Tangent ∞-categories and Goodwillie calculus

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Inspired by:

Goodwillie Calculus: Analyze functors between $\infty$-categories by analogy with ordinary calculus / differential geometry:

$\infty$-category $\mathcal{C}$ $\leftrightarrow$ smooth manifold

functor $F : \mathcal{C} \to \mathcal{D}$ $\leftrightarrow$ smooth map

How far does this analogy go?
Definition (Lurie, Higher Algebra 7.3.1)

The tangent bundle on a (differentiable) ∞-category \( \mathcal{C} \) is the ∞-category

\[
\mathcal{T}\mathcal{C} := \text{Exc}(\text{Top}^\text{fin}_*, \mathcal{C})
\]

of excisive functors \( \text{Top}^\text{fin}_* \to \mathcal{C} \), together with the projection map

\[
p_{\mathcal{C}} : \mathcal{T}\mathcal{C} \to \mathcal{C}; \quad L \mapsto L(*)..
\]

\[
\mathcal{T}_X \text{Top} \simeq \text{Sp}_X \quad \text{(spectra parameterized over a space } X \text{)}
\]

\[
\mathcal{T}_R \text{CRingSp} \simeq \text{Mod}_R \quad \text{(modules over a commutative ring spectrum } R \text{)}
\]

In general, the tangent space to \( \mathcal{C} \) at \( X \) is: \( \mathcal{T}_X \mathcal{C} \simeq \text{Sp}(\mathcal{C}/X) \).
The tangent bundle on a (differentiable) $\infty$-category $\mathcal{C}$ is the $\infty$-category

$$T\mathcal{C} := \text{Exc}(\text{Top}^\text{fin}_*, \mathcal{C}).$$

The total derivative of a (finitary) functor $F : \mathcal{C} \to \mathcal{D}$ is

$$TF : T\mathcal{C} \to T\mathcal{D}; \quad L \mapsto P_1(FL).$$

These constructions form a tangent bundle functor $T : \text{Cat}^\text{diff}_\infty \to \text{Cat}^\text{diff}_\infty$ on the $\infty$-category of differentiable $\infty$-categories and finitary functors.

Goal of this Talk

Make precise the analogy between $T$ and the ordinary tangent bundle functor on the category $\text{Mfld}$ of smooth manifolds and smooth maps.
A notion of tangent category was developed by:

- Jiří Rosický, Abstract tangent functors, *Diagrammes* 12 (1984);
- Robin Cockett and Geoff Cruttwell, Differential structure, tangent structure and SDG, *Applied Categorical Structures* 22 (2014);

in order to axiomatize the categorical properties of the tangent bundle functor

$$T : \text{Mfld} \to \text{Mfld},$$

and to highlight connections to other “tangent” structures, including

- Zariski tangent spaces in algebraic geometry;
- synthetic differential geometry (SDG);
- “differential” structures in computer science and logic.

We give an equivalent definition from

Definition (Leung)

**Weil**: the category of augmented commutative \(\mathbb{N}\)-algebras (semirings / rigs) with:

- objects: \(\mathbb{N}[x_1, \ldots, x_n]/(x_ix_j \mid (i, j) \in R)\) for an equivalence relation \(R\) on \(\{1, \ldots, n\}\) for some \(n \geq 0\);
- morphisms: maps of commutative \(\mathbb{N}\)-algebras that commute with the augmentation.

Examples:

- \(\mathbb{N}\)
- \(W = \mathbb{N}[x]/(x^2)\)
- \(W \otimes W = \mathbb{N}[x, y]/(x^2, y^2)\) (the coproduct in Weil)
- \(W^2 = \mathbb{N}[x, y]/(x^2, xy, y^2)\) (the product in Weil)
- \(W^{n_1} \otimes \cdots \otimes W^{n_k}\) (every object of Weil is isomorphic to one of these)
Definition (Leung (for 1-categories); Bauer-Burke-C (for ∞-categories))

