Tangent ∞-categories and Goodwillie calculus

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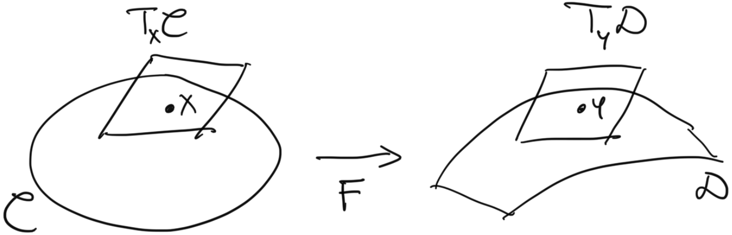
Inspired by:

Kristine Bauer, Brenda Johnson, Christina Osborne, Emily Riehl and Amelia Tebbe, Directional derivatives and higher order chain rules for abelian functor calculus, *Topology Appl.* 235 (2018)

Goodwillie Calculus: Analyze functors between ∞-categories by analogy with ordinary calculus / differential geometry:

∞-category $\mathcal{C}$	$\leftrightsquigarrow$	smooth manifold
functor $F : \mathcal{C} \to \mathcal{D}$	$\leftrightsquigarrow$	smooth map



How far does this analogy go?

Definition (Lurie, Higher Algebra 7.3.1)

The tangent bundle on a (differentiable) ∞-category $\mathcal{C}$ is the ∞-category

$$T\mathcal{C} := \mathrm{Exc}(\mathrm{Top}_*^{\mathrm{fin}}, \mathcal{C})$$

of excisive functors $\mathrm{Top}_*^{\mathrm{fin}} \to \mathcal{C}$, together with the projection map

$$p_{\mathcal{C}} : T\mathcal{C} \to \mathcal{C}; \quad L \mapsto L(*).$$

$T_X\mathrm{Top} \simeq \mathrm{Sp}_X$ (spectra parameterized over a space X)

$T_R\mathrm{CRing}_{\mathrm{Sp}} \simeq \mathrm{Mod}_R$ (modules over a commutative ring spectrum R)

In general, the tangent space to $\mathcal{C}$ at X is: $T_X\mathcal{C} \simeq \mathrm{Sp}(\mathcal{C}_{/X})$.

Definition

The tangent bundle on a (differentiable) ∞-category $\mathcal{C}$ is the ∞-category

$$T\mathcal{C} := \mathrm{Exc}(\mathrm{Top}_*^{\mathrm{fin}}, \mathcal{C}).$$

The total derivative of a (finitary) functor $F : \mathcal{C} \to \mathcal{D}$ is

$$TF : T\mathcal{C} \to T\mathcal{D}; \quad L \mapsto P_1(FL).$$

These constructions form a tangent bundle functor $T : \mathcal{C}\mathrm{at}_\infty^{\mathrm{diff}} \to \mathcal{C}\mathrm{at}_\infty^{\mathrm{diff}}$ on the ∞-category of differentiable ∞-categories and finitary functors.

Goal of this Talk

Make precise the analogy between T and the ordinary tangent bundle functor on the category Mfld *of smooth manifolds and smooth maps.*

A notion of tangent category was developed by:

- Jiří Rosický, Abstract tangent functors, *Diagrammes* 12 (1984);
- Robin Cockett and Geoff Cruttwell, Differential structure, tangent structure and SDG, *Applied Categorical Structures* 22 (2014);

in order to axiomatize the categorical properties of the tangent bundle functor

$$T : \mathsf{Mfld} \to \mathsf{Mfld},$$

and to highlight connections to other “tangent” structures, including

- Zariski tangent spaces in algebraic geometry;
- synthetic differential geometry (SDG);
- “differential” structures in computer science and logic.

We give an equivalent definition from

- Poon Leung, Classifying tangent structures using Weil algebras, *Theory and Applications of Categories* 32(9), (2017).

Definition (Leung)

Weil: the category of augmented commutative $\mathbb{N}$-algebras (semirings / rigs) with:

- objects: $\mathbb{N}[x_1, \ldots, x_n]/(x_i x_j \mid (i,j) \in R)$ for an equivalence relation R on $\{1, \ldots, n\}$ for some $n \geq 0$;
- morphisms: maps of commutative $\mathbb{N}$-algebras that commute with the augmentation.

