THE GALOIS COHOMOLOGY OF $\mathcal{O}_C$

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These are notes taken in preparation for a talk for Babytop during Spring 2024, which covered the [BSSW24] computation of the rational homotopy groups of the $K(n)$-local sphere using methods from p-adic Hodge theory. My talk aimed to cover part of §4 of this paper, on the Galois cohomology of the ring of integers in the completion of the algebraic closure of a local field. I do not know much about anything involving the adjective p-adic, so I would like to apologize in advance for any confusion caused. Any mistakes are due to me, and unless otherwise indicated all results here are due to [BSSW24].

I would like to thank Andy Senger for a lot of help in understanding the contents of the paper, and Merrick Cai, Daishi Kiyohara, Frank Lu, and Dylan Pentland for helping me fill in some background I needed to understand what was going on.

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1. Introduction

So far in the seminar, we have spent a lot of time introducing the background needed to understand and prove the following theorem of interest.

Theorem 1.0.1 ([BSSW24], Thm. B)**.** *For every integer $s \geq 0$, the natural map $W := W(\overline{\mathbb{F}}_p) \hookrightarrow W(\overline{\mathbb{F}}_p)[\![u_1, \ldots, u_{n-1}]\!] =: A$ induces a rational isomorphism*

$$H^s_{\text{cts}}(\mathbb{G}_n, W) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \xrightarrow{\sim} H^s_{\text{cts}}(\mathbb{G}_n, A) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p.$$

For the topologically-inclined, this provides a way, at least rationally, to simplify the input to the Devinatz-Hopkins spectral sequence

$$H^s_{\text{cts}}(\mathbb{G}_n, A_t) \Longrightarrow \pi_{t-s} L_{K(n)} \mathbb{S},$$

which is what will eventually allow us to deduce the landmark computation of the rational homotopy groups of the $K(n)$-local sphere.

Theorem 1.0.2 ([BSSW24], Thm. A)**.** *There is an isomorphism of graded $\mathbb{Q}$-algebras*

$$\mathbb{Q} \otimes \pi_* L_{K(n)} \mathbb{S} \simeq \Lambda_{\mathbb{Q}_p}(\zeta_1, \zeta_2, \ldots, \zeta_n).$$

Using the titular two towers isomorphism that Andy discussed, we can approach a proof of Theorem 1.0.1 via methods of p-adic geometry. It turns out that a crucial ingredient will be to have a good understanding of the pro-étale cohomology

$$H_*(X_{\text{proét}}, \hat{\mathcal{O}}^+)$$

of a rigid analytic space X over a local field K of mixed characteristic $(0, p)$. Today, we'll start working towards understanding this for the simplest case, $X = \operatorname{Spa} K$, in which case this pro-étale cohomology can actually be described as something perhaps more familiar: Galois cohomology. We have

$$H^*(\operatorname{Spa} K, \hat{\mathcal{O}}^+) = H^*_{\text{cts}}(\operatorname{Gal}(\overline{K}/K), \mathcal{O}_C),$$

where C is the completion of the algebraic closure of K. To spoil the punchline, we will eventually obtain the following theorem.

Theorem 1.0.3 ([BSSW24], Thm. C)**.** *Let K be a local field of characteristic $(0, p)$, and let C be the completion of an algebraic closure $\overline{K}$ of K. Let $\mathcal{O}_K \subset K$ and $\mathcal{O}_C \subset C$ denote the valuation rings. There is an isomorphism of graded $\mathcal{O}_K$-modules*

$$H^*_{\text{cts}}(\operatorname{Gal}(\overline{K}/K), \mathcal{O}_C) \simeq \mathcal{O}_K[\epsilon] \oplus T,$$

where ϵ is a degree 1 element with $\epsilon^2 = 0$, and T is p^N-torsion for some absolute *constant N.*

In my talk, I will begin by reviewing some Galois theory and ramification theory, and then give an overview of the argument to prove Theorem 1.0.3. Next week, Tristan will fill in the holes that I have left behind to complete the proof.

2. Some ramification theory

We'll begin by defining some of our main objects of study.

Definition 2.0.1. A *nonarchimedean field* is a field K which is complete with respect to the topology induced from a nontrivial nonarchimedean valuation $|\ | : K \to \mathbb{R}_{\geq 0}$. Given a nonarchimedean field K, we define its *valuation ring* or *ring of integers*

$$\mathcal{O}_K := \{\alpha \in K : |\alpha| \leq 1\},$$

which is a local ring; denote its residue field by κ. The characteristic of a nonarchimedean field refers to the pair $(\operatorname{char} K, \operatorname{char} \kappa)$.

