Stable stems and the Chow-Novikov t-structure in motivic stable homotopy category

Zhouli Xu

MIT

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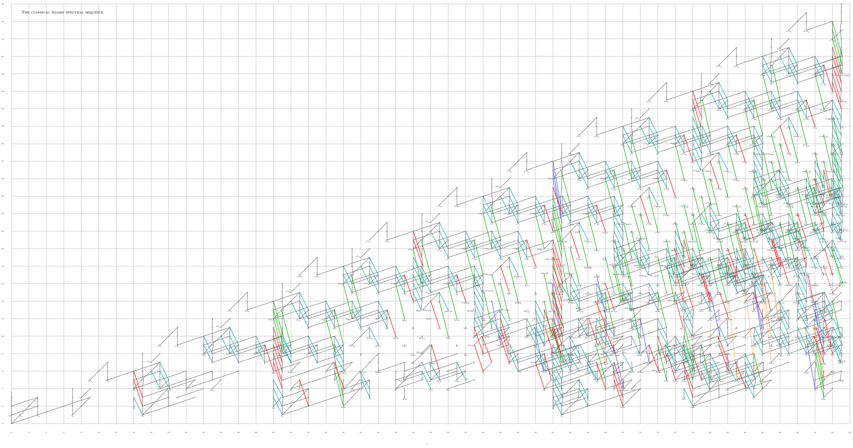
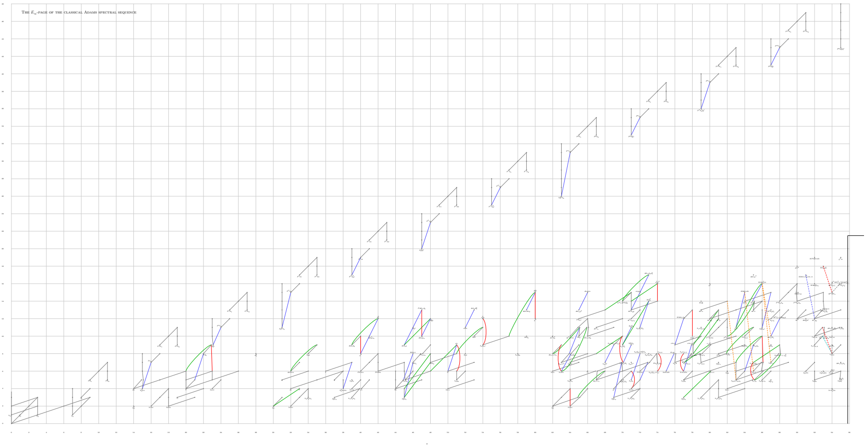
The Classical Adams Spectral Sequence



The Classical Adams E_∞-page



Adams differentials

Theorem (Isaksen - Wang - X.)

Up to 6 differentials, we have complete info on E_∞ through 90.

- *71-stem, d_5 on $h_1 p_1$,*
- *77-stem, d_2 on $x_{77,7}$,*
- *83-stem, d_9 on $h_6 g + h_2 e_2$,*
- *85-stem, d_9 on $x_{85,6}$,*
- *87-stem, d_6 on $\Delta h_1 H_1$,*
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$$d_2(x_{77,7}) = h_0^3 x_{76,6}.$$

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$$d_2(x_{77,7}) = h_0^3 x_{76,6}.$$

Theorem (Burklund - Isaksen - X.)

$$d_5(h_1 p_1) = 0.$$

Hopf classes and Kervaire classes

- h_j: Hopf classes
- $h_0,\ h_1,\ h_2,\ h_3$ survive and detect $2,\ \eta,\ \nu,\ \sigma$.
- Adams: $d_2(h_j) = h_0 h_{j-1}^2$, for $n \geqslant 4$.

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- h_j^2: Kervaire classes
- $h_j^2,\ 0 \leqslant j \leqslant 5$ survive and detect θ_j.
- Hill-Hopkins-Ravenel: For $j \geqslant 7$, h_j^2 supports a nonzero differential.
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Question

What are the differentials that $h_j^2, \; j \geqslant 7$ support?

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Easy to show for all j,

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Theorem (Burklund - X.)

- $d_4(h_j^2) = 0$, *for all* j.