Examples:

- $\mathbb{N}$
- $W = \mathbb{N}[x]/(x^2)$
- $W \otimes W = \mathbb{N}[x, y]/(x^2, y^2)$ (the coproduct in Weil)
- $W^2 = \mathbb{N}[x, y]/(x^2, xy, y^2)$ (the product in Weil)
- $W^{n_1} \otimes \cdots \otimes W^{n_k}$ (every object of Weil is isomorphic to one of these)

Definition (Leung (for 1-categories); Bauer-Burke-C (for ∞-categories))

A *tangent structure* on an ∞-category $\mathcal{X}$ is a (strong) monoidal functor

$$T^\bullet : (\mathrm{Weil}, \otimes, \mathbb{N}) \to (\mathrm{End}(\mathcal{X}), \circ, \mathrm{Id})$$

that preserves the following pullback squares in Weil: for any $A \in \mathrm{Weil}$ and $n, m > 0$:

$$\begin{array}{ccc} A \otimes W^{n+m} & \longrightarrow & A \otimes W^m \\ \downarrow & & \downarrow \\ A \otimes W^n & \longrightarrow & A \end{array} \qquad \begin{array}{ccc} W^2 & \xrightarrow[y \mapsto y]{x \mapsto xy} & W \otimes W \\ \downarrow & & \downarrow {\scriptstyle x \mapsto x,\ y \mapsto 0} \\ \mathbb{N} & \longrightarrow & W \end{array}$$

- The value of $T^\bullet$ on an arbitrary Weil-algebra $W^{n_1} \otimes \cdots \otimes W^{n_k}$ is determined by the endofunctor $T = T^W : \mathcal{X} \to \mathcal{X}$.
- The value of $T^\bullet$ on the augmentation $\epsilon : W \to \mathbb{N}$ determines a natural transformation $T^\epsilon : T^W \to T^{\mathbb{N}}$, the projection $p : T \to \mathrm{Id}$.

Examples

1. $\mathcal{X} = \mathrm{Mfld}$: $T : \mathrm{Mfld} \to \mathrm{Mfld}$ *is the usual tangent bundle functor with the pullbacks:*

$$\begin{array}{ccc} TM \times_M TM & \longrightarrow & T(TM) \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle T(p_M)} \\ M & \xrightarrow{0} & TM \end{array}$$

In particular: $T(T_xM) \cong T_xM \times T_xM$.

2. $\mathcal{X} =$ *commutative rings:* $T(R) = R[x]/(x^2)$
3. $\mathcal{X} =$ *schemes over* $\mathrm{Spec}\,\mathbf{k}$: $T(S) = \underline{\mathrm{Hom}}_{\mathbf{k}}(\mathrm{Spec}\,\mathbf{k}[x]/(x^2), S)$
4. $\mathcal{X} =$ *category of 'microlinear' objects in a model of SDG*
5. $\mathcal{X} =$ *model for the differential* λ*-calculus*
6. $\mathcal{X} =$ *comm. ring spectra:* $T(R) := R \oplus R$ *(square-zero extension)*
7. *(?)* $\mathcal{X} =$ *derived schemes*
8. *(?)* $\mathcal{X} =$ *derived smooth manifolds*

An ∞-category $\mathcal{C}$ is differentiable if it has finite limits and sequential colimits, which commute.

$\mathcal{C}at_\infty^{\mathrm{diff}}$: the ∞-category of (small) differentiable ∞-categories and sequential-colimit-preserving functors.

Theorem (Bauer-Burke-C)

There is a tangent structure on $\mathcal{C}at_\infty^{\mathrm{diff}}$ for which:

- $T : \mathcal{C}at_\infty^{\mathrm{diff}} \to \mathcal{C}at_\infty^{\mathrm{diff}}; \quad \mathcal{C} \mapsto \mathrm{Exc}(\mathrm{Top}_*^{\mathrm{fin}}, \mathcal{C})$;
- *the projection* $p : T\mathcal{C} \to \mathcal{C}$ *is*

$$L \mapsto L(*);$$

- *the zero section* $0 : \mathcal{C} \to T\mathcal{C}$ *is*

$$X \mapsto \mathrm{const}_X;$$

- *the fibrewise addition* $+ : T\mathcal{C} \times_{\mathcal{C}} T\mathcal{C} \to T\mathcal{C}$ *is given by*

$$(L, L') \mapsto L(-) \times_{L(*)=L'(*)} L'(-).$$

Lemma ('Universality of Vertical Lift')

For a differentiable ∞-category $\mathcal{C}$, there is a pullback square (in $\mathcal{C}at_{\infty}^{\text{diff}}$):

$$\begin{array}{ccc} T\mathcal{C} \times_{\mathcal{C}} T\mathcal{C} & \longrightarrow & T(T\mathcal{C}) \\ \downarrow & & \downarrow {\scriptstyle T(p)} \\ \mathcal{C} & \xrightarrow{\;0\;} & T\mathcal{C} \end{array}$$

For any object $X \in \mathcal{C}$: $T(T_X\mathcal{C}) \simeq T_X\mathcal{C} \times T_X\mathcal{C}$. (In fact, the tangent bundle on any stable ∞-category splits in this way.)