We note that if L is any Galois extension of a nonarchimedean field K, then the valuation on K extends uniquely to a valuation on L, and the completion $\hat{L}$ is nonarchimedean field with a continuous action of $\operatorname{Gal}(L/K)$. So we can consider the continuous Galois cohomology

$$H^*_{\text{cts}}(\operatorname{Gal}(L/K), \mathcal{O}_{\hat{L}}) := H^*\left(\mathcal{O}_{\hat{L}}^{h \operatorname{Gal}(L/K)}\right),$$

the group cohomology of the Galois group acting on $\mathcal{O}_{\hat{L}}$. Our goal will be to gain an understanding of these groups when K is a local field.

Definition 2.0.2. A *local field* is a nonarchimedean field K such that

- the valuation on K is discrete, and
- the residue field κ is perfect.

We will be concerned with local fields of characteristic $(0,p)$, the prototypical examples of which are $\mathbb{Q}_p$ and finite extensions of it; we also have slightly more exotic examples like $W(\overline{\mathbb{F}}_p)[\frac{1}{p}]$. Our study of extensions of these local fields will be via *ramification theory*, which roughly studies how the prime ideals in the ring of integers of a field split under field extensions.

First, let K be a local field of characteristic $(0,p)$. Then the ring of integers $\mathcal{O}_K$ is a discrete valuation ring, and so we can choose a uniformizer π_K and write any element $\alpha \in K^\times$ as

$$\alpha = \pi_K^{v_K(\alpha)} u,$$

where u is a unit. This defines a surjective map $v_K : K^\times \to \mathbb{Z}$ (which is independent of the choice of uniformizer), known as the relative valuation map.

Definition 2.0.3. Let K be a local field of characteristic $(0,p)$. The *absolute ramification index* of K is

$$e_K := v_K(p).$$

One way to understand what this number describes is by thinking about ideals. If K is a local field of characteristic $(0,p)$, it is in fact an extension of $\mathbb{Q}_p$, and we have an inclusion of discrete valuation rings

$$\mathbb{Z}_p \subset \mathcal{O}_K.$$

p is a uniformizer for $\mathbb{Z}_p$, and so the ideal $p\mathbb{Z}_p$ is the unique maximal ideal of $\mathbb{Z}_p$. However, this does not usually remain the case when we extend to K; by definition, we have

$$p\mathcal{O}_K = (\pi_K)^{e_K}.$$

So the absolute ramification index tells us how the maximal ideal of $\mathbb{Z}_p$ splits as a product of copies of the maximal ideal in $\mathcal{O}_K$, upon extending from $\mathbb{Q}_p$ to K.

One may notice that we can do something similar for any extension L/K, where K and L are both local of characteristic $(0,p)$. We can consider the inclusion of the rings of integers $\mathcal{O}_K \subset \mathcal{O}_L$, and see how the maximal ideal of $\mathcal{O}_K$ splits upon extending to $\mathcal{O}_L$; that is, find the integer $e_{L/K}$ that gives us

$$\pi_K \mathcal{O}_L = (\pi_L)^{e_{L/K}}.$$

This leads to the definition of the relative ramification index.

Definition 2.0.4. Let K be a local field of characteristic $(0,p)$, and let L be a finite extension of K. Then it turns out that L is also a local field of characteristic $(0,p)$. The *relative ramification index* of L/K is

$$e_{L/K} := v_L(\pi_K).$$

Question 2.0.5. The above definition makes sense when L is an arbitrary extension of K, as long as they are both local. Later, we will make a definition of this relative ramification divisor for arbitrary extensions, but I wonder if these definitions end up agreeing.

Now, recall that our goal is to eventually study $\overline{K}/K$, which is an infinite extension, and which in particular need not be local. So we will need a way to make sense of ramification in infinite extensions. We will accomplish this by studying certain filtrations of the Galois group.

Definition 2.0.6. Let L/K be a finite Galois extension. We define an exhaustive decreasing filtration on the Galois group $\mathrm{Gal}(L/K)$, denoted $\mathrm{Gal}(L/K)_u$ by the real numbers $u \geq -1$, by

$$\mathrm{Gal}(L/K)_u = \{\sigma \in \mathrm{Gal}(L/K) : v_L(\sigma(\alpha) - \alpha) \geq u + 1 \text{ for all } \alpha \in \mathcal{O}_L\}.$$

We call this the *lower numbering filtration* on $\mathrm{Gal}(L/K)$.

It turns out that this filtration packages a lot of ramification-theoretic information nicely, as perhaps illustrated by the following proposition.

Proposition 2.0.7. *Let L/K be a finite Galois extension of local fