Global group laws and the equivariant Quillen theorem

Markus Hausmann

Bonn University

June 30th, 2020

Complex oriented ring spectra

- E complex oriented ring spectrum
- $y \in E^*(\mathbb{C}P^\infty)$ Euler class of the universal complex line bundle
- Then $E^*(\mathbb{C}P^\infty) \cong E^*[|y|]$
- The tensor product of line bundles give a comultiplication

$$E^*[|y|] \cong E^*(\mathbb{C}P^\infty) \xrightarrow{\Delta} E^*(\mathbb{C}P^\infty \times \mathbb{C}P^\infty) \cong E^*[|y_1, y_2|]$$

- The element

$$F(y_1, y_2) = \Delta(y) = \sum_{i,j \in \mathbb{N}} a_{i,j} \cdot y_1^i y_2^j$$

defines a formal group law over E^*, i.e.,

- $F(y_1, 0) = y_1$; $F(0, y_2) = y_2$ (counitality)
- $F(y_1, y_2) = F(y_2, y_1)$ (cocommutativity)
- $F(y_1, F(y_2, y_3)) = F(F(y_1, y_2), y_3)$ (coassociativity)

Quillen's theorem

- MU complex bordism spectrum
- L Lazard ring (carrying the universal formal group law)

Theorem (Quillen, '69)

For $E = MU$, the associated formal group law is the universal one, i.e., the map

$$L \to MU^*$$

is an isomorphism.

Starting point for chromatic homotopy theory

What happens equivariantly?

- A abelian compact Lie group.
- $S(\mathcal{U}_A) \times_{\mathbb{T}} \mathbb{C} \to \mathbb{C}P(\mathcal{U}_A)$ universal A-equivariant complex line bundle. ($\mathcal{U}_A$ a complete complex A-universe)
- Thom space again A-homeomorphic to $\mathbb{C}P(\mathcal{U}_A)$.
- ϵ trivial 1-dimensional A-representation.

Definition: Equivariant orientations

A complex orientation of an A-ring spectrum E is a class $t \in \widetilde{E}^2(\mathbb{C}P(\mathcal{U}_A))$ which restricts to an $RO(A)$-graded unit in

$$\widetilde{E}^2(\underbrace{\mathbb{C}P(\epsilon \oplus V)}_{\cong S^V}) \cong \widetilde{E}^2(S^V)$$

for every character V of A, and to the element $1 \in E^0 \cong \widetilde{E}^2(S^2)$ for $V = \epsilon$.

Algebraic structure of $E^*(\mathbb{C}P(\mathcal{U}_A))$?

- $y(\epsilon) \in E^*(\mathbb{C}P(\mathcal{U}_A))$ universal Euler class (coordinate)
- Action by $A^* = \mathrm{Hom}(A, \mathbb{T})$ via tensor products with characters. In particular: Have elements $y(V) = V \cdot y(\epsilon)$ for $V \in A^*$.
- Complete:

$$E^*(\mathbb{C}P(\mathcal{U}_A)) \cong \lim_{V_1,\ldots,V_n \in A^*} E^*(\mathbb{C}P(V_1 \oplus \ldots \oplus V_n))$$
$$\cong \lim_{V_1,\ldots,V_n \in A^*} \underbrace{E^*(\mathbb{C}P(\mathcal{U}_A))/y(V_1)y(V_2)\cdots y(V_n)}_{E^*\text{-basis: } 1,\ y(V_1),\ y(V_1)y(V_2),\ y(V_1)y(V_2)\cdots y(V_{n-1})}$$

- The tensor product of line bundles gives a comultiplication

$$E^*(\mathbb{C}P(\mathcal{U}_A)) \xrightarrow{\Delta} E^*(\mathbb{C}P(\mathcal{U}_A) \times \mathbb{C}P(\mathcal{U}_A))$$
$$\cong E^*(\mathbb{C}P(\mathcal{U}_A)) \hat{\otimes}_{E^*} E^*(\mathbb{C}P(\mathcal{U}_A))$$

Definition (Cole-Greenlees-Kriz '00)

An A-equivariant formal group law is a quintuple

$$(k, R, \Delta, \ell, y(\epsilon))$$

- k = ground ring
- R = complete, augmented, topological k-algebra
- $\Delta\colon R \to R\hat{\otimes}_k R$ comultiplication: counital, cocommutative, coassociative
- $\ell = A^*$-action on R, compatible with Δ, such that

$$R \cong \lim_{V_1,\ldots,V_n \in A^*} (R/y(V_1)\cdots y(V_n)))$$

- $y(\epsilon) \in R$ regular generator of augmentation ideal (coordinate).

E complex oriented A-ring spectrum. Then

$$(E^*, E^*(\mathbb{C}P(\mathcal{U}_A)), \Delta, \ell, y(\epsilon))$$

forms an A-equivariant formal group law.

Euler classes of equivariant formal group laws

$(k, R, \Delta, \ell, y(\epsilon))$ A-equivariant formal group law, $V \in A^*$.

- e_V = augmentation of $y(V)$ 'Euler class of V'.
- Topologically: Euler classes $e_V \in E^*$ for bundles over a point.

Example

- If e_V is invertible for all $V \neq \epsilon$, then $R \cong \mathrm{map}(A^*, k[|y|])$.
- If $e_V = 0$ for all $V \in A^*$, then $R \cong k[|y|]$.

The equivariant Lazard ring

Proposition (Cole-Greenlees-Kriz)

There exists a universal A-equivariant formal group law, defined over an A-equivariant Lazard ring L_A.

MU_A (stable) A-equivariant complex bordism (tom Dieck, '70). Get a map

$$\varphi_A \colon L_A \to MU_A^*$$

Theorem (Greenlees, '01)

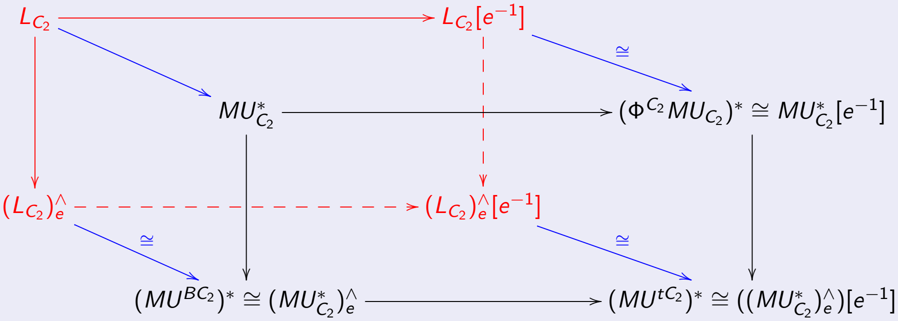
A finite abelian. Then

$$\varphi_A \colon L_A \to MU_A^*$$

is surjective, with Euler-torsion and infinitely Euler-divisible kernel.

Sketch of the argument for $A = C_2$

The Tate square for MU_{C_2} yields a pullback square $\square$



Problem: Unclear that $\square$ is a pullback square.

Upshot: Main issue is controlling the Euler torsion in L_A.

An equivariant Quillen theorem

Theorem (Strickland '02)

Presentation of $MU^*_{C_2}$ *and section* $MU^*_{C_2} \to L_{C_2}$ *of* φ_{C_2}.

Theorem (Hanke-Wiemeler '17)

φ_{C_2} *is an isomorphism.*

Theorem (H.)

φ_A *is an isomorphism for every abelian compact Lie group* A.

Main tool: Global homotopy theory (Schwede)

Slogan

Global spectrum = compatible collection of A-spectra for all A

$$\begin{array}{ccc}
E \text{ global spectrum} & \dashrightarrow & \text{cohomology theory } E^* \text{ on orbispaces} \\
\big\downarrow & & \big\downarrow \text{ evaluate on } (- /\!\!/ A) \\
E_A \ A\text{-spectrum} & \dashrightarrow & \text{cohomology theory } E_A^* \text{ on } A\text{-spaces}
\end{array}$$

Remark

$E_A^* \cong E^*(* /\!\!/ A)$

Examples

$\mathbb{S}_{gl}, KU_{gl}$, Borel theories, MU_{gl}

- ◇ E complex oriented global ring spectrum
  (⇒ E_A complex oriented for all A)
- ◇ $\mathbb{C} /\!\!/ \mathbb{T} \to * /\!\!/ \mathbb{T}$ universal global complex line bundle

Question

What is the algebraic structure of $E^*(* /\!\!/ \mathbb{T}) = E^*_{\mathbb{T}}$?

Have:

- ◇ universal Euler class $e \in E^*_{\mathbb{T}}$.
- ◇ comultiplication $m^* \colon E^*_{\mathbb{T}} \to E^*_{\mathbb{T}\times\mathbb{T}}$.

Problems/differences:

- ◇ $E^*_{\mathbb{T}}$ is generally not complete.
  Example: $KU^*_{\mathbb{T}} \cong RU(\mathbb{T}) \cong \mathbb{Z}[t^{\pm 1}]$
- ◇ $E^*_{\mathbb{T}\times\mathbb{T}}$ is generally not a tensor product of two copies of $E^*_{\mathbb{T}}$.

$\Rightarrow$ Need to consider all of $E^*, E^*_{\mathbb{T}}, E^*_{\mathbb{T}\times\mathbb{T}}, \ldots$ as a functor

$$\underline{E}^* \colon \mathrm{tori}^{op} \to \text{commutative rings},$$

together with $e \in E^*_{\mathbb{T}}$.

New question

What are the properties of the pair $(\underline{E}^*, e)$?

