The homotopy of the motivic image-of-J spectrum

joint with Eva Belmont and Dan Isaksen

Hana Jia Kong

Institute for Advanced Study

November 27, 2021

We work 2-primarily:
Most of things are either 2-local or 2-complete without notational indications.

Classical story

- J-homomorphism:

Classical story

- J-homomorphism:
  - an orthogonal map $\mathbb{R}^n \to \mathbb{R}^n \rightsquigarrow$ a self map $S^n \to S^n$.

Classical story

- J-homomorphism:
  - an orthogonal map $\mathbb{R}^n \to \mathbb{R}^n \rightsquigarrow$ a self map $S^n \to S^n$.
  - compatible maps $O(n) \to \Omega^n S^n$ for all n.

Classical story

- J-homomorphism:
  - an orthogonal map $\mathbb{R}^n \to \mathbb{R}^n \rightsquigarrow$ a self map $S^n \to S^n$.
  - compatible maps $O(n) \to \Omega^n S^n$ for all n.
  - $J_* : \pi_*(O) \to \pi_*(S)$.

Classical story

- J-homomorphism:
  - an orthogonal map $\mathbb{R}^n \to \mathbb{R}^n \rightsquigarrow$ a self map $S^n \to S^n$.
  - compatible maps $O(n) \to \Omega^n S^n$ for all n.
  - $J_* : \pi_*(O) \to \pi_*(S)$.
- Image-of-J spectrum:

$$j := \mathrm{fib}(\mathbf{ko} \xrightarrow{\psi^3 - 1} \Sigma^4 \mathbf{ksp}).$$

Classical story

- J-homomorphism:
  - an orthogonal map $\mathbb{R}^n \to \mathbb{R}^n \rightsquigarrow$ a self map $S^n \to S^n$.
  - compatible maps $O(n) \to \Omega^n S^n$ for all n.
  - $J_* : \pi_*(O) \to \pi_*(S)$.
- Image-of-J spectrum:

$$j := \mathrm{fib}(\mathbf{ko} \xrightarrow{\psi^3 - 1} \Sigma^4 \mathbf{ksp}).$$

- $\mathbf{ko}$: connective real K-theory spectrum.

Classical story

- J-homomorphism:
  - an orthogonal map $\mathbb{R}^n \to \mathbb{R}^n \rightsquigarrow$ a self map $S^n \to S^n$.
  - compatible maps $O(n) \to \Omega^n S^n$ for all n.
  - $J_* : \pi_*(O) \to \pi_*(S)$.
- Image-of-J spectrum:

$$j := \mathrm{fib}(\mathbf{ko} \xrightarrow{\psi^3 - 1} \Sigma^4 \mathbf{ksp}).$$

- $\mathbf{ko}$: connective real K-theory spectrum.
- $\mathbf{ksp}$: connective quaternionic K-theory spectrum.

Classical story

- J-homomorphism:
  - an orthogonal map $\mathbb{R}^n \to \mathbb{R}^n \rightsquigarrow$ a self map $S^n \to S^n$.
  - compatible maps $O(n) \to \Omega^n S^n$ for all n.
  - $J_* : \pi_*(O) \to \pi_*(S)$.
- Image-of-J spectrum:

$$j := \mathrm{fib}(\mathbf{ko} \xrightarrow{\psi^3 - 1} \Sigma^4 \mathbf{ksp}).$$

- $\mathbf{ko}$: connective real K-theory spectrum.
- $\mathbf{ksp}$: connective quaternionic K-theory spectrum.
- ψ^3: the Adams operation.

Homotopy groups

- Bott periodicity:

Homotopy groups

- Bott periodicity:
$\mathbb{Z} \times BO, \quad O, \quad O/U, \quad U/Sp, \quad \mathbb{Z} \times BSp, \quad Sp, \quad Sp/U, \quad U/O.$

Homotopy groups

- Bott periodicity:
  $\mathbb{Z} \times BO, \quad O, \quad O/U, \quad U/Sp, \quad \mathbb{Z} \times BSp, \quad Sp, \quad Sp/U, \quad U/O.$
- $\Sigma^4\mathbf{ksp}$ is the 4th connective cover of $\mathbf{ko}$.

Homotopy groups

- Bott periodicity:
  $\mathbb{Z} \times BO, \quad O, \quad O/U, \quad U/Sp, \quad \mathbb{Z} \times BSp, \quad Sp, \quad Sp/U, \quad U/O.$
- $\Sigma^4 \mathbf{ksp}$ is the 4th connective cover of $\mathbf{ko}$.
- Homotopy groups:

	0	1	2	3	4	5	6	7	8	9	10	11
$\pi_* \mathbf{ko}$	$\mathbb{Z}$	$\mathbb{F}_2$	$\mathbb{F}_2$	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	$\mathbb{F}_2$	$\mathbb{F}_2$	0

Homotopy groups

- Bott periodicity:
  $\mathbb{Z} \times BO, \quad O, \quad O/U, \quad U/Sp, \quad \mathbb{Z} \times BSp, \quad Sp, \quad Sp/U, \quad U/O.$
- $\Sigma^4\mathbf{ksp}$ is the 4th connective cover of $\mathbf{ko}$.
- Homotopy groups:

	0	1	2	3	4	5	6	7	8	9	10	11
$\pi_*\mathbf{ko}$	$\mathbb{Z}$	$\mathbb{F}_2$	$\mathbb{F}_2$	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	$\mathbb{F}_2$	$\mathbb{F}_2$	0
$\pi_*\Sigma^4\mathbf{ksp}$	0	0	0	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	$\mathbb{F}_2$	$\mathbb{F}_2$	0

.

homotopy of j

$$\cdots \to \pi_n(j) \to \pi_n(\mathbf{ko}) \to \pi_n(\Sigma^4\mathbf{ksp}) \to \pi_{n-1}(j) \to \cdots$$

homotopy of j

$$\cdots \to \pi_n(j) \to \pi_n(\mathbf{ko}) \to \pi_n(\Sigma^4\mathbf{ksp}) \to \pi_{n-1}(j) \to \cdots$$

	0	1	2	3	4	5	6	7	8	9	10
$\pi_*\mathbf{ko}$	$\mathbb{Z}$	$\mathbb{F}_2$	$\mathbb{F}_2$	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	$\mathbb{F}_2$	$\mathbb{F}_2$
					$\downarrow \times 8$				$\downarrow \times 16$		
$\pi_*\Sigma^4\mathbf{ksp}$	0	0	0	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	$\mathbb{F}_2$	$\mathbb{F}_2$

homotopy of j

$$\cdots \to \pi_n(j) \to \pi_n(\mathbf{ko}) \to \pi_n(\Sigma^4\mathbf{ksp}) \to \pi_{n-1}(j) \to \cdots$$

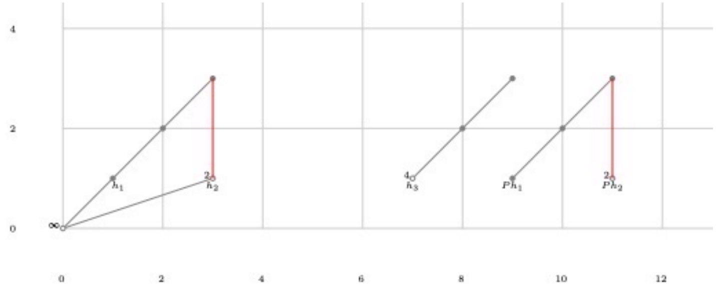
	0	1	2	3	4	5	6	7	8	9	10
$\pi_*\mathbf{ko}$	$\mathbb{Z}$	$\mathbb{F}_2$	$\mathbb{F}_2$	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	$\mathbb{F}_2$	$\mathbb{F}_2$
					↓ ×8				↓ ×16		
$\pi_*\Sigma^4\mathbf{ksp}$	0	0	0	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	$\mathbb{F}_2$	$\mathbb{F}_2$.
π_*j	$\mathbb{Z}$	$\mathbb{F}_2$	$\mathbb{F}_2$	$\mathbb{Z}/8$	0	0	0	$\mathbb{Z}/16$	$\mathbb{F}_2$	$(\mathbb{F}_2)^2$	

homotopy of j

$$\cdots \to \pi_n(j) \to \pi_n(\mathbf{ko}) \to \pi_n(\Sigma^4\mathbf{ksp}) \to \pi_{n-1}(j) \to \cdots$$

	0	1	2	3	4	5	6	7	8	9	10
$\pi_*\mathbf{ko}$	$\mathbb{Z}$	$\mathbb{F}_2$	$\mathbb{F}_2$	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	$\mathbb{F}_2$	$\mathbb{F}_2$
					$\times 8$				$\times 16$		
$\pi_*\Sigma^4\mathbf{ksp}$	0	0	0	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	$\mathbb{F}_2$	$\mathbb{F}_2$
$\pi_* j$	$\mathbb{Z}$	$\mathbb{F}_2$	$\mathbb{F}_2$	$\mathbb{Z}/8$	0	0	0	$\mathbb{Z}/16$	$\mathbb{F}_2$	$(\mathbb{F}_2)^2$	

- Adams–Novikov α-family:



Motivic analogue

What is the motivic analogue?

Motivic analogue

What is the motivic analogue?

- $\mathrm{SH}(k)$: motivic stable homotopy category over base field k.

Motivic analogue

What is the motivic analogue?

- $\mathrm{SH}(k)$: motivic stable homotopy category over base field k.
- (Bachmann–Hopkins) Motivice $j := \mathrm{fib}(\mathbf{ko} \xrightarrow{\psi^3 - 1} \Sigma^4 \mathbf{ksp})$

Motivic analogue

What is the motivic analogue?

- $\mathrm{SH}(k)$: motivic stable homotopy category over base field k.
- (Bachmann–Hopkins) Motivice $j := \mathrm{fib}(\mathbf{ko} \xrightarrow{\psi^3-1} \Sigma^4\mathbf{ksp})$

$$\Rightarrow j := \mathrm{fib}(\mathbf{ko} \xrightarrow{\psi^3-1} \Sigma^{4,2}\mathbf{ksp})$$

Motivic analogue

What is the motivic analogue?

- $\mathrm{SH}(k)$: motivic stable homotopy category over base field k.
- (Bachmann–Hopkins) Motivice $j := \mathrm{fib}(\mathbf{ko} \xrightarrow{\psi^3 - 1} \Sigma^4 \mathbf{ksp})$

$$\Rightarrow j := \mathrm{fib}(\mathbf{ko} \xrightarrow{\psi^3 - 1} \Sigma^{4,2} \mathbf{ksp})$$

- $\mathbf{ko}$: very effective motivic Hermitian K-theory spectrum.

Motivic analogue

What is the motivic analogue?

- $\mathrm{SH}(k)$: motivic stable homotopy category over base field k.
- (Bachmann–Hopkins) Motivice $j := \mathrm{fib}(\mathbf{ko} \xrightarrow{\psi^3-1} \Sigma^4\mathbf{ksp})$

$$\Rightarrow j := \mathrm{fib}(\mathbf{ko} \xrightarrow{\psi^3-1} \Sigma^{4,2}\mathbf{ksp})$$

- $\mathbf{ko}$: very effective motivic Hermitian K-theory spectrum.
- $\Sigma^{4,2}\mathbf{ksp}$: 2nd very effective cover of $\mathbf{ko}$.