A tangent structure on an ∞-category $\mathcal{X}$ is a (strong) monoidal functor

$$T^\bullet : (\text{Weil}, \otimes, \mathbb{N}) \to (\text{End}(\mathcal{X}), \circ, \text{Id})$$

that preserves the following pullback squares in Weil: for any $A \in \text{Weil}$ and $n, m > 0$:

$$
\begin{array}{c}
A \otimes W^{n+m} \rightarrow A \otimes W^m \\
\downarrow \quad \downarrow \quad \downarrow \quad \downarrow
\end{array}
\quad
\begin{array}{c}
W^2 \rightarrow W \otimes W \\
\downarrow \quad \downarrow \quad \downarrow \quad \downarrow
\end{array}
\quad
\begin{array}{c}
A \otimes W^n \rightarrow A \\
\downarrow \quad \downarrow \quad \downarrow \quad \downarrow
\end{array}
\quad
\begin{array}{c}
\mathbb{N} \rightarrow W \\
\downarrow \quad \downarrow \quad \downarrow \quad \downarrow
\end{array}
$$

The value of $T^\bullet$ on an arbitrary Weil-algebra $W^{n_1} \otimes \cdots \otimes W^{n_k}$ is determined by the endofunctor $T = T^W : \mathcal{X} \to \mathcal{X}$.

The value of $T^\bullet$ on the augmentation $\epsilon : W \to \mathbb{N}$ determines a natural transformation $T^\epsilon : T^W \rightarrow T^\mathbb{N}$, the projection $p : T \rightarrow \text{Id}$.
### Examples

1. $X = \text{Mfld}: T : \text{Mfld} \to \text{Mfld}$ is the usual tangent bundle functor with the pullbacks:

   $TM \times_M TM \to T(TM)$

   $\begin{array}{ccc}
   TM \times M & \to & T(TM) \\
   p \downarrow & & \downarrow T(p_M) \\
   M & \xrightarrow{0} & TM
   \end{array}$

   In particular: $T(T_xM) \cong T_xM \times T_xM$.

2. $X = \text{commutative rings}: T(R) = R[x]/(x^2)$

3. $X = \text{schemes over } \text{Spec } \mathbb{k}: T(S) = \underline{\text{Hom}}_{\mathbb{k}}(\text{Spec } \mathbb{k}[x]/(x^2), S)$

4. $X = \text{category of ‘microlinear’ objects in a model of SDG}$

5. $X = \text{model for the differential } \lambda\text{-calculus}$

6. $X = \text{comm. ring spectra}: T(R) := R \oplus R$ (square-zero extension)

7. (?) $X = \text{derived schemes}$

8. (?) $X = \text{derived smooth manifolds}$
An $\infty$-category $\mathcal{C}$ is **differentiable** if it has finite limits and sequential colimits, which commute.

$\text{Cat}_{\text{diff}}^{\infty}$: the $\infty$-category of (small) differentiable $\infty$-categories and sequential-colimit-preserving functors.

**Theorem (Bauer-Burke-C)**

There is a tangent structure on $\text{Cat}_{\text{diff}}^{\infty}$ for which:

- $T : \text{Cat}_{\text{diff}}^{\infty} \to \text{Cat}_{\text{diff}}^{\infty}$; $\mathcal{C} \mapsto \text{Exc}(\text{Top}_{\text{fin}}^{\ast}, \mathcal{C})$;
- the **projection** $p : T\mathcal{C} \to \mathcal{C}$ is
  $$L \mapsto L(\ast);$$
- the **zero section** $0 : \mathcal{C} \to T\mathcal{C}$ is
  $$X \mapsto \text{const}_X;$$
- the **fibrewise addition** $+ : T\mathcal{C} \times_{\mathcal{C}} T\mathcal{C} \to T\mathcal{C}$ is given by
  $$(L, L') \mapsto L(-) \times_{L(\ast) = L'(\ast)} L'(\ast).$$
Lemma (‘Universality of Vertical Lift’)

For a differentiable $\infty$-category $\mathcal{C}$, there is a pullback square (in $\mathbf{Cat}^{\text{diff}}$):

$$
\begin{array}{ccc}
T\mathcal{C} \times\mathcal{C} & \sim & T(T\mathcal{C}) \\
\downarrow & & \downarrow T(p) \\
\mathcal{C} & \sim & T\mathcal{C}
\end{array}
$$

For any object $X \in \mathcal{C}$: $T(T_X\mathcal{C}) \sim T_X\mathcal{C} \times T_X\mathcal{C}$. (In fact, the tangent bundle on any stable $\infty$-category splits in this way.)

