Cohomology of extended powers and free infinite loop spaces

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Joint work with Chad Giusti, Lorenzo Guerra, David Pengelley and Paolo Salvatore.
Expanded versions of these slides along with some relevant papers, but without the material on derived divided powers, are at:
`https://pages.uoregon.edu/dps/VNUS2019/`

Cohomology of extended powers

The extended powers construction

$$D_n(X) = {X^n}_{h\mathcal{S}_n} = (E\mathcal{S}_n \times X^n)/\mathcal{S}_n$$

was introduced by Steenrod and plays a number of important roles in algebraic topology.

For example, the group completion of $\coprod_n {X^n}_{h\mathcal{S}_n}$ is the free infinite loopspace on X_+.

Today I will focus on the cohomology ring structure of these spaces.

Homology and cohomology of symmetric groups

The ranks of homology groups of BS_n has been understood since work of Kudo-Araki, Dyer-Lashof, and Nakaoka.

The ring structure of the cohomology of BS_∞ at the prime two was calculated by Nakaoka in 1960.

The coproduct structure for $H_*(B\mathcal{S}_n)$ was given by Cohen-Lada-May. It is difficult to work with because of a need to apply Adem relations.

Through the '80s and '90s, Hưng, Adem, Milgram, McGinnis and Feshbach studied individual symmetric groups to better understand cohomology ring and Steenrod structure.

Theorem 5.4

$$H^*(\mathcal{S}_{12}) \cong P[\sigma_1, \sigma_2, \sigma_3, c_3, \sigma_4, \sigma_5, \sigma_6, d_6, d_7, d_9](x_5, x_7, x_8)/\langle R\rangle$$

where deg $\sigma_i = i$, deg $c_3 = 3$, deg $d_i = i$, deg $x_i = i$ *and* R *is the set of relations*

$$x_7^2 + \sigma_4 x_5^2 + \sigma_2\sigma_4 x_8 + (\sigma_6 c_3 + \sigma_2\sigma_4 c_3)x_5 + \sigma_4^2 + c_3^2 + \sigma_2^2\sigma_4 d_6 + \sigma_6\sigma_2 d_6,$$
$$x_8^2 + d_6 x_5^2 + c_3 d_6 x_7 + (d_9\sigma_2 + c_3 d_6\sigma_2)x_5 + d_6^2\sigma_2^2 + c_3^2\sigma_4 d_6 + d_9 c_3\sigma_2,$$
$$x_5x_7 + \sigma_2 x_5^2 + \left[\sigma_2^2 + \sigma_4\right]x_8 + \sigma_2^2 c_3 x_5 + \sigma_2^3 d_6 + \sigma_6 c_3^2 + \sigma_6 d_6 + \sigma_2\sigma_4 c_3^2,$$
$$x_5x_8 + c_3 x_5^2 + \left[c_3^2 + d_6\right]x_7 + c_3^2\sigma_2 x_5 + c_3^3\sigma_4 + d_9\sigma_2^2 + d_9\sigma_4 + c_3 d_6\sigma_2^2,$$
$$x_5^3 + c_3\sigma_2 x_5^2 + x_7x_8 + c_3\sigma_2^2 x_8 + \sigma_2 c_3^2 x_7 + \left(\sigma_2^2 d_6 + \sigma_4 c_3^2\right)x_5 + \sigma_2^3 d_9 + c_3^3\sigma_6 + \sigma_6 d_9,$$

$$d_9\sigma_1, d_9\sigma_3, d_9\sigma_5$$
$$d_7\sigma_3, d_7x_5, d_7\sigma_1c_3, d_7(x_7 + \sigma_4c_3), d_7(\sigma_5 + \sigma_4\sigma_1), d_7(\sigma_6 + \sigma_4\sigma_2),$$
$$d_7(x_8 + d_6\sigma_2), d_7(d_9 + d_6c_3),$$