Motivic analogue

What is the motivic analogue?

- $\mathrm{SH}(k)$: motivic stable homotopy category over base field k.
- (Bachmann–Hopkins) Motivice $j := \mathrm{fib}(\mathbf{ko} \xrightarrow{\psi^3 - 1} \Sigma^4 \mathbf{ksp})$

$$\Rightarrow j := \mathrm{fib}(\mathbf{ko} \xrightarrow{\psi^3 - 1} \Sigma^{4,2} \mathbf{ksp})$$

- $\mathbf{ko}$: very effective motivic Hermitian K-theory spectrum.
- $\Sigma^{4,2}\mathbf{ksp}$: 2nd very effective cover of $\mathbf{ko}$.
- ψ^3: motivic Adams operation.

Motivic analogue

- How do j and the motivic sphere compare?

Motivic analogue

- How do j and the motivic sphere compare?
- Computational tools for the motivic sphere:
  - The motivic Adams spectral sequence.
  - The effective slice spectral sequence.
  - ....

motivic Adams spectral sequence

- Motivic bigraded spheres $S^{s,w}$:

motivic Adams spectral sequence

- Motivic bigraded spheres $S^{s,w}$:
  - $S^{1,1}$: Tate circle $\mathbb{A}^1 - \{0\}$;
    $S^{1,0}$: topological circle.

motivic Adams spectral sequence

- Motivic bigraded spheres $S^{s,w}$:
  - $S^{1,1}$: Tate circle $\mathbb{A}^1 - \{0\}$;
    $S^{1,0}$: topological circle.
  - s: stem;
  - w: weight;

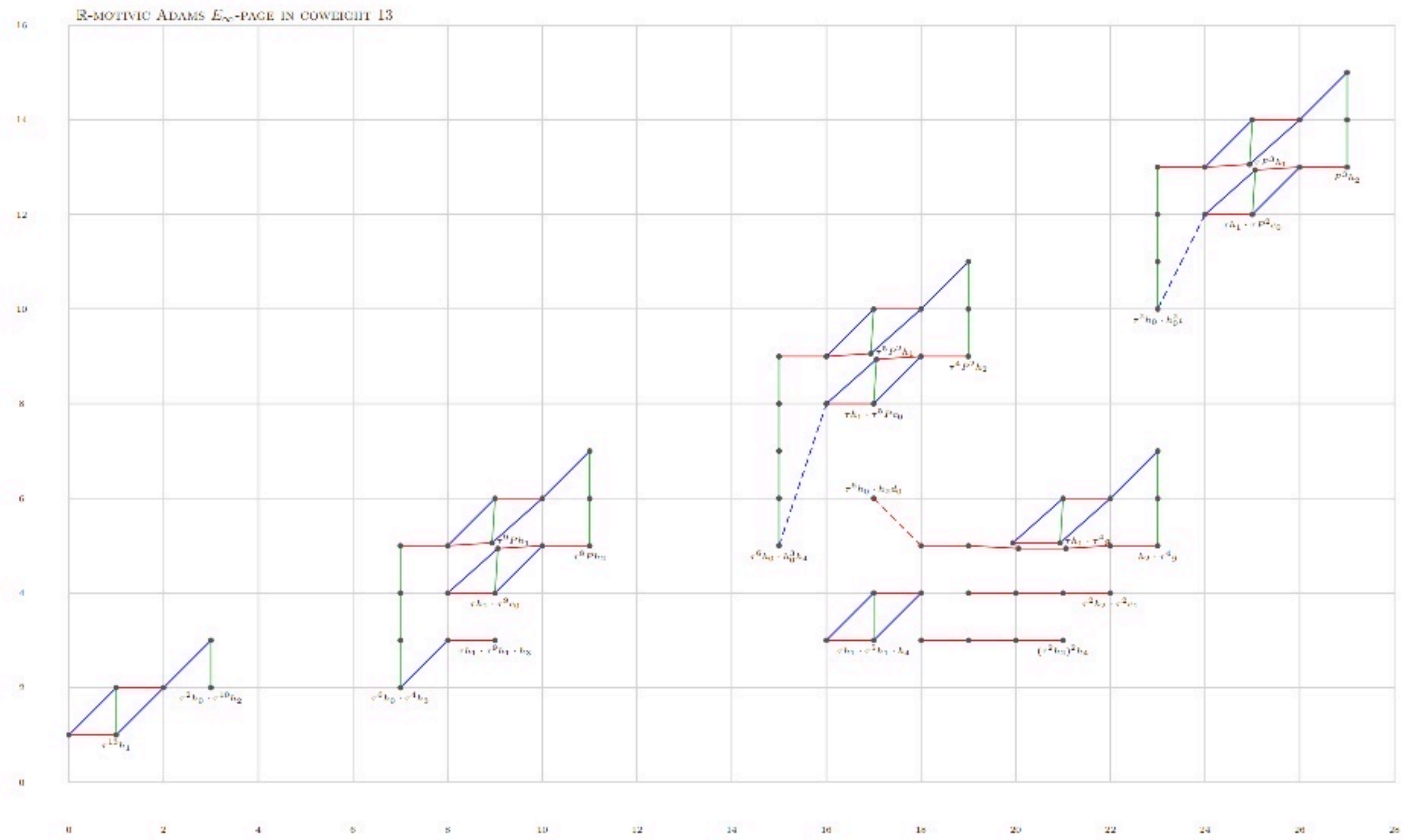
motivic Adams spectral sequence

- Motivic bigraded spheres $S^{s,w}$:
  - $S^{1,1}$: Tate circle $\mathbb{A}^1 - \{0\}$;
    $S^{1,0}$: topological circle.
  - s: stem;
  - w: weight;
- mASS: $E_2^{s,t,w} = \mathrm{Ext}_A^{s,t,w}(H\mathbb{F}_{2**}, H\mathbb{F}_{2**}) \Rightarrow \pi_{t-s,w}(S^{0,0})_2^\wedge$.

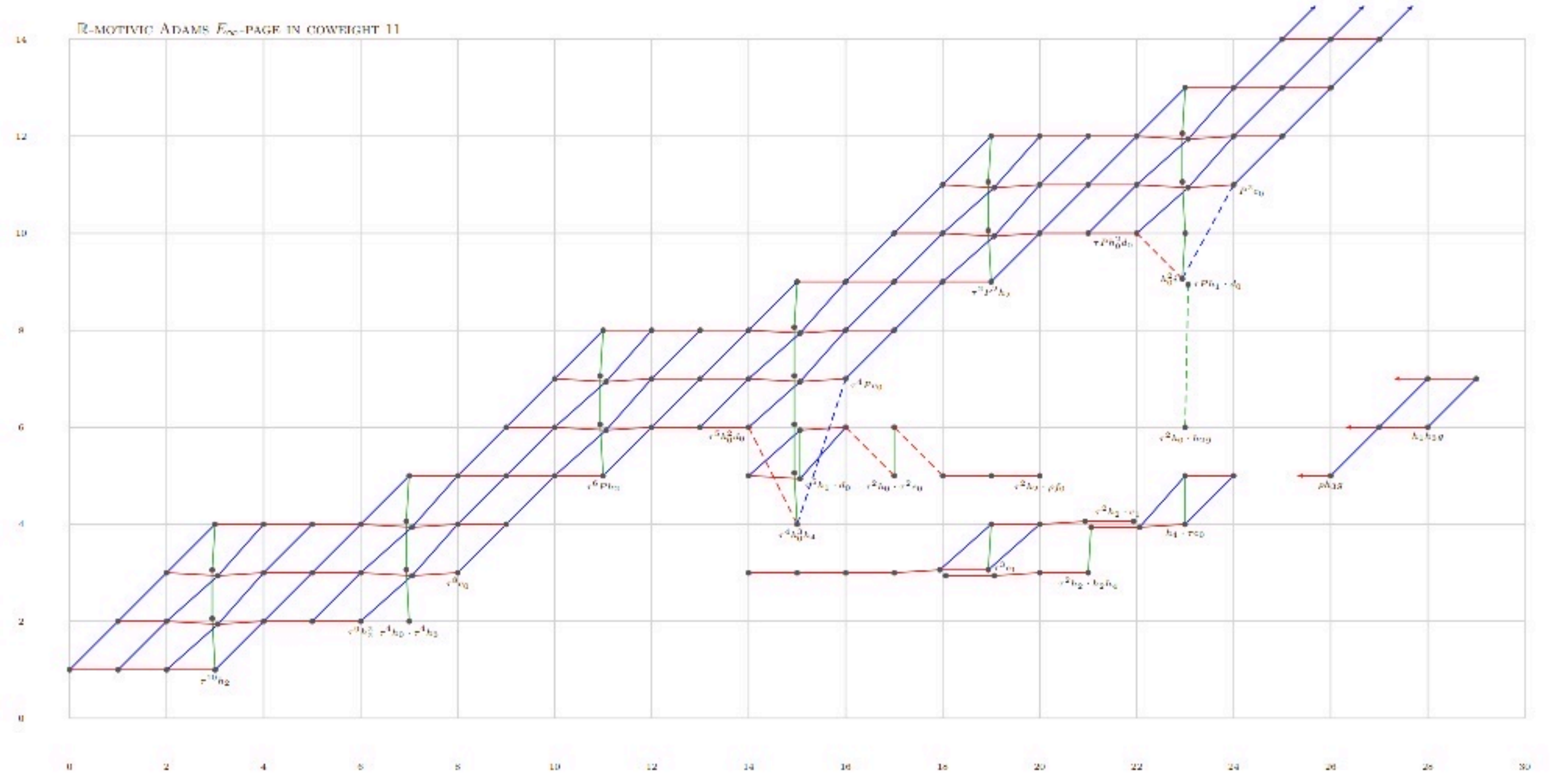
motivic Adams spectral sequence

- Motivic bigraded spheres $S^{s,w}$:
  - $S^{1,1}$: Tate circle $\mathbb{A}^1 - \{0\}$;
    $S^{1,0}$: topological circle.
  - s: stem;
  - w: weight;
- mASS: $E_2^{s,t,w} = \mathrm{Ext}_A^{s,t,w}(H\mathbb{F}_{2**}, H\mathbb{F}_{2**}) \Rightarrow \pi_{t-s,w}(S^{0,0})_2^{\wedge}$.
- Organize by coweight $s - w$.

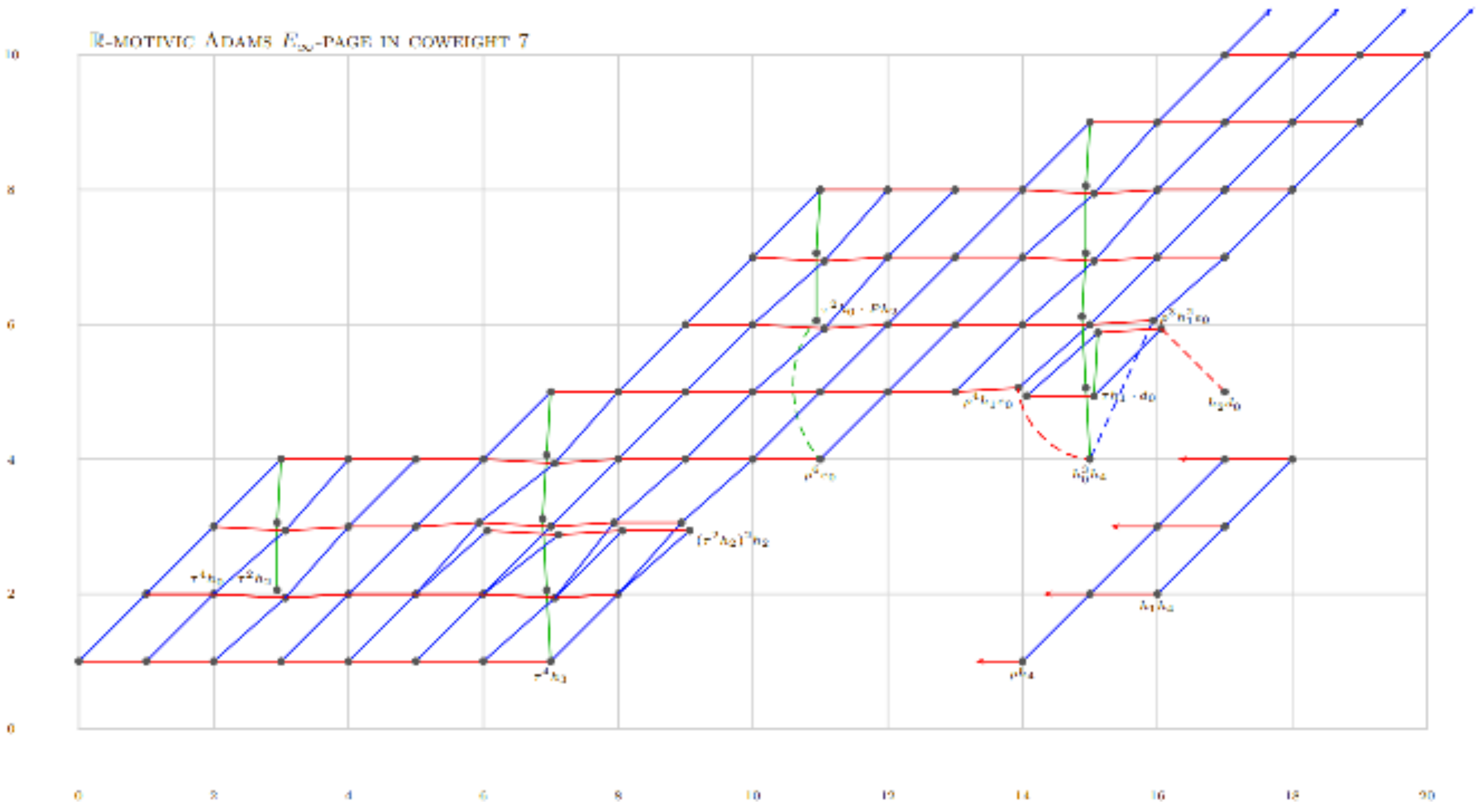
ℝ-motivic Adams spectral sequence



ℝ-motivic Adams spectral sequence



ℝ-motivic Adams spectral sequence



η-periodic phenomena

Approximation: η-inverted sphere.

- $\pi_* \eta^{-1} S$:

η-periodic phenomena

Approximation: η-inverted sphere.

- $\pi_* \eta^{-1} S$:
  Andrews–Miller,
  Guillou–Isaksen, G. Wilson,
  Ormsby–Röndigs,
  Bachmann–Hopkins.

η-periodic phenomena

Approximation: η-inverted sphere.

- $\pi_* \eta^{-1} S$:
  Andrews–Miller,
  Guillou–Isaksen, G. Wilson,
  Ormsby–Röndigs,
  Bachmann–Hopkins.

- There is a "Image of J"-style pattern.

Computational tool

Motivic effective slice spectral sequence on $\mathrm{SH}(k)$:

- $\mathrm{SH}^{\mathsf{eff}}(k) \subset \mathrm{SH}(k)$: the smallest triangulated full subcategory containing $\{\Sigma^\infty_+ X, X \in \mathrm{Sm}_k\}$.

Computational tool

Motivic effective slice spectral sequence on $\mathrm{SH}(k)$:

- $\mathrm{SH}^{\mathsf{eff}}(k) \subset \mathrm{SH}(k)$: the smallest triangulated full subcategory containing $\{\Sigma^\infty_+ X, X \in \mathrm{Sm}_k\}$.
- Have adjunction $i_q \colon \Sigma^{q,0}\mathrm{SH}^{\mathsf{eff}}(k) \rightleftarrows \mathrm{SH}(k) : r_q$.

Computational tool

Motivic effective slice spectral sequence on $\mathrm{SH}(k)$:

- $\mathrm{SH}^{\mathsf{eff}}(k) \subset \mathrm{SH}(k)$: the smallest triangulated full subcategory containing $\{\Sigma^\infty_+ X, X \in \mathrm{Sm}_k\}$.
- Have adjunction $i_q \colon \Sigma^{q,0}\mathrm{SH}^{\mathsf{eff}}(k) \rightleftarrows \mathrm{SH}(k) : r_q$.

$\rightsquigarrow$ qth effective cover $f_q := i_q \circ r_q$.

Computational tool

Motivic effective slice spectral sequence on $\mathrm{SH}(k)$:

- $\mathrm{SH}^{\mathsf{eff}}(k) \subset \mathrm{SH}(k)$: the smallest triangulated full subcategory containing $\{\Sigma^{\infty}_{+} X, X \in \mathrm{Sm}_k\}$.
- Have adjunction $i_q \colon \Sigma^{q,0}\mathrm{SH}^{\mathsf{eff}}(k) \rightleftarrows \mathrm{SH}(k) : r_q$.

  $\rightsquigarrow$ qth effective cover $f_q := i_q \circ r_q$.
- For any $X \in \mathrm{SH}(k)$, we have a tower and associated graded slices:

Computational tool

Motivic effective slice spectral sequence on $\mathrm{SH}(k)$:

- $\mathrm{SH}^{\mathsf{eff}}(k) \subset \mathrm{SH}(k)$: the smallest triangulated full subcategory containing $\{\Sigma^\infty_+ X, X \in \mathrm{Sm}_k\}$.
- Have adjunction $i_q \colon \Sigma^{q,0}\mathrm{SH}^{\mathsf{eff}}(k) \rightleftarrows \mathrm{SH}(k) : r_q$.

  $\rightsquigarrow$ qth effective cover $f_q := i_q \circ r_q$.
- For any $X \in \mathrm{SH}(k)$, we have a tower and associated graded slices:

$$\begin{array}{ccccccccc} \cdots \longrightarrow & f_{q+1}X & \longrightarrow & f_qX & \longrightarrow & f_{q-1}X & \longrightarrow \cdots \\ & \uparrow & & \uparrow & & & \\ & s_{q+1}X & & s_qX & & & \end{array}.$$

Computational tool

Motivic effective slice spectral sequence on $\mathrm{SH}(k)$:

- $\mathrm{SH}^{\mathsf{eff}}(k) \subset \mathrm{SH}(k)$: the smallest triangulated full subcategory containing $\{\Sigma^{\infty}_{+} X, X \in \mathrm{Sm}_k\}$.
- Have adjunction $i_q : \Sigma^{q,0}\mathrm{SH}^{\mathsf{eff}}(k) \rightleftarrows \mathrm{SH}(k) : r_q$.

  $\rightsquigarrow$ qth effective cover $f_q := i_q \circ r_q$.
- For any $X \in \mathrm{SH}(k)$, we have a tower and associated graded slices:

$$\begin{array}{ccccccccc} \cdots \longrightarrow & f_{q+1}X & \longrightarrow & f_q X & \longrightarrow & f_{q-1}X & \longrightarrow \cdots \\ & \uparrow & & \uparrow & & & \\ & s_{q+1}X & & s_q X & & & \end{array}.$$

- Effective slice spectral sequence: $\pi_{s,w} s_f(X) \implies \pi_{s,w} X$.

Computational tool

Motivic effective slice spectral sequence on $\mathrm{SH}(k)$:

- $\mathrm{SH}^{\mathsf{eff}}(k) \subset \mathrm{SH}(k)$: the smallest triangulated full subcategory containing $\{\Sigma^\infty_+ X, X \in \mathrm{Sm}_k\}$.
- Have adjunction $i_q \colon \Sigma^{q,0}\mathrm{SH}^{\mathsf{eff}}(k) \rightleftarrows \mathrm{SH}(k) \colon r_q$.

  $\rightsquigarrow$ qth effective cover $f_q := i_q \circ r_q$.
- For any $X \in \mathrm{SH}(k)$, we have a tower and associated graded slices:

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & f_{q+1}X & \longrightarrow & f_qX & \longrightarrow & f_{q-1}X & \longrightarrow & \cdots \\ & & \uparrow & & \uparrow & & & & \\ & & s_{q+1}X & & s_qX & & & & \end{array} .$$

- Effective slice spectral sequence: $\pi_{s,w}s_f(X) \implies \pi_{s,w}X$.
  $|d_r|_{(s,f,w)} = (-1, +r, 0)$; in particular $|d_r|_{\mathsf{coweight}} = -1$.

Example: slices of $\mathbf{ko}$

- (Ananyevskiy–Röndigs–Østvær) Effective slices of $\mathbf{ko}$:

Example: slices of $\mathbf{ko}$

- (Ananyevskiy–Röndigs–Østvær) Effective slices of $\mathbf{ko}$:

$$s_*(\mathbf{ko}) \simeq \mathrm{H}\mathbb{Z} \otimes \{\mathbb{Z}[\alpha_1, v_1^2]/2\alpha_1\}.$$

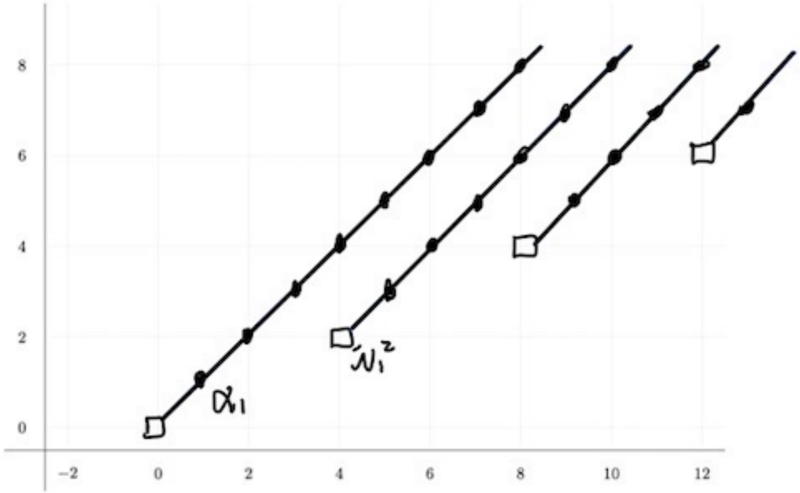
- $|\alpha_1| = (1,1), |v_1^2| = (4,2)$.