Proof.

The pullback consists of functors $L : (\text{Top}_*^{\text{fin}})^2 \to \mathcal{C}$ that are excisive in each variable separately, and reduced in one variable. Such functors split as $L_0(Y) \times L_1(X \wedge Y)$ for excisive functors $L_0, L_1 : \text{Top}_*^{\text{fin}} \to \mathcal{C}$. □

Any constructions that can be done in an arbitrary tangent (∞-)category can now be applied to our tangent structure on $\mathcal{C}at_\infty^{\mathrm{diff}}$.

1. A differential object in a tangent ∞-category $\mathcal{X}$ is a commutative monoid M together with a suitable splitting $TM \simeq M \times M$. In Mfld these are the vector spaces $\mathbb{R}^n$. In $\mathcal{C}at_\infty^{\mathrm{diff}}$ these are the stable ∞-categories.
2. A differential bundle in $\mathcal{X}$ is a bundle of commutative monoids $q : E \to M$ together with a suitable pullback

$$\begin{array}{ccc} E \times_M E & \longrightarrow & TE \\ \downarrow & & \downarrow{\scriptstyle T(q)} \\ M & \xrightarrow{0} & TM \end{array}$$

In Mfld these are the vector bundles (of locally constant rank). In $\mathcal{C}at_\infty^{\mathrm{diff}}$ they are some kind of bundles of stable ∞-categories?

- Robin Cockett and Geoff Cruttwell, Differential bundles and fibrations for tangent categories, *Cah. Topol. Géom. Différ. Catég.* 59 (2018).

3. There are notions of connections, curvature and cohomology in a tangent category. What do these look like in $\mathcal{C}at_\infty^{\text{diff}}$?

   - Robin Cockett and Geoff Cruttwell, Connections in tangent categories, *Theory and Applications of Categories* 32 (2017).
   - Geoff Cruttwell and Rory Lucyshyn-Wright, A simplicial foundation for differential and sector forms in tangent categories, *J. Homotopy Relat. Struct.* 13 (2018).

4. An *n*-jet at $x : * \to \mathcal{C}$ is an equivalence class of morphisms $\mathcal{C} \to \mathcal{D}$ where $f \sim g$ if:
   - $f(x) \simeq y \simeq g(x)$ for some point y in $\mathcal{D}$;
   - f, g induce equivalent maps

$$T_x^n f \simeq T_x^n g : T_x^n \mathcal{C} \to T_y^n \mathcal{D}.$$

In Mfld, this recovers the usual notion of *n*-jets of smooth maps.

Theorem (Bauer-Burke-C)

In $\mathcal{C}at_\infty^{\text{diff}}$, functors $F, G : \mathcal{C} \to \mathcal{D}$ determine the same n-jet at $X \in \mathcal{C}$ if and only if

$$P_n^x F \simeq P_n^x G.$$

So n-jets correspond to Goodwillie's n-excisive functors.

What about the Taylor tower?

The Taylor tower of $F : \mathcal{C} \to \mathcal{D}$ is a sequence

$$F \to \cdots \to P_2F \to P_1F \to P_0F$$

of 2-morphisms in the $(\infty, 2)$-category $\mathbf{Cat}_\infty^{\mathbf{diff}}$ of differentiable ∞-categories and sequential-colimit-preserving functors (and natural transformations).

Definition

A tangent structure on an $(\infty, 2)$-category $\mathbf{X}$ is a (strong) monoidal functor

$$T : (\mathrm{Weil}, \otimes, \mathbb{N}) \to (\mathrm{End}(\mathbf{X}), \circ, \mathrm{Id})$$

that preserves the relevant pullbacks. (More generally, we can define tangent structures on any object of an arbitary $(\infty, 2)$-category $\mathbf{C}$.)

Theorem (Bauer-Burke-C)

The tangent structure on $\mathcal{C}at_\infty^{\text{diff}}$ is the restriction of a tangent structure on the $(\infty, 2)$-category $\mathbf{Cat}_\infty^{\mathbf{diff}}$.