$$x_7\sigma_1 + x_5(\sigma_1\sigma_2 + \sigma_3) + c_3\left(\sigma_2\sigma_3 + \sigma_2^2\sigma_1 + \sigma_1\sigma_4 + \sigma_5\right),$$
$$x_7\sigma_3 + x_5(\sigma_5 + \sigma_1\sigma_4) + c_3(\sigma_1\sigma_6 + \sigma_3\sigma_4 + \sigma_1\sigma_2\sigma_4),$$
$$x_7\sigma_5 + x_5\sigma_1\sigma_6 + c_3(\sigma_3\sigma_6 + \sigma_1\sigma_2\sigma_6),$$
$$x_8\sigma_1 + d_6(\sigma_3 + \sigma_1\sigma_2),$$
$$x_8\sigma_3 + d_6(\sigma_5 + \sigma_1\sigma_4),$$
$$x_8\sigma_5 + d_6\sigma_1\sigma_6.$$

Main results (at two)

Theorem (Giusti-Guerra-Salvatore-S)

- *The direct sum of cohomology* $\bigoplus_n H^*(B\mathcal{S}_n; \mathbb{F}_2)$ *is the free divided-powers component Hopf ring primitively generated by classes* $\gamma_\ell \in H^{2^\ell - 1}(B\mathcal{S}_{2^\ell})$.
- *The direct sum over n of the cohomology of* $D_n X$ *is the free divided powers Hopf ring over this ring* ($X = pt.$) *generated by the cohomology ring of* X.

All of the information from the previous slide follows from the first statement.

We also explicitly treat Steenrod structure. We start with algebra and move to geometry, though our insight flowed in the other direction. We will end with applications and further directions.

Part one: algebra

1. What is... a divided power Hopf ring structure?
2. Graphical representation of Hopf ring monomial basis
3. Multiplication
4. Extended powers and free infinite loop spaces

Hopf ring structure

Definition

A Hopf ring is a ring object in the category of coalgebras.

Explicitly, a Hopf ring is vector space V with two multiplications, one comultiplication, and an antipode $(\odot, \cdot, \Delta, S)$ such that the first multiplication forms a Hopf algebra with the comultiplication and antipode, the second multiplication forms a bialgebra with the comultiplication, and these structures satisfy the distributivity relation

$$\alpha \cdot (\beta \odot \gamma) = \sum_{\Delta\alpha = \sum a' \otimes a''} (a' \cdot \beta) \odot (a'' \cdot \gamma).$$

Hopf ring structure

A free algebra (or magma) on two products would be very large, but the distributivity relation cuts things down considerably.

$$\alpha \cdot (\beta \odot \gamma) = \sum_{\Delta\alpha = \sum a' \otimes a''} (a' \cdot \beta) \odot (a'' \cdot \gamma).$$

Consequence: every element can be reduced to Hopf monomials $m_1 \odot m_2 \odot \cdots \odot m_i$, where the m_i are monomials in the $\cdot$-product alone.

Hopf ring structure

Hopf rings occur "categorically" in topology, as the homology of infinite loop spaces which represent ring spectra. But Strickland and Turner saw these structures in (generalized) group cohomology of symmetric groups.

Theorem (Strickland-Turner)

For any ring-theory E^*, the cohomology of symmetric groups $\bigoplus_n E^*(B\mathcal{S}_n)$ forms a (derived) Hopf ring where

- *The $\cdot$ product is the standard (cup) product (with zero products between distinct summands).*
- *The coproduct Δ is induced by the standard covering $p : B\mathcal{S}_n \times B\mathcal{S}_m \to B\mathcal{S}_{n+m}$.*
- *The product $\odot$ is the transfer associated to p.*

These operations are familiar in the case of K-theory.

Hopf ring structure

Definition

A divided powers Hopf ring generated by a finite set $a_1, \cdots a_k$ is the Hopf ring generated under the two products by variables $a_{i[n]}$ with $1 \leq i \leq k$ and $n \geq 1$ with coproducts determined by

$$\Delta a_{i[n]} = \sum_{k+\ell=n} a_{i[k]} \otimes a_{i[\ell]},$$

and $\odot$-products

$$a_{i[n]} \odot a_{i[m]} = \binom{n+m}{n} a_{i[n+m]}.$$

It is better (but not as quick) to describe free divided powers Hopf rings as a left adjoint to a forgetful functor to rings.

Component divided powers Hopf ring structure

A component Hopf ring decomposes as $\bigoplus_n R_n$, where the $\cdot$ is zero between summands. This is immediate for $\bigoplus_n H^*(B\mathcal{S}_n)$.

The divided powers structure is for the "plus product" $\odot$, giving operations $H^*(B\mathcal{S}_m) \to H^*(B\mathcal{S}_{km})$. We typically denote it $x \mapsto x_{[k]}$ (reserving superscripts for exponentiation of $\cdot$). Recall that divided powers over $\mathbb{F}_2$ is exterior, as $x \odot x = 2x_{[2]} = 0$.

The divided powers operation commutes with $\cdot$ on Δ-primitives, and satisfies the coproduct formula $\Delta x_{[n]} = \sum_{i+j=n} x_{[i]} \otimes x_{[j]}$.

Unpacking the structure and graphical representation

The Hopf monomial basis of a Hopf ring can represented graphically.

In our setting, generators γ_ℓ are represented by blocks. We give them width $2^{\ell-1}$, so that the width corresponds to half the component.
We choose height so that area is equal to degree, namely $2^\ell - 1$.

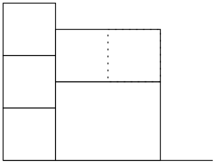
The cup $\cdot$ product is represented by stacking vertically, and the transfer $\odot$ product is represented by stacking horizontally. Divided power is denoted by repeated stacking horizontally with dashed lines.

The unit class 1_m on the $B\mathcal{S}_m$ component is indicated by an "empty space."

Graphical representation

$$\gamma_1^3 \odot \gamma_{1[2]} \cdot \gamma_2 \odot 1_2$$

is represented by:



These graphical representations for Hopf ring monomials are called "skyline diagrams."

Additive basis

To form an additive basis, inductively:

- Take transfer products (without repeats) of basis elements for previous symmetric groups.
- On 2^ℓ components, take all $\cdot$ products of $\gamma_{i[2^j]}$ with $i + j = \ell$ (that is, a basis for the polynomial algebra on these).

These latter form a basis for the $\odot$-indecompsibles of this Hopf ring, from which everything else is built through $\odot$. (For experts: under restriction to transitive elementary abelian subgroup, these map to Dickson algebras, isomorphically as algebras but not over the Steenrod algebra.)

Hopf ring generators

	$B\mathcal{S}_2$	$B\mathcal{S}_4$	$B\mathcal{S}_6$	$B\mathcal{S}_8$	$B\mathcal{S}_{10}$	$B\mathcal{S}_{12}$	$B\mathcal{S}_{14}$	$B\mathcal{S}_{16}$
15								γ_4
14								$\gamma_{3[2]}$
13								
12								$\gamma_{2[4]}$
11								
10								
9						$\gamma_{2[3]}$		
8								$\gamma_{1[8]}$
7				γ_3			$\gamma_{1[7]}$	
6				$\gamma_{2[2]}$		$\gamma_{1[6]}$		
5					$\gamma_{1[5]}$			
4				$\gamma_{1[4]}$				
3		γ_2	$\gamma_{1[3]}$					
2		$\gamma_{1[2]}$						
1	γ_1							

Multiplication

The main payoff of our techniques are much better control of mulitplication, as well as new insight into Steenrod structure.

Multiplication requires repeated use of Hopf ring distributivity, though we will be able to streamline things graphically.

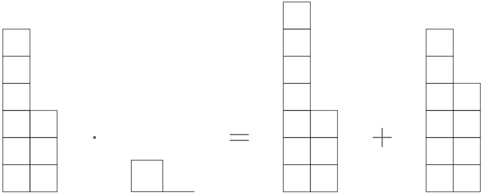
$$
\begin{aligned}
(\gamma_{1[2]} \odot 1_2) \cdot (\gamma_{1[2]}^2 \odot 1_2) &= \sum_{\Delta \gamma_{1[2]} \odot 1_2 = \sum a_i \otimes b_i} (a_i \cdot \gamma_{1[2]}^2) \odot (b_i \cdot 1_2) \\
&= (\gamma_{1[2]} \cdot \gamma_{1[2]}^2) \odot (1_2 \cdot 1_2) + (\gamma_1 \odot 1_2 \cdot \gamma_{1[2]}^2) \odot (\gamma_1 \cdot 1_2) \\
&= \gamma_{1[2]}^3 \odot 1_2 + \gamma_1^3 \odot \gamma_1^2 \odot \gamma_1
\end{aligned}
$$

Multiplication

Graphically, transfer product corresponds to placing two column Skyline diagrams next to each other and merging columns with the same constituent blocks, with a coefficient of zero if any of those column widths share a one in their dyadic expansion.

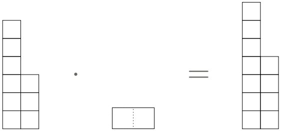
For cup product, we start with two column diagrams and consider all possible ways to split each into columns, along either original boundaries of columns or along the vertical lines of full height internal to the rectangles representing $\gamma_{\ell,n}$. We then match columns of each in all possible ways up to automorphism, and stack the resulting matched columns to get a new set of columns – a Tetris-like game.

Multiplication

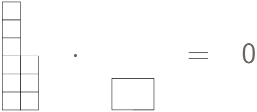


$$(\gamma_1^i \odot \gamma_1^j) \cdot (\gamma_1 \odot 1_2) = \gamma_1^{i+1} \odot \gamma_1^j + \gamma_1^i \odot \gamma_1^{j+1}$$

Multiplication

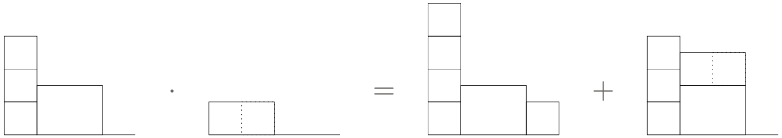


$$(\gamma_1^i \odot \gamma_1^j) \cdot \gamma_{1[2]} = \gamma_1^{i+1} \odot \gamma_1^{j+1}$$



$$(\gamma_1^i \odot \gamma_1^j) \cdot \gamma_2 = 0$$

Multiplication



$$(\gamma_1^3 \odot \gamma_2 \odot 1_2) \cdot (\gamma_{1[2]} \odot 1_4) \quad = \quad \gamma_1^4 \odot \gamma_2 \odot 1_2 + \gamma_1^3 \odot \gamma_{1[2]}\gamma_2 \odot 1_2$$

Exercises

Calculate products such as

- $(\gamma_1 \odot 1_2) \cdot \gamma_{1[2]}^2$
- $(\gamma_1 \odot 1_2) \cdot (\gamma_1^2 \odot \gamma_1)$
- $(\gamma_1 \odot 1_2) \cdot \gamma_2$
- $(\gamma_1^3 \odot \gamma_2 \odot 1_2) \cdot (\gamma_{1[2]} \odot 1_4)$.

Or (start to) give a basis of $B\mathcal{S}_{12}$

Or ponder this presentation as you like.

Talking with your neighbors is strongly encouraged.

Ring presentations and beyond

If one wants generators and relations for an individual symmetric groups, finding such from an additive basis with multiplication rules is readily algorithmic, by proceeding by degree.

Such are necessarily complicated; the Adem-Mcginnis-Milgram presentation for $B\mathcal{S}_{12}$ shown previously is minimal, for example.

We readily recover Feshbach's generators. His relations are inductively defined - it would be nice to have a "closed form."

There are many other open questions, starting with for example computing the prime ideal spectrum, as needed for applications to the study of support varieties.

Extended powers

Theorem (Giusti-Guerra-Salvatore-S)

- *The direct sum of cohomology* $\bigoplus_n H^*(B\mathcal{S}_n; \mathbb{F}_2)$ *is the free divided-powers component Hopf ring primitively generated by classes* $\gamma_\ell \in H^{2^\ell - 1}(B\mathcal{S}_{2^\ell})$.
- *The direct sum over* n *of the cohomology of* $D_n X$ *is the free divided powers Hopf ring over this ring* ($X = pt.$) *generated by the cohomology ring of* X.

Graphical basis for the cohomology of extended powers: skyline diagrams where each column is labeled by a cohomology class of X (which is at least implicitly repeated twice in each column).

Hopf ring structure essentially as before, but includes multiplication of labels when columns are stacked.

The cohomology of $B\mathcal{S}_\infty$

The cohomology of $B\mathcal{S}_\infty$ is the inverse limit, which by explicit calculation or application of homological stability can be represented by skyline diagrams with "infinite room."

Hopf ring structure of finite approximations can be used to calculate products. We notice that relations "die" in the inverse system. In fact, as Nakaoka first show, the limit is a polynomial ring.

We have shown that there is no consistent Hopf ring structure on the limit, so in this case the finite approximations illuminate the infinite, instead of the other way around.

The cohomology of $B\mathcal{S}_\infty$

Exercise:

a. Define and use a width filtration to show that "single columns" generate the cohomology of $B\mathcal{S}_\infty$ (under cup product).

b. Show that the cohomology is polynomial (as proven by Nakaoka), and find a set of generators.

The cohomology of QX

Barratt-Priddy showed that the cohomology of QX is that of $\mathcal{C}_\infty X$, a quotient of $D_\infty X$. (Quillen and Segal then showed this follows from a stronger group completion theorem.)

Analysis of skyline diagrams with infinite room, with columns labeled by $H^*(X)$ shows that the cohomology of QX is generated by columns of width one with at least one block type occurring an odd number of times or with a label which is not a square in $H^*(X)$, modulo the relation that if a label satisfies $x^{2^n} = 0$ then a column with that label will also have such a relation.

Steenrod structure

Because transfers are stable maps, our transfer product has a Cartan formula - there are two!

Thus, Steenrod operations on all cohomology of symmetric groups is thus determined by that on the $\gamma_{i[j]}$.

Definition

- The height of a skyline diagram is the largest number of blocks stacked in a column. (This is not the degree of that column.)
- The effective scale of a column is the index of the largest block-type which occurs. (The largest i for which $\gamma_{i,2^j}$ appears in the column.) The effective scale of a skyline diagram is the minimum of the effective scales of its columns.
- We say a skyline diagram is not full width if it has an empty space. (That is, it represents a non-trivial transfer product of some monomial with some 1_k.)

Steenrod structure

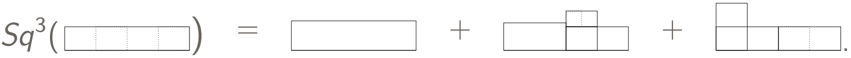
Theorem

$Sq^i\gamma_{\ell[2^k]}$ is the sum of all full-width skyline diagrams of total degree $2^k(2^\ell - 1) + i$, height one or two, and effective scale at least ℓ.

We call monomials represented by such skyline diagrams the outgrowth monomials of $\gamma_{\ell[2^k]}$.

For example,

$$Sq^3\gamma_{2[4]} = \gamma_4 + \gamma_3 \odot \gamma_2\gamma_{1[2]} \odot \gamma_2 + {\gamma_2}^2 \odot \gamma_2 \odot \gamma_{2[2]}.$$



Part two: geometry

1. Configuration space models of structures in homology and cohomology

2. Further directions: derived divided powers (aka "transfer Kudo-Araki") operations

Configuration models

Let $\mathrm{Conf}_n(\mathbb{R}^m)$ denote the configuration space of $n$distinct ordered points in $\mathbb{R}^m$ - that is the subspace of $(x_1, \ldots, x_n)$ where $x_i \neq x_j$ when $i \neq j$.

We denote by $\overline{\mathrm{Conf}}_n(\mathbb{R}^m)$ the quotient $\mathrm{Conf}_n(\mathbb{R}^m)/\mathcal{S}_n$, which is the space of unlabeled configurations of n points in $\mathbb{R}^m$.

Configuration models

Familiar group-theoretic constructions have geometric representatives.

The inclusion $\mathcal{S}_n \times \mathcal{S}_m \hookrightarrow \mathcal{S}_{n+m}$ is given by "'stacking configurations next to each other."

Explicitly, if $f : M \to \overline{\mathrm{Conf}}_i(\mathbb{I}^\infty)$ and $g : N\overline{\mathrm{Conf}}_j(\mathbb{I}^\infty)$ represent homology classes x and y then $x * y$ is represented by a map $M \times N \to \overline{\mathrm{Conf}}_{i+j}(\mathbb{I}^\infty)$ sending (m, n) to

$$\frac{1}{3}(f(m) - v) \bigsqcup \frac{1}{3}(g(n) + v),$$

for a fixed unit vector v.

Homology of symmetric groups

This makes $\coprod B\mathcal{S}_n = \coprod \overline{\mathrm{Conf}}_n(\mathbb{R}^\infty)$ an H-space (in fact, A_∞).

There is a choice of in which direction one stacks. By making continuous families of such choices, in the case of the product of a class with itself, we get "higher products" or operations.

Definition (Kudo-Araki, Dyer-Lashof)

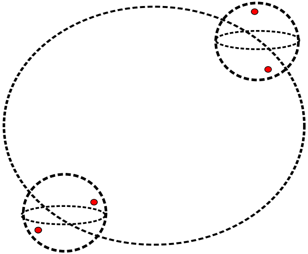
Let $f : M \to \overline{\mathrm{Conf}}_n(\mathbb{I}^\infty)$ represent a homology class x.
Then $q_i(x)$ is represented by a map from $S^i \times_{\mathcal{S}_2} (M \times M)$ to $\overline{\mathrm{Conf}}_{2n}(\mathbb{I}^\infty)$ sending (v, m_1, m_2) to $\frac{1}{3}(f(m) - v) \coprod \frac{1}{3}(g(n) + v)$.

Homology of symmetric groups

These operations can be defined using the transfer of $\mathcal{S}_n \wr \mathcal{S}_2 \hookrightarrow \mathcal{S}_{2n}$ (along with a little equivariant homology calculation).

These are non-trivial even (especially!) on the non-zero class ι in $H_0(B\mathcal{S}_1)$, in which case the $q_i(\iota)$ are the homology of $B\mathcal{S}_2 = \mathbb{R}P^\infty$.

One can of course compose these operations, with the following a picture of $q_1(q_2(\iota))$.



Homology of symmetric groups

These operations are close cousins to Steenrod operations.

- There is an additive - but highly mixing of degrees - isomorphism between **mod-two** homology of symmetric groups and cohomology of Eilenberg-MacLane spaces. (Below we might make note of explicit isomorphisms between Fox-Neuwirth resolutions and the iterated bar construction for C_2.)

- The diagonal on X yields a multiplicaton on its Spanier-Whitehead dual, and its Kudo-Araki operations reflect the Steenrod operations of X.

- (Adem) For $m > n$, $q_m \circ q_n = \sum_i \binom{i-n-1}{2i-m-n} q_{m+2n-2i} \circ q_i$.