Example: slices of $\mathbf{ko}$

- (Ananyevskiy–Röndigs–Østvær) Effective slices of $\mathbf{ko}$:

$$s_*(\mathbf{ko}) \simeq \mathrm{H}\mathbb{Z} \otimes \{\mathbb{Z}[\alpha_1, v_1^2]/2\alpha_1\}.$$

- $|\alpha_1| = (1,1), |v_1^2| = (4,2)$.



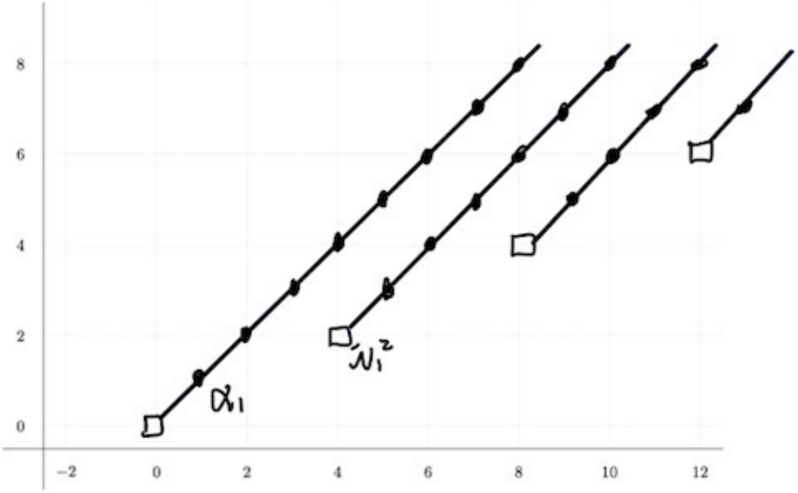
Example: slices of $\mathbf{ko}$

- (Ananyevskiy–Röndigs–Østvær) Effective slices of $\mathbf{ko}$:

$$s_*(\mathbf{ko}) \simeq \mathrm{H}\mathbb{Z} \otimes \{\mathbb{Z}[\alpha_1, v_1^2]/2\alpha_1\}.$$

- $|\alpha_1| = (1,1), |v_1^2| = (4,2)$.

$$s_1 = \Sigma^{1,1} H\mathbb{F}_2$$

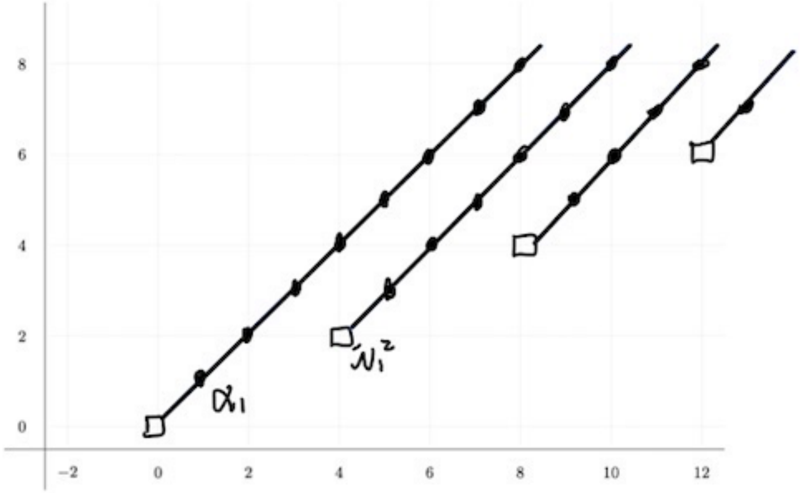


Example: slices of $\mathbf{ko}$

- (Ananyevskiy–Röndigs–Østvær) Effective slices of $\mathbf{ko}$:

$$s_*(\mathbf{ko}) \simeq \mathrm{H}\mathbb{Z} \otimes \{\mathbb{Z}[\alpha_1, v_1^2]/2\alpha_1\}.$$

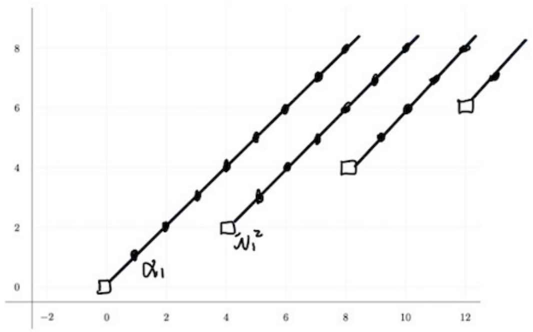
- $|\alpha_1| = (1,1), |v_1^2| = (4,2)$.



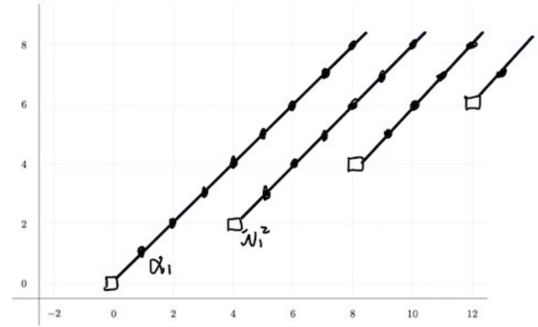
$s_1 = \Sigma^{1,1} H\mathbb{F}_2$
$s_2 = \Sigma^{2,2} H\mathbb{F}_2 \vee \Sigma^{4,2} H\mathbb{Z}$

Example: $\mathbb{R}$-motivic ESSS E_1 of $\mathbf{ko}$

Take $\pi_{*,*}$ of the slices to obtain E_1-page.



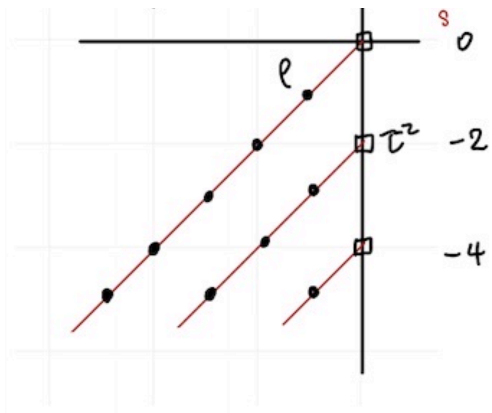
Example: $\mathbb{R}$-motivic ESSS E_1 of $\mathbf{ko}$



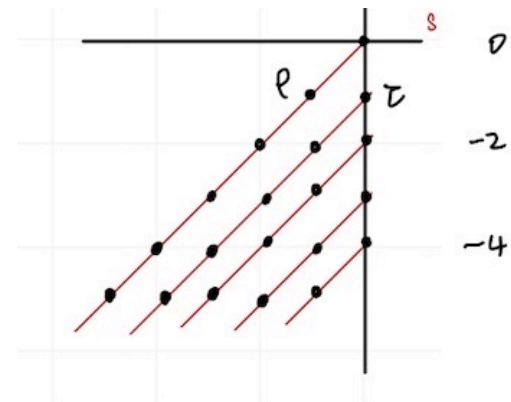
Take $\pi_{*,*}$ of the slices to obtain E_1-page.

$\mathbb{R}$-motivic coefficient rings:

$\mathbf{H}\mathbb{Z}_{*,*}$

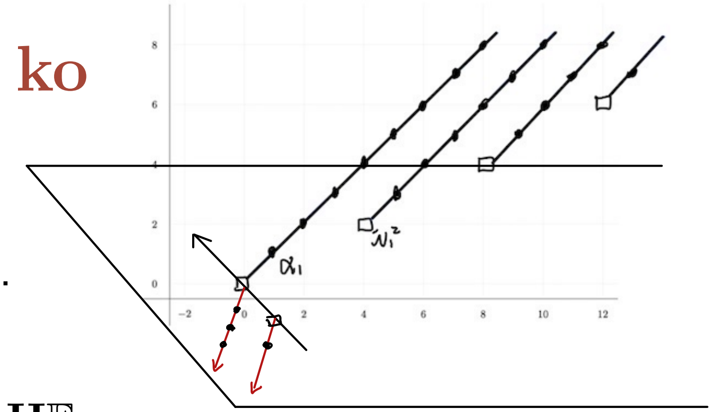


$\mathbf{H}\mathbb{F}_{2*,*}$



$|\tau| = (0, -1), |\rho| = (-1, -1)$

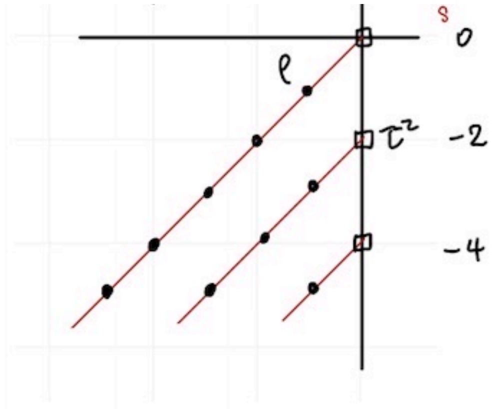
Example: $\mathbb{R}$-motivic ESSS E_1 of $\mathbf{ko}$



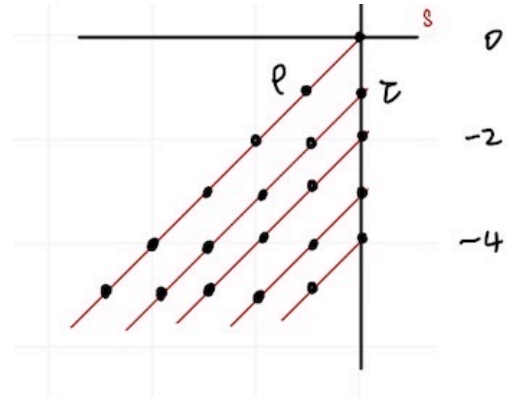
Take $\pi_{*,*}$ of the slices to obtain E_1-page.

