ is contractible, conditions 2 and 3 are trivially satisfied.
2. The Riemann surfaces operad $\mathcal{M}$ with $\mathcal{M}(n) = \coprod_{g\geq 0} \mathcal{M}_{g,n+1} \simeq \coprod_{g\geq 0} B\Gamma_{g,n+1}$

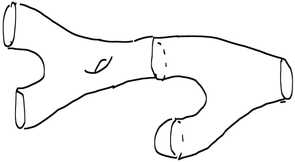

Figure: $\gamma : \mathcal{M}_{0,2+1} \times [\mathcal{M}_{1,2+1} \times \mathcal{M}_{0,0+1}] \longrightarrow \mathcal{M}_{1,2+1}$

Main Theorem

Theorem (B., Bobkova, Ponto, Tillmann, Yeakel)

Suppose $\mathcal{O}$ is an OHS. Then,

$$\mathcal{G} : \mathcal{O} - \textit{algebras} \longrightarrow \Omega^{\infty} - \textit{spaces}$$

is a functor with image Ω^{∞}-spaces with a compatible Ω^{∞}-map

$$\mathcal{G}\mathbb{O}(*) \times \mathcal{G}X \longrightarrow \mathcal{G}X,$$

where the source is given the product Ω^{∞}-space structure.

Proof sketch: Step 1 - Operad replacement

Let $\mathcal{O}$ be an OHS.
Then the product operad $\widetilde{\mathcal{O}} := \mathcal{O} \times \mathcal{C}_\infty$ is an OHS with compatible maps of operads

$$\mathcal{O} \xleftarrow{\pi_1} \widetilde{\mathcal{O}} \xrightarrow{\pi} \mathcal{C}_\infty.$$

Then, any $\mathcal{O}$-algebra is an $\widetilde{\mathcal{O}}$-algebra.
W.L.O.G we assume a compatible map $\pi : \mathcal{O} \longrightarrow \mathcal{C}_\infty$.
For any space X, there is a map of $\mathcal{O}$-algebras

$$\tau \times \pi : \mathbb{O}(X) \longrightarrow \mathbb{O}(*) \times \mathbb{C}_\infty(X),$$

where τ is induced by $X \longrightarrow *$ and the target has the diagonal action of $\mathcal{O}$.

Step 2 - Group completion of free $\mathcal{O}$-algebras

Claim: For any based space X,

$$\mathcal{G}(\tau)\times\mathcal{G}(\pi)\colon \mathcal{G}(\mathbb{O}(X)) \longrightarrow \mathcal{G}(\mathbb{O}(*))\times\mathcal{G}(\mathbb{C}_\infty(X))$$

is a weak homotopy equivalence.
"Proof": By Whitehead theorem e.t.s isomorphism in homology.
By the group completion theorem e.t.s

$$\tau_\infty\times\pi_\infty\colon \mathbb{O}_\infty(X)\longrightarrow \mathbb{O}_\infty(*)\times\mathbb{C}_\infty(X)$$

induces isomorphism in homology.
Filtering by *arity* in the operad and taking filtration quotients reduces to show that for each n and Σ_n space Y

$$\bar{D}_\infty\times(\pi_\infty\times 1_Y)\colon \mathcal{O}_\infty(n)\times_{\Sigma_n} Y\longrightarrow \mathcal{O}_\infty(0)\times(\mathcal{C}_\infty(n)\times_{\Sigma_n} Y)$$

is a homology isomorphism.
This follows by **homological stability of $\mathcal{O}$.**

Step 3: A functor from $\mathcal{O}$-algebras to Ω^∞ spaces

Claim: The assignment $X \mapsto |\mathcal{G}B_\bullet(\mathbb{C}_\infty, \mathbb{O}, X)|$ defines a functor from $\mathcal{O}$-algebras to Ω^∞-spaces.
Proof: Recall (May): there is a map of monads

$$\alpha \colon \mathbb{C}_\infty \longrightarrow \Omega^\infty\Sigma^\infty;$$

and for every based space Z, the map

$$\alpha \colon \mathbb{C}_\infty Z \longrightarrow \Omega^\infty\Sigma^\infty Z$$

is a group completion.

For any map of $\mathcal{O}$-algebras $f : X \longrightarrow Y$ the following diagram commutes. (The vertical arrows are equivalences and the horizontal ones are induced by f.)