Homology of symmetric groups

Theorem (Nakaoka; Kudo-Araki; Cohen-Lada-May)

$H_*(\coprod_n B\mathcal{S}_n)$, as a ring under $*$ is the polynomial algebra generated by the nonzero class $\iota \in H_0(B\mathcal{S}_1)$ and $q_I(\iota) \in H_*(B\mathcal{S}_{2^k})$ for I strongly admissible.

Our first definition of the key γ_ℓ classes in cohomology is as the linear duals of the $q_{1,\ldots,1}$ classes in this monomial basis.

Cohomology and geometry

Through Poincaré duality, Thom cochains or other technology, a proper submanifold (subvariety) of a manifold represents a cohomology class.

Theorem

The class γ_ℓ is represented by the subvariety of $\overline{Conf}_n(\mathbb{R}^\infty)$ in which all 2^ℓ points share their first coordinate.

For odd primes, we have analogues of all of the above results. In particular, there are γ_ℓ classes in which p^ℓ points share two coordinates, or one complex coordinate. (But these aren't all the Hopf ring generators.)

Cohomology and geometry

Digression: the one-point compactifications of $\mathrm{Conf}_n(\mathbb{R}^d)$ admit Fox-Neuwirth cell structures defined by looking at coordinates shared by consecutive points in the dictionary order on the configuration.

These are $\mathcal{S}_n$-equivariant. By Alexander duality, these chain complexes yield (Fox-Neuwirth) resolutions of the trivial module for $k\mathcal{S}_n$. The γ_ℓ have simple representatives in terms of these resolutions.

Ayala and Hepworth noticed that the boundary category of this cell structure is Θ_n.

Hopf ring structure and geometry

For such representatives, these Hopf ring structures are interpreted as follows

- The $\cdot$ product is the cup product, which as always is defined by intersection, which means imposing both conditions (simultaneously, on the same collection of points, and keeping in mind the configurations are unordered).
- The coproduct Δ means restricting conditions to configurations which have been "stacked".
- The product $\odot$ means taking conditions on m and n points and getting a condition on $m + n$ points by finding disjoint subsets which satisfy those conditions.

We re-discovered these structures through this geometry.

Divided powers structure

The divided powers structure geometrically corresponds to "repeating a condition."

That is, if W is a submanifold of X which represents a cohomology class, its divided power in $D_n X$ corresponds to having all labels of the configuration in W.

One then uses the transfer associated to $D_n \circ D_m \to D_{nm}$ to define a divided powers operation on the cohomology of all extended powers.

We can alternately use wreath products/ operad insertion. We found these in the process of finding a geometric proof of the first Adem relation.

Derived divided powers structure

Instead of imposing conditions only on labels, we could impose conditions on the configuration as well.

Given any $\gamma \in H^i(B\mathcal{S}_n)$ and some $x \in H^*(X)$ we consider again the cohomology class with condition γ on the points in the configuration and condition associated to x on the labels. This was denoted by having x label all the columns in the skyline diagram for γ.

Shifting perspective, we can view this as an operation - $x_{[\gamma]} \in H^*(D_n X)$. The previous $x_{[k]}$ is $x_{[1_k]}$ in this notation.

These can be viewed as "derived divided power operations" or "transfer Kudo-Araki operations."

Derived divided powers structure

Theorem

$\gamma_{n[\gamma_1]} = \gamma_{n+1}$.

Theorem

The cohomology of extended powers of X is generated by that of X, as a Hopf ring with divided powers and the derived divided powers operation associated to $\gamma_1 \in H^1(B\mathcal{S}_2)$.

Note that all of the divided powers are generated in this setting by the (derived) operation associated to $1_2 \in H^0(B\mathcal{S}_2)$.

This is very much work-in-progress: we haven't developed the theory of these operations yet, and in particular do not know the relations.

Odd primes, and beyond

At odd primes, the cohomology of extended powers on X is only free over the cohomology of symmetric groups when we take coefficients in $\mathbb{F}_p \oplus sgn$.