Lemma

For every torus A and split character $V \in A^*$ (i.e., $V: A \to \mathbb{T}$ has a section), the Euler class $e_V = V^*(e) \in E_A^*$ is a regular generator of the kernel of the restriction map

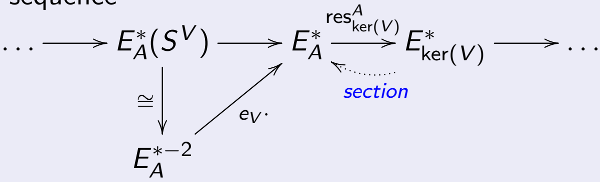
$$E_A^* \to E_{\ker(V)}^*.$$

Proof

- $\diamond$ Note: $S(V) \cong A/\ker(V)$

$\Rightarrow$ A-cofiber sequence $(A/\ker(V))_+ \to S^0 \to S^V$.

$\Rightarrow$ long exact sequence



Observation

For every torus A, the A-equivariant formal group law

$$(E_A^*, E_A^*(\mathbb{C}P(\mathcal{U}_A)), \Delta, \ell, y(\epsilon))$$

can be reconstructed from $(\underline{E}^*, e)$, using only the lemma.

- Uses a completion theorem

$$E_A^*(\mathbb{C}P(\mathcal{U}_A)) \cong \lim_{V_1,\ldots,V_n \in A^*} (E^*_{A\times\mathbb{T}}/y(V_1)\cdots y(V_n)),$$

  with $y(V) = e_{(V,\mathrm{id}_{\mathbb{T}})} \in E^*_{A\times\mathbb{T}}$.
- Δ and ℓ are induced by the global functoriality, from the maps

$$id_A \times m \colon A \times \mathbb{T} \times \mathbb{T} \to A \times \mathbb{T}$$

  and

$$\begin{pmatrix} \mathrm{id}_A & V \\ 0 & \mathrm{id}_{\mathbb{T}} \end{pmatrix} : A \times \mathbb{T} \to A \times \mathbb{T}.$$

Global group laws

This motivates:

Definition

A global group law is a pair (X, e) of

- a functor $X\colon \mathrm{tori}^{op} \to$ commutative rings, and
- an element $e \in X(\mathbb{T})$,

such that for every torus A and split character $V \in A^*$ the element $e_V = V^*(e) \in X(A)$ is a regular generator of the kernel of the restriction map

$$X(A) \to X(\ker(V)).$$

Lemma

There exists an initial global group law (L_{gl}, e).

How does L_{gl} relate to L_A?

There is an adjunction

$$F\colon \{\text{global group laws}\} \rightleftarrows \{\text{A-equivariant formal group laws}\} : G$$

given by

$$F(X,e) = (X(A), X(A\times \mathbb{T})^{\wedge}, \Delta, \ell, e_{(\epsilon,\mathrm{id}_{\mathbb{T}})})$$

and

$$G(k,R,\Delta,\ell,y(\epsilon)) = (\mathbb{T}^n \mapsto R^{\hat{\otimes}_k n}, y(\epsilon)).$$

Corollary

$L_{gl}(A) \cong L_A$.

Regularity of Euler classes

Corollary

If A is a torus and $V \in A^*$ is split, then $e_V \in L_A$ is a regular element.

With more work we can show:

Theorem (H.)

If A is a torus and $V \in A^*$ is non-trivial, then $e_V \in L_A$ is a regular element.

Corollary

If A is a torus, then L_A is an integral domain.

Using this, it is not hard to show that

$$\varphi_A \colon L_A \to MU_A^*$$

is an isomorphism for every torus A. The general case follows by embedding into tori.

A global Quillen theorem

Theorem (H.)

The pair $(\underline{MU}^*_{gl}, e)$ *is the universal global group law.*

Slogan: Global group laws are an uncompleted version of formal group laws.

Examples of global group laws

- F ordinary formal group law over k, then $\mathbb{T}^n \mapsto k[|y_1, \ldots, y_n|]$ with functoriality via F is a global group law. This defines a fully-faithful embedding of formal group laws into global group laws. (topologically: Borel theories)
- additive group: $\mathbb{G}_a(\mathbb{T}^n) = \mathbb{Z}[e_1, \ldots, e_n]$
- multiplicative group: $\mathbb{G}_m(\mathbb{T}^n) = \mathbb{Z}[t_1^{\pm 1}, \ldots, t_n^{\pm 1}]$ (topologically: KU_{gl})
- Elliptic curve C over a scheme S, then $\mathbb{T}^n \mapsto C^{\times_S n}$ is a 'global group' (topologically: ??)
- Universal constructions: e.g., initial global group law (MU_{gl}), a free global group law on two coordinates ($MU_{gl} \wedge MU_{gl}$) , or a free global group law satisfying $e_{V^{-1}} = -e_V$, ...

Thank you for your attention!