Step 3: A functor from $\mathcal{O}$-algebras to Ω^∞ spaces

$$\begin{array}{ccc}
|\mathcal{G}B_\bullet(\mathbb{C}_\infty, \mathbb{O}, X)| & \longrightarrow & |\mathcal{G}B_\bullet(\mathbb{C}_\infty, \mathbb{O}, Y)| \\
\downarrow & & \downarrow \\
|\mathcal{G}B_\bullet(\Omega^\infty\Sigma^\infty, \mathbb{O}, X)| & \longrightarrow & |\mathcal{G}B_\bullet(\Omega^\infty\Sigma^\infty, \mathbb{O}, Y)| \\
\uparrow & & \uparrow \\
|\mathcal{G}\Omega^\infty B_\bullet(\Sigma^\infty, \mathbb{O}, X)| & \longrightarrow & |\mathcal{G}\Omega^\infty B_\bullet(\Sigma^\infty, \mathbb{O}, Y)| \\
\uparrow & & \uparrow \\
|\Omega^\infty B_\bullet(\Sigma^\infty, \mathbb{O}, X)| & \longrightarrow & |\Omega^\infty B_\bullet(\Sigma^\infty, \mathbb{O}, Y)| \\
\downarrow & & \downarrow \\
\Omega^\infty|B_\bullet(\Sigma^\infty, \mathbb{O}, X)| & \longrightarrow & \Omega^\infty|B_\bullet(\Sigma^\infty, \mathbb{O}, Y)|
\end{array}$$

Step 4: Group completion of $\mathcal{O}$-algebras

We have seen that for any based space X
$\mathcal{G}(\mathbb{O}(X)) \simeq \mathcal{G}(\mathbb{O}(*)) \times \mathcal{G}(\mathbb{C}_\infty(X))$
For an $\mathcal{O}$-algebra X we have a homotopy fibration sequence

$$\mathcal{G}\mathbb{O}(*) \longrightarrow |\mathcal{G}B_\bullet(\mathbb{O}, \mathbb{O}, X)| \longrightarrow |\mathcal{G}B_\bullet(\mathbb{C}_\infty, \mathbb{O}, X)|$$

applying it to the product $\mathcal{O}$-algebra $\mathbb{O}(*) \times X$ allows to conclude that

$$\mathcal{G}X \simeq |\mathcal{G}B_\bullet(\mathbb{C}_\infty, \mathbb{O}, \mathbb{O}(*) \times X)|.$$

which we saw to be an Ω^∞-space.

Examples and applications: Surface operads

Oriented surfaces and diffeomorphisms: $\mathcal{S}$

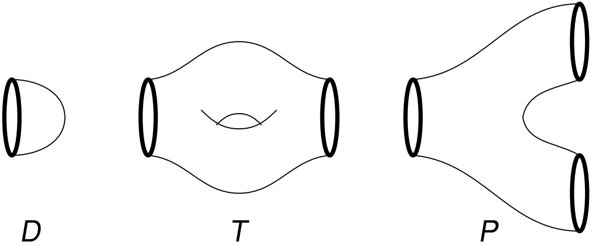

Figure: Orientable Atomic Surfaces

$$\mathcal{S}(n) = \coprod_{g \geq 0} B\mathcal{S}_{g,n+1}.$$

By Madsen-Weiss $\mathcal{GS}(0) \simeq \mathbb{Z} \times B\Gamma_\infty^+ \simeq \Omega^\infty \mathbf{MT}SO(2)$.

Examples and applications: Surface operads

Nonorientable surfaces and diffeomorphisms: $\mathcal{N}$ Let $N = \mathbb{R}P^2 \setminus (D^2 \coprod D^2)$. Let $N_{k,n+1}$ be a surface of nonorientable genus k with one outgoing and n incoming boundary components built out of D, P, S^1 and N.

$$\mathcal{N}(n) \simeq \coprod_{k \geq 0} B\mathcal{N}_{k,n+1}.$$

Homology stability results of Wahl give that $\mathcal{N}$ is a n OHS and

$$\mathcal{G}\mathcal{N}(0) \simeq \mathbb{Z} \times B\mathcal{N}_\infty^+ \simeq \Omega^\infty \mathbf{MT}O(2),$$

where $\mathcal{N}_\infty = \lim_{k \longrightarrow \infty} \pi_0 \mathrm{Diff}(N_{k,1}, \partial)$ denotes the infinite mapping class group.

Examples and applications: Manifold operads

Let $W_{g,j+1}$ be the connected sum of g copies of $S^k \times S^k$ with $j+1$ open disks removed .
Let $\theta \colon B \longrightarrow BO(2k)$ be the k-th connected cover and fix a bundle map $\ell_W \colon TW \longrightarrow \theta^* \gamma_{2k}$.
We construct a graded operad with

$$\mathcal{W}_g^{2k}(j) \simeq \mathcal{M}_k^{\theta}(W_{g,j+1}, \ell_{W_{g,j+1}})$$

By homological stability results of Galatius and Randal-Williams we have that for $2k \geq 2$ the operad $\mathcal{W}^{2k}$ is an OHS and

$$\Omega B_0 \mathcal{W}^{2k}(0) \simeq \left(\operatorname*{hocolim}_{g \longrightarrow \infty} \mathcal{M}_k^{\theta}(W_{g,1}, \ell_{W_{g,1}}) \right)^+ \simeq \Omega_0^{\infty} \mathbf{MT}\theta.$$