The cohomology of symmetric groups themselves is not a free divided powers Hopf ring, because of Bocksteins. (But we do have a nice presentation.)

There is an "even backbone" - the divided powers Hopf ring with a single derived operation associated to $\gamma_1 \in H^{2(p-1)}(B\mathcal{S}_p)$. These lift to MU-cohomology.

Problem

Define an "inverse Bockstein" operation over which the odd-primary cohomology would be freely generated (as a Hopf ring with a single derived operation).

Part three: further directions

1. Alternating groups
2. Cohomology of symmetric groups over the integers
3. Homology-cohomology pairing
4. Support varieties
5. Characteristic classes for surface bundles - calculations and geometry
6. The unstable Adams SS for QS^0
7. Chromatic cohomology of symmetric groups

Alternating groups

Giusti and I completed an analogous - but much, much more intricate - calculation of the mod-two cohomology of alternating groups, with a presentation as an almost-Hopf ring.

Needed all available techniques, but in the end we "beat the computer" and found errors in previously published results.

Ring structure itself is much less clear than for symmetric groups (we know that there are no analogues of Feshbach generators).

And we ought to work out

The cohomology of symmetric groups over the integers: May originally proved some results about the Bockstein SS. The (in)decomposible exact sequence for transfer product illuminates things greatly.

The pairing between homology and cohomology is “block diagonal” by partition (of columns in skyline diagram vs. product structure in Kudo-Araki algebra). But the blocks are not diagonal. What are they? Conjecturally, pairing coincides with that of admissible basis and Milnor basis and for the Steenrod algebra and its dual.

Application to connect with modular representation theory

Given a representation M of G, consider $\mathrm{Ext}_{kG}(M, M)$. This is a module over the cohomology of G. The *support variety* of M is the variety associated to the annihilator ideal of this module.

Problem

Understand the prime ideal spectrum of the cohomology of symmetric groups, and calculate some support varieties!

And more broadly, what more can we say about modular representations of symmetric groups now that we better understand some key Ext groups?

Characteristic classes for surface bundles

Galatius-Madsen-Tillmann-Weiss: $B\mathrm{Diff}(\Sigma_\infty) \simeq MTSO(2)$.

There is a surjection in cohomology that of $Q\mathbb{C}P^\infty_+$ to that of $MTSO(2)$. In his thesis, Galatius used this to make some calculations, but has noted the possibility of better understanding of ring structure if one could better understand the cohomology of $Q\mathbb{C}P^\infty_+$ – which we now do!

The cohomology of symmetric groups yields characteristic classes of finite covering spaces. Our configuration space models represent these geometrically by embedding a covering space in a product with a Euclidean space, and "recording conditions." There are similar descriptions of Mumford classes. Can we find geometry for all stable characteristic classes of surfaces?

Barrier: passing from characteristic classes covering spaces to those of iterated loop spaces is not even clear!

Applications in homotopy theory

Dana Fry is revisiting Wellington's calculations of the UASS for QS^0.

Time for another attack on chromatic cohomology of symmetric groups? We now have

- HKR theory, including evenness
- Hopf ring structure, with transfer-indecomposibles calculated by Strickland for E-theory
- Input from ordinary cohomology
  - an "even backbone" to cohomology which lifts to MU
  - restrictions to elementary abelian subgroups
  - an ability to calculate Margolis homology, assuming a concrete conjecture about Margolis homology of Dickson algebras
- New "transfer Kudo-Araki operations" which seem to have MU-analogues.

Some takeaways

1. Divided powers Hopf rings are manageable!

2. Transfer maps associated to $\mathcal{S}_n \times \mathcal{S}_m \hookrightarrow \mathcal{S}_{n+m}$ and $\mathcal{S}_n \wr \mathcal{S}_m \hookrightarrow \mathcal{S}_{nm}$, coefficients in the sign representation, and derived divided powers operations are new key structures.

3. There are rich (complex/ algebro) geometric representatives for the "even-degree backbone" of the cohomology.

And many possible further directions!