$\mathbb{R}$-motivic coefficient rings:

$\mathbf{H}\mathbb{Z}_{*,*}$

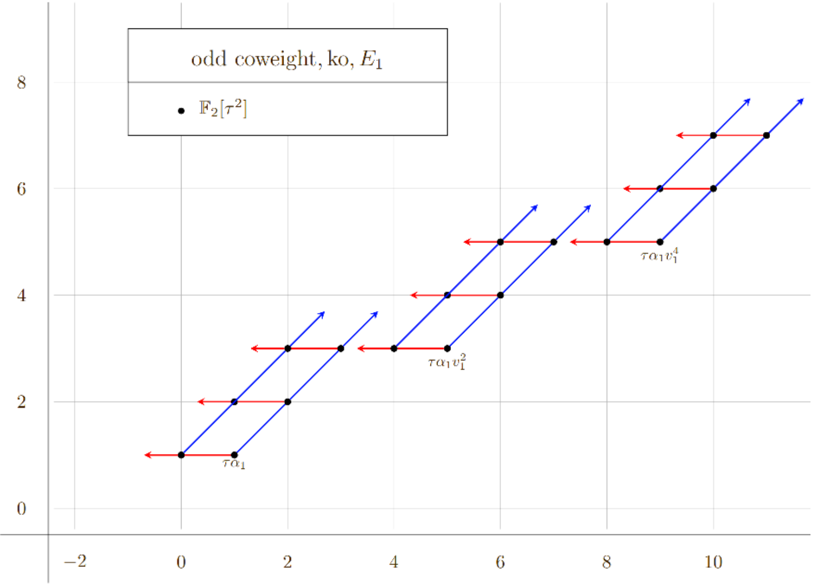


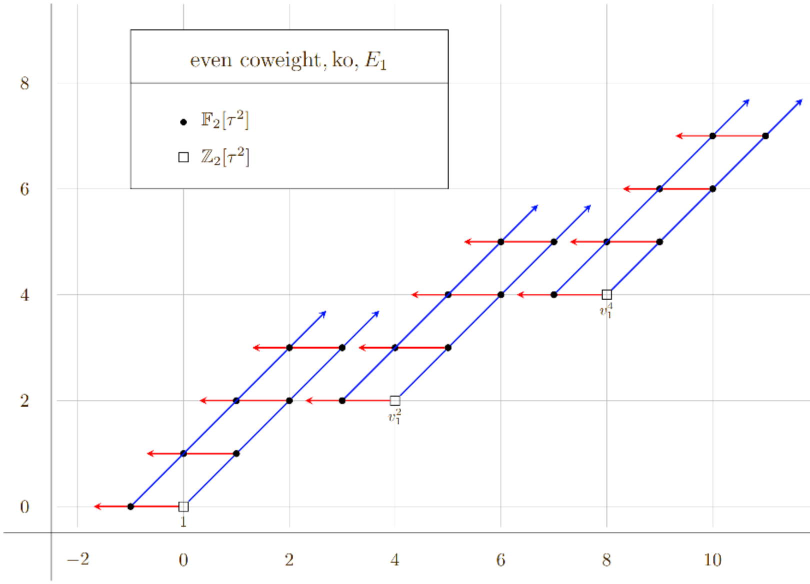
$\mathbf{H}\mathbb{F}_{2\,*,*}$



$|\tau| = (0, -1), |\rho| = (-1, -1)$

Charts: $\mathbb{R}$-motivic ESSS E_1 of $\mathbf{ko}$





Computation of j

Compute j using effective slice spectral sequence.

Computation of j

Compute j using effective slice spectral sequence.

- Slices

Computation of j

Compute j using effective slice spectral sequence.

- Slices
- Differentials

Computation of j

Compute j using effective slice spectral sequence.

- Slices
- Differentials

- Hidden extensions

Computation of j

Compute j using effective slice spectral sequence.

- Slices
- Differentials

- Hidden extensions
- Relevance & convergence

Computation of j

Compute j using effective slice spectral sequence.

- Slices
- Differentials

- Hidden extensions
- Relevance & convergence

Theorem (Belmont–Isaksen–K.)

The bigraded homotopy groups of $\mathbb{R}$-motivic j can be explicitly calculated via the slice spectral sequence.

Computation of j

Compute j using effective slice spectral sequence.

- Slices from slices of $\mathbf{ko}$ and $\Sigma^{4,2}\mathbf{ksp}$.
- Differentials
- Hidden extensions
- Relevance & convergence

Theorem (Belmont–Isaksen–K.)

The bigraded homotopy groups of $\mathbb{R}$-motivic j can be explicitly calculated via the slice spectral sequence.

Computation of j

Compute j using effective slice spectral sequence.

- Slices — from slices of $\mathbf{ko}$ and $\Sigma^{4,2}\mathbf{ksp}$.
- Differentials — connecting homomorphism/ η-periodic result.
- Hidden extensions
- Relevance & convergence

Theorem (Belmont–Isaksen–K.)

The bigraded homotopy groups of $\mathbb{R}$-motivic j can be explicitly calculated via the slice spectral sequence.

Computation of j

Compute j using effective slice spectral sequence.

- Slices — from slices of $\mathbf{ko}$ and $\Sigma^{4,2}\mathbf{ksp}$.
- Differentials — connecting homomorphism/ η-periodic result.
- Hidden extensions — compare $\mathbf{ko}$ and $\Sigma^{4,2}\mathbf{ksp}$.
- Relevance & convergence

Theorem (Belmont–Isaksen–K.)

The bigraded homotopy groups of $\mathbb{R}$-motivic j can be explicitly calculated via the slice spectral sequence.

Computation of j

Compute j using effective slice spectral sequence.

- Slices — from slices of $\mathbf{ko}$ and $\Sigma^{4,2}\mathbf{ksp}$.
- Differentials — connecting homomorphism/ η-periodic result.
- Hidden extensions — compare $\mathbf{ko}$ and $\Sigma^{4,2}\mathbf{ksp}$.
- Relevance & convergence — $\Rightarrow \pi_{**}j_2^\wedge$.

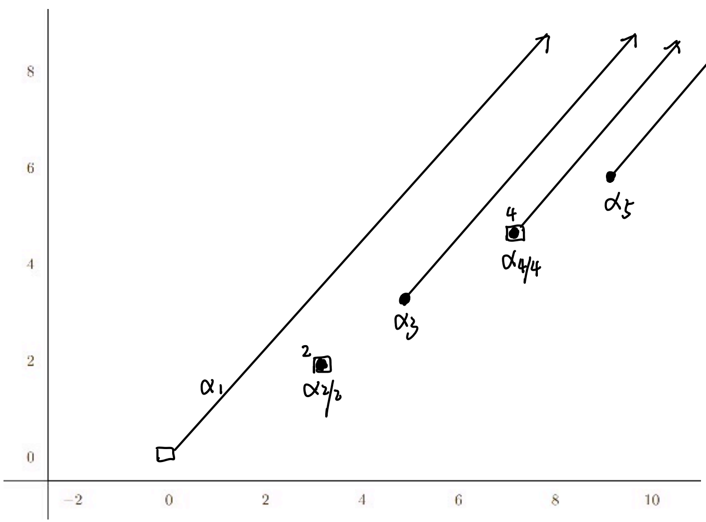
Theorem (Belmont–Isaksen–K.)

The bigraded homotopy groups of $\mathbb{R}$-motivic j can be explicitly calculated via the slice spectral sequence.

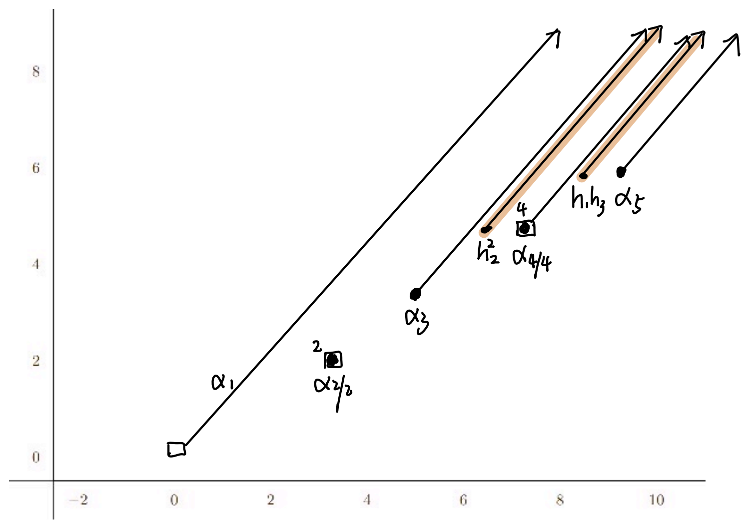
Slices of j

Effective slices: captures the α-family in ESSS of the motivic sphere.