Proof.

The pullback consists of functors $L : (\text{Top}_{\text{fin}}^*)^2 \to \mathcal{C}$ that are excisive in each variable separately, and reduced in one variable. Such functors split as $L_0(Y) \times L_1(X \land Y)$ for excisive functors $L_0, L_1 : \text{Top}_{\text{fin}}^* \to \mathcal{C}$. \hfill \qed
Any constructions that can be done in an arbitrary tangent (∞-)category can now be applied to our tangent structure on $\mathcal{C}at_{\text{diff}}^\infty$.

1. A differential object in a tangent $\infty$-category $\mathcal{X}$ is a commutative monoid $M$ together with a suitable splitting $TM \simeq M \times M$. In Mfld these are the vector spaces $\mathbb{R}^n$. In $\mathcal{C}at_{\text{diff}}^\infty$ these are the stable $\infty$-categories.

2. A differential bundle in $\mathcal{X}$ is a bundle of commutative monoids $q : E \to M$ together with a suitable pullback

$$E \times_M E \to TE$$

$$\downarrow \quad \downarrow T(q)$$

$$M \quad 0 \to TM$$

In Mfld these are the vector bundles (of locally constant rank). In $\mathcal{C}at_{\text{diff}}^\infty$ they are some kind of bundles of stable $\infty$-categories.

There are notions of **connections**, **curvature** and **cohomology** in a tangent category. What do these look like in $\mathcal{C}_{\text{at}_{\text{diff}}}^\infty$?


An $n$-jet at $x : \ast \to \mathcal{C}$ is an equivalence class of morphisms $\mathcal{C} \to \mathcal{D}$ where $f \sim g$ if:

- $f(x) \simeq y \simeq g(x)$ for some point $y$ in $\mathcal{D}$;
- $f, g$ induce equivalent maps

$$T_x^n f \simeq T_x^n g : T_x^n \mathcal{C} \to T_y^n \mathcal{D}.$$ 

In Mfld, this recovers the usual notion of $n$-jets of smooth maps.

**Theorem (Bauer-Burke-C)**

In $\mathsf{Cat}^{\mathsf{diff}}_{\infty}$, functors $F, G : \mathcal{C} \to \mathcal{D}$ determine the same $n$-jet at $X \in \mathcal{C}$ if and only if

$$P^x_n F \simeq P^x_n G.$$ 

So $n$-jets correspond to Goodwillie’s $n$-excisive functors.

What about the Taylor tower?
The **Taylor tower** of \( F : \mathcal{C} \to \mathcal{D} \) is a sequence

\[
F \to \cdots \to P_2F \to P_1F \to P_0F
\]

of 2-morphisms in the \((\infty, 2)\)-category \( \text{Cat}^{\text{diff}}_\infty \) of differentiable \( \infty \)-categories and sequential-colimit-preserving functors (and natural transformations).

**Definition**

A **tangent structure** on an \((\infty, 2)\)-category \( \mathbf{X} \) is a (strong) monoidal functor

\[
T : (\text{Weil}, \otimes, \mathbb{N}) \to (\text{End}(\mathbf{X}), \circ, \text{Id})
\]

that preserves the relevant pullbacks. (More generally, we can define tangent structures on any object of an arbitrary \((\infty, 2)\)-category \( \mathbf{C} \).)

**Theorem (Bauer-Burke-C)**

*The tangent structure on \( \text{Cat}^{\text{diff}}_\infty \) is the restriction of a tangent structure on the \((\infty, 2)\)-category \( \text{Cat}^{\text{diff}}_\infty \).*