Work in progress: For j large enough,

- $d_5(h_j^2) = 0$,
- $d_6(h_j^2) = 0$.

Hopf classes, Kervaire classes and ???

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What about $h_j^3, \ h_j^4, \cdots$?

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Theorem (Burklund - X.)

- $d_4(h_j^3) = h_0^3 g_{j-2} +$ *possible other classes, for* $j \geqslant 6$.

The open case θ_6

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- *For any $\alpha \in \langle \theta_4, 2, \theta_5 \rangle$, if $\eta\alpha = 0$, and $0 \in \langle \alpha, \eta, 2 \rangle$, then θ_6 exists.*

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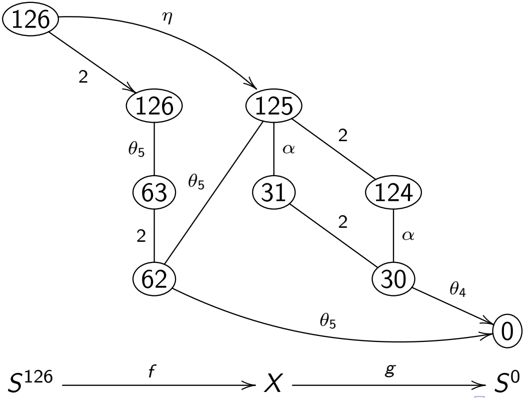

Cellular Motivic Stable Homotopy Category $\mathrm{SH}(k)_{\mathrm{cell}}$

- $S^{1,0}$: simplicial sphere S^1
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- MGL: algebraic cobordism spectrum
- Motivic analogue of classical computational tools exist!

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Theorem (Voevodsky)

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Theorem (Gheorghe - Wang - X.)

$$\widehat{\mathbb{1}}/\tau\text{-}\mathbf{Mod}_{\mathrm{cell}} \simeq \mathbf{Stable}(\mathrm{BP}_*\mathrm{BP}\text{-}\mathbf{Comod})$$

Alternative proofs: Krause, and Pstrągowski.

Isomorphism of spectral sequences

Theorem (Gheorghe - Wang - X.)

$$\textbf{algNovikovSS}(\mathrm{BP}_*) \cong \textbf{motAdamsSS}(\widehat{\mathbb{1}}/\tau).$$

$$\begin{array}{ccc}
\mathrm{Ext}^{s,2w}_{\mathrm{BP}_*\mathrm{BP}}(\mathbb{F}_p, I^{a-s}/I^{a-s+1}) & \xrightarrow{\cong} & \mathrm{Ext}^{a,2w-s+a,w}_{A}(\mathbb{F}_p[\tau], \mathbb{F}_p) \\
\Downarrow \ \textbf{Algebraic Novikov SS} & & \Downarrow \ \textbf{Motivic Adams SS} \\
\mathrm{Ext}^{s,2w}_{\mathrm{BP}_*\mathrm{BP}}(\mathrm{BP}_*, \mathrm{BP}_*) & \xrightarrow{\cong} & \pi_{2w-s,w}(\widehat{\mathbb{1}}/\tau).
\end{array}$$

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- Invert τ.

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- What about the non-cellular part?

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- ξ: virtual vector bundle over X,

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This defines a t-structure.

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Question

What is this heart?

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Motivic Adams and Adams-Novikov spectral sequences

- $\widehat{\mathbb{1}}$: $H\mathbb{F}_p$-completed sphere,
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Work in progress: (Bachmann - Kong - Wang - X.)
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Motivic Adams and Adams-Novikov spectral sequences

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- $\widehat{\mathbb{1}}^{t=n}$: cellular spectrum such that

$$\mathrm{BPGL}_{*,*}\widehat{\mathbb{1}}^{t=n} = \text{Chow-Novikov degree n part of } \mathrm{BPGL}_{*,*}.$$

Computing Stable Stems over k

Apply the motivic Adams spectral sequences:

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Stable Stems over $\mathbb{R}$ at $p = 2$