j



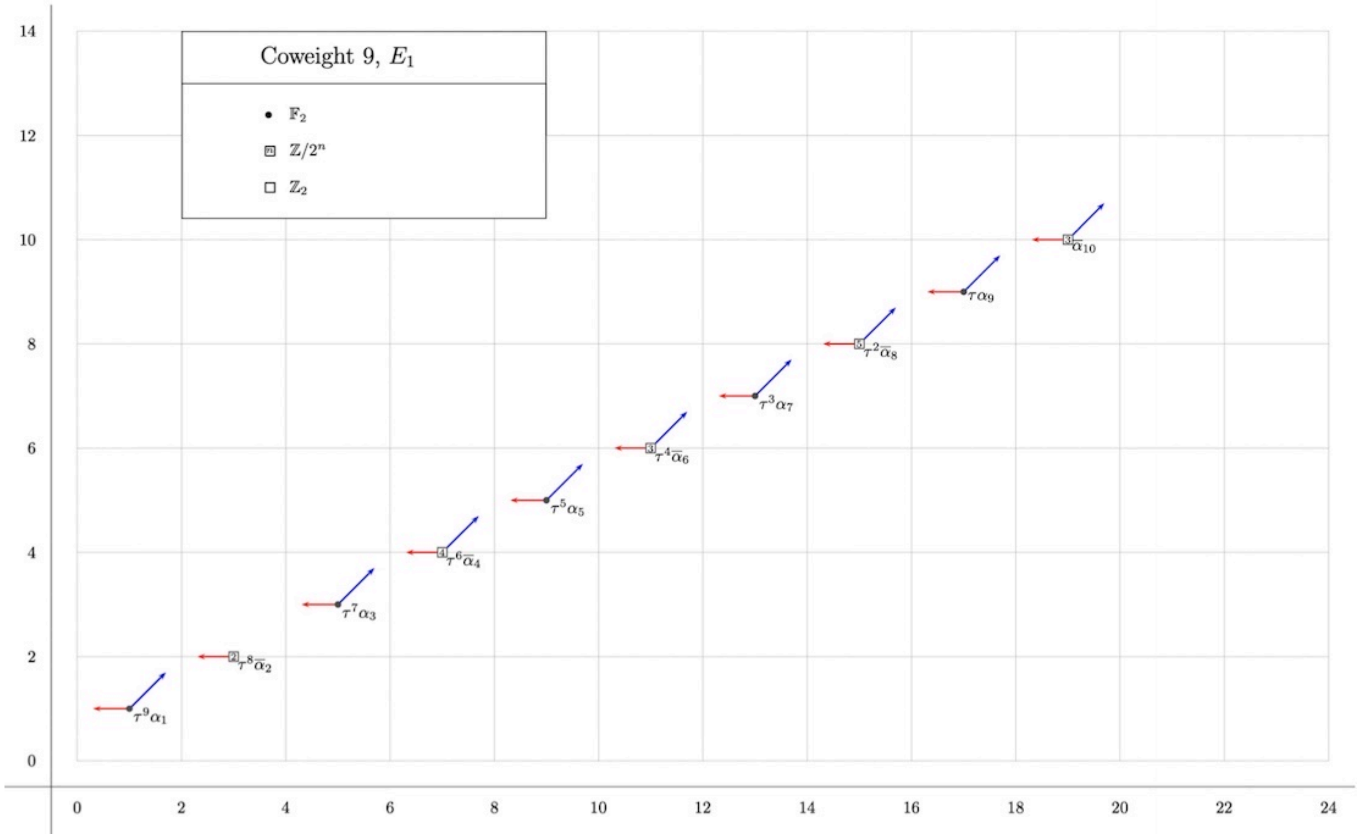
S



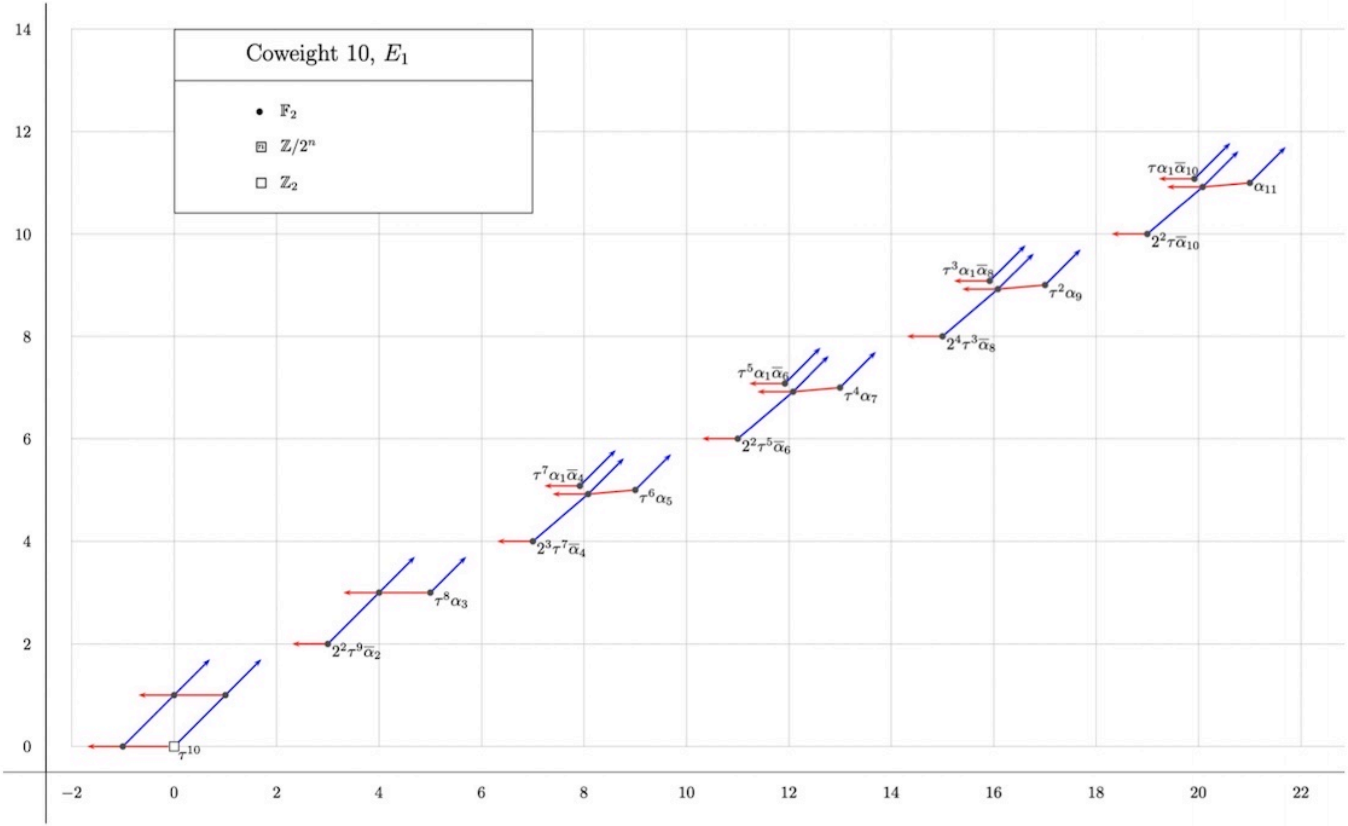
E_1-page of j

E_1-page: captures the α-family in ESSS of the $\mathbb{R}$-sphere.

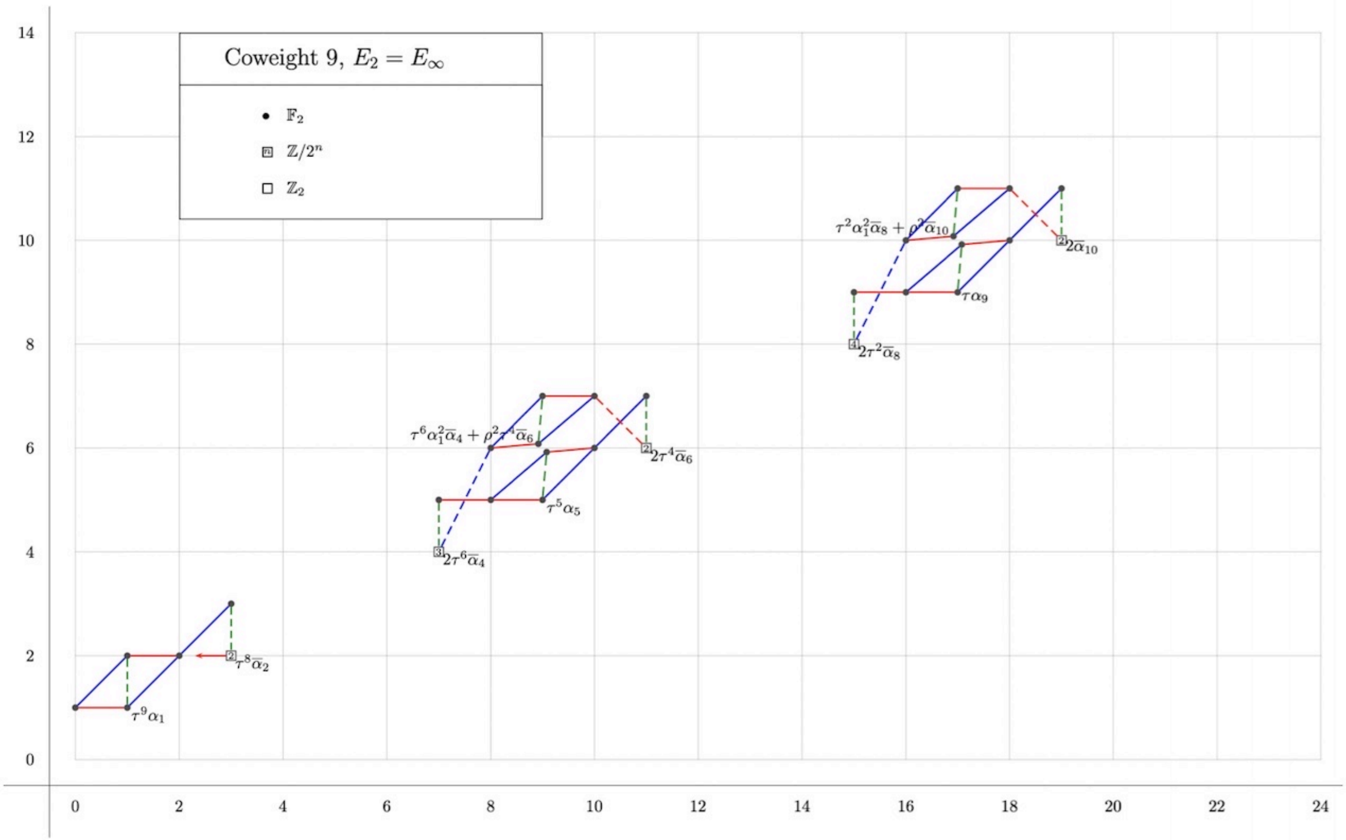
Charts: E_1-page, odd coweight



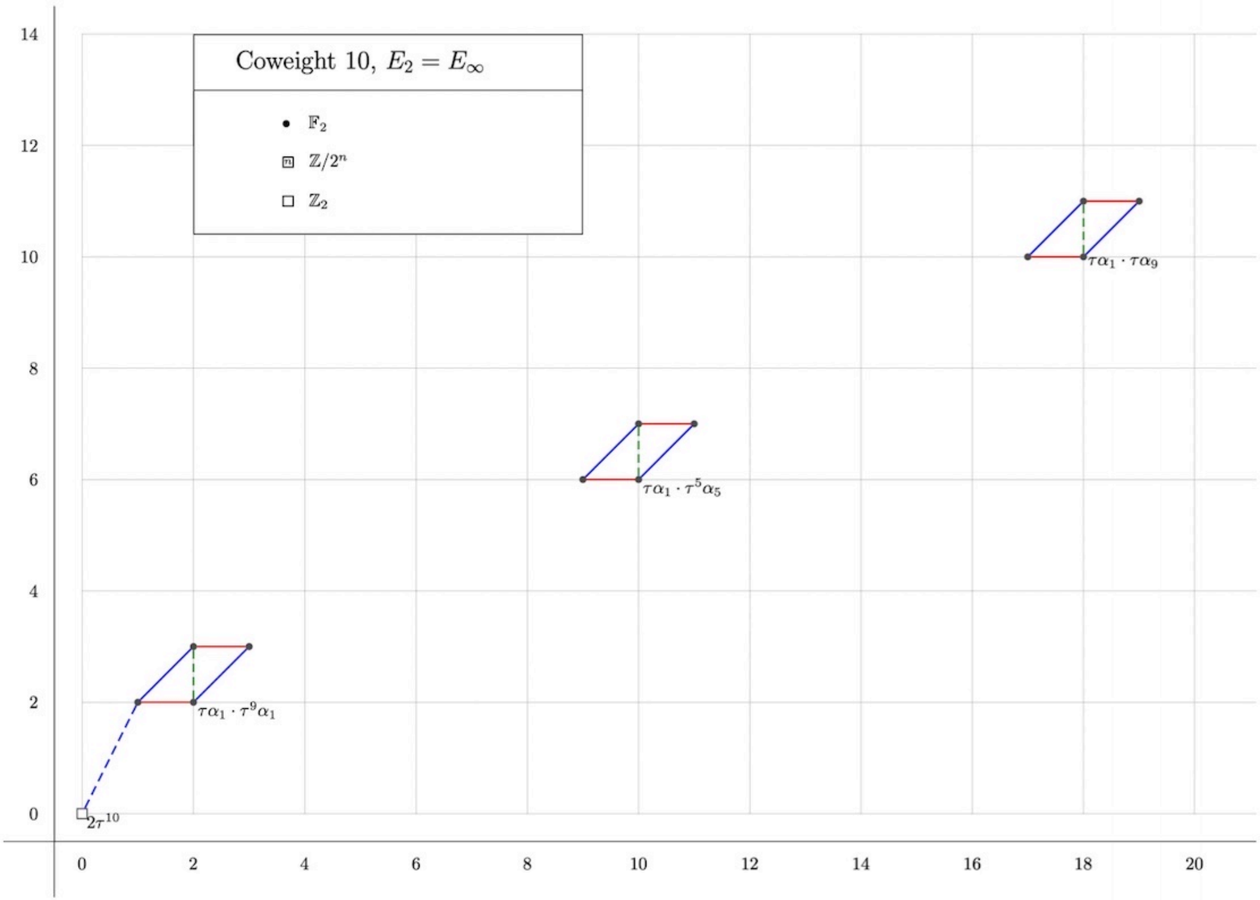
Charts: E_1-page, even coweight



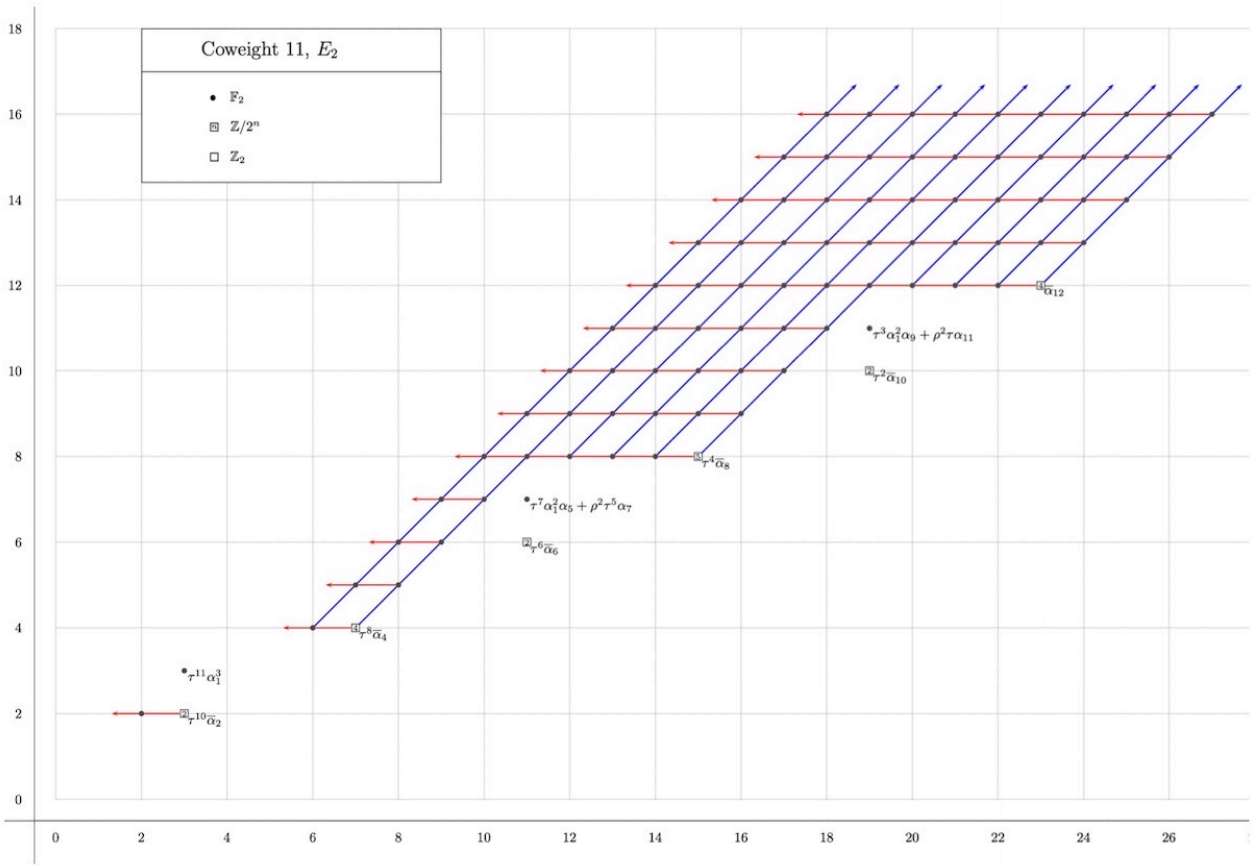
Charts: E_2-page



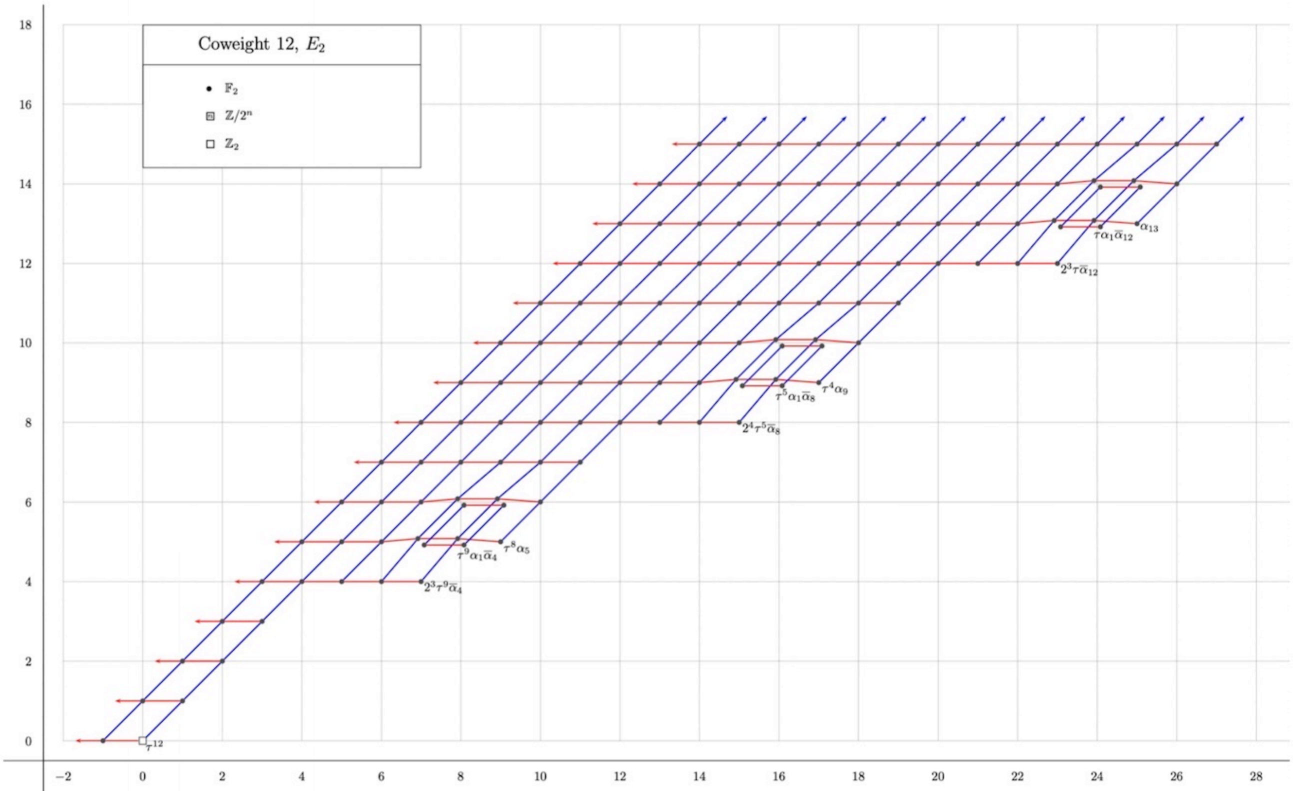
Charts: E_2-page



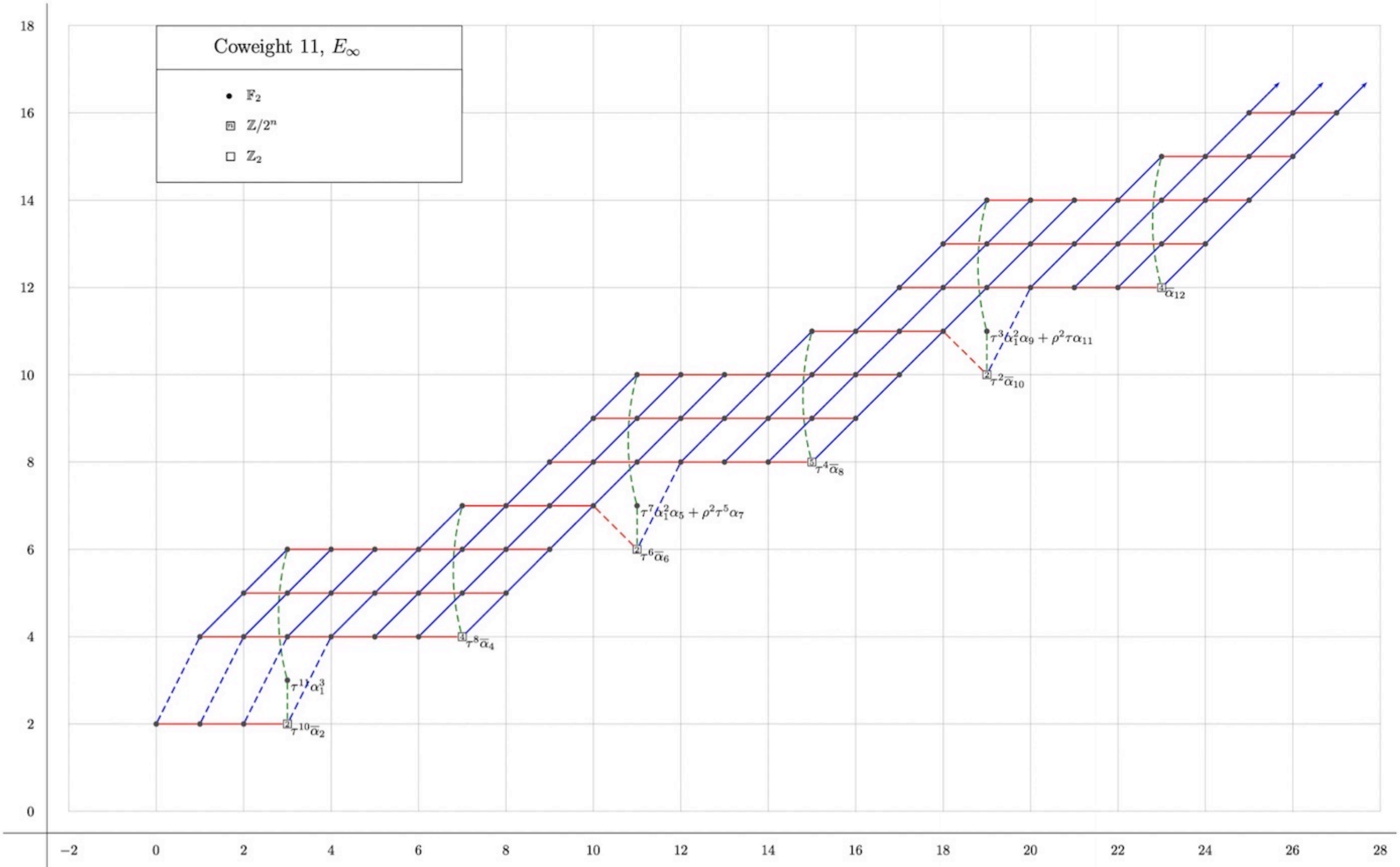
Charts: E_2-page



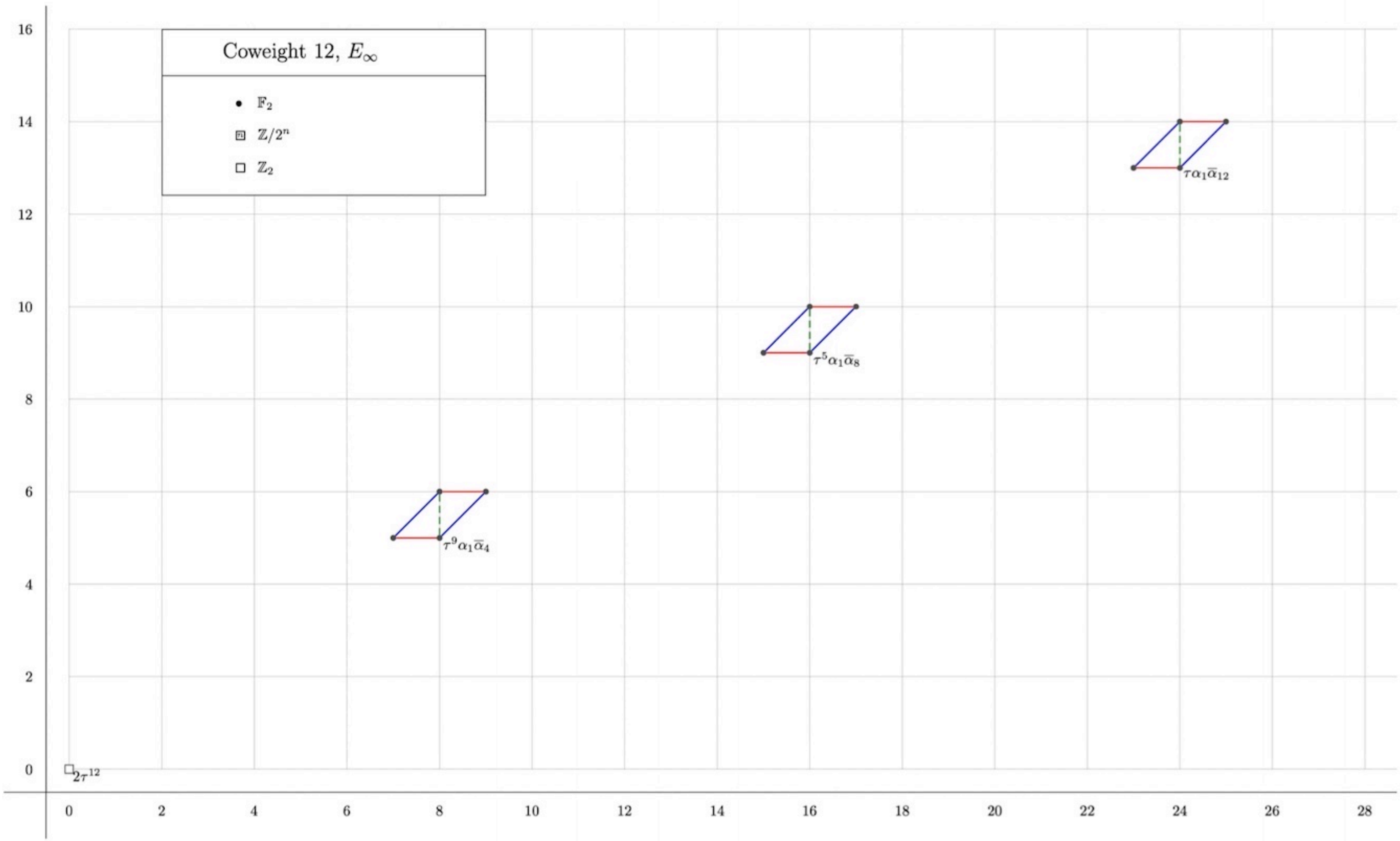
Charts: E_2-page



Charts: E_∞-page



Charts: E_∞-page



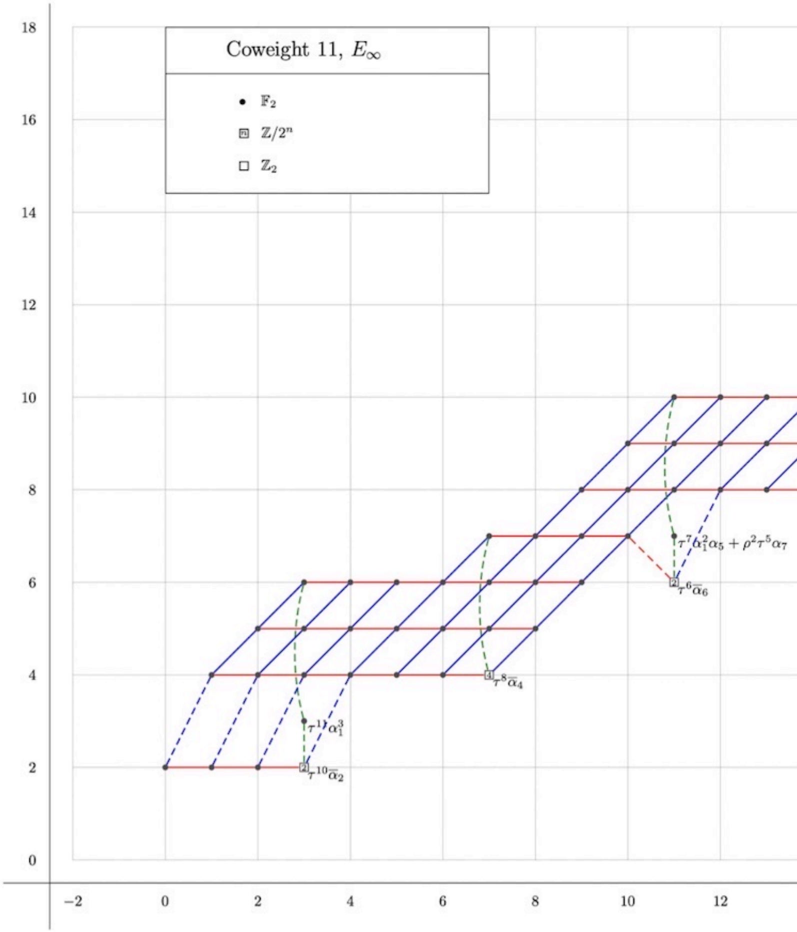
E_∞-page

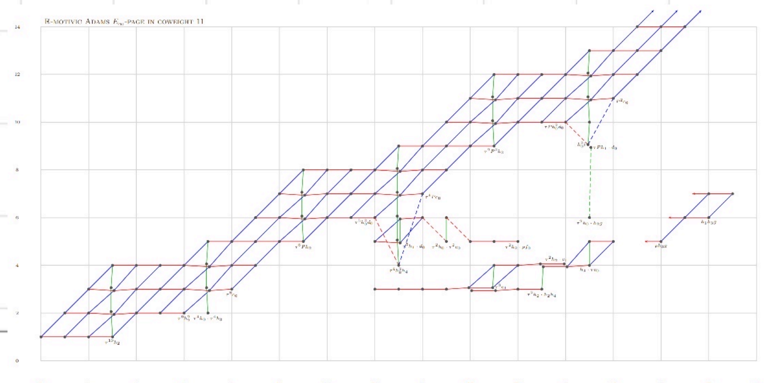