Theorem (Hu-Kriz, Hill)

$$\mathrm{BPGL}_{*,*} = \mathbb{Z}_2 \left[\begin{array}{cccccc} \rho, & & & & & \\ v_0, & \tau^2 v_0, & \tau^4 v_0, & \tau^6 v_0, & \tau^8 v_0, & \cdots \\ v_1, & & \tau^4 v_1, & & \tau^8 v_1, & \cdots \\ v_2, & & & & \tau^8 v_2, & \cdots \\ \cdots & & & & & \end{array} \right] \Big/ \left[\begin{array}{rl} v_0 & = 2 \\ \rho v_0 & = 0 \\ \rho^3 v_1 & = 0 \\ \rho^7 v_2 & = 0 \\ \cdots & \end{array} \right]$$

and the generators satisfy the further relations

$$\tau^{2^{i+1} \cdot j} v_i \cdot \tau^{2^{k+1} \cdot l} v_k = \tau^{2^{i+1}(j + 2^{k-i} l)} v_i v_k$$

when $i \leqslant k$, as if the class τ were an element in this ring.

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$$= \Sigma^{-8,-8}BP_*/(2, v_1, v_2) \oplus (\Sigma^{0,-4}BP_* + \Sigma^{2,-3}BP_*/2).$$

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- $\widehat{\mathbb{1}}^{t=2} \simeq \Sigma^{-2,-2}\widehat{\mathbb{1}}/(\rho, \tau, 2)$.
- $\widehat{\mathbb{1}}^{t=3} \simeq \Sigma^{-3,-3}\widehat{\mathbb{1}}/(\rho, \tau, 2, v_1)$.
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- $\widehat{\mathbb{1}}^{t=7} \simeq \Sigma^{-7,-7}\widehat{\mathbb{1}}/(\rho, \tau, 2, v_1, v_2)$.
- $\widehat{\mathbb{1}}^{t=8} \simeq \Sigma^{-8,-8}\widehat{\mathbb{1}}/(\rho, \tau, 2, v_1, v_2) \vee X$, X = cofiber of:

$$\Sigma^{-2,-3}\widehat{\mathbb{1}}/(\rho, \tau) \xrightarrow{(v_0, v_1)} \Sigma^{-2,-3}\widehat{\mathbb{1}}/(\rho, \tau) \vee \Sigma^{0,-4}\widehat{\mathbb{1}}/(\rho, \tau)$$

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More purely algebraic parts!

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$$\begin{cases} \Sigma^{-1,-1}\mathrm{BP}_*/2 & \text{if } q \equiv 3 \bmod 4, \\ \Sigma^{-1,-1}\mathrm{BP}_*/4 & \text{if } q \equiv 5 \bmod 8, \\ \Sigma^{-1,-1}\mathrm{BP}_*/8 \cdot 2^m & \text{if } q \equiv 1 \bmod 8. \end{cases}$$

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Proposition (Wilson-Østvær)

- $d_2(\tau g) = u h_0^2 g$, *if* $q \equiv 5 \bmod 8$,
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$$\begin{cases} 0 \leqslant n \leqslant 20 & \text{if } q \equiv 1 \bmod 4, \\ 0 \leqslant n \leqslant 18 & \text{if } q \equiv 3 \bmod 4. \end{cases}$$

Proposition (Wilson-Østvær)

- $d_2(\tau g) = u h_0^2 g$, *if* $q \equiv 5 \bmod 8$,
- $d_2(\tau g) = 0$, *if* $q \equiv 1 \bmod 8$,
- $d_2(\tau^2 g) = \rho \tau h_0^2 g$ *or 0, if* $q \equiv 3 \bmod 4$.

Stable Stems over $\mathbb{F}_q$ at $p = 2$

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Proposition (Wilson-Østvær)

- $d_2(\tau g) = uh_0^2 g$, *if* $q \equiv 5 \bmod 8$,
- $d_2(\tau g) = 0$, *if* $q \equiv 1 \bmod 8$,
- $d_2(\tau^2 g) = \rho\tau h_0^2 g$ *or 0, if* $q \equiv 3 \bmod 4$.

Using Chow-Novikov degree $\leqslant 3$ part of $\pi_{*,*}\mathrm{BPGL}$,

Proposition (Bachmann-Kong-Wang-X.)

When $q \equiv 3 \bmod 4$, $d_2(\tau^2 g) = 0$.

Thank you!