- η-periodic pattern appears at coweight $4n - 1$.

E_∞-page

- η-periodic pattern appears at coweight $4n-1$.
- E_∞-page matches the pattern in the mASS of the $\mathbb{R}$-sphere.

Charts: E_∞-page

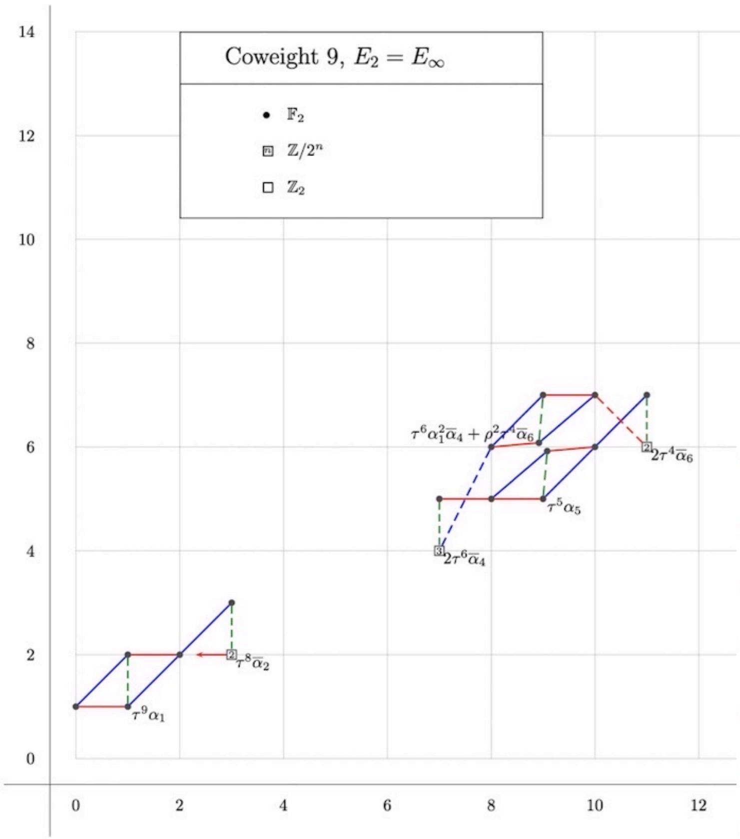


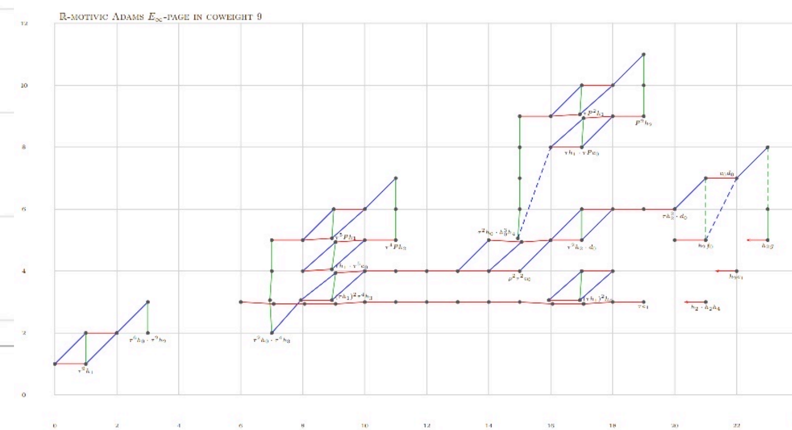


E_∞-page

- η-periodic pattern appears at coweight $4n-1$.
- E_∞-page matches the pattern in the mASS of the $\mathbb{R}$-sphere.
- E_∞-page helps analyze ESSS differentials for the $\mathbb{R}$-sphere.

Charts: E_∞-page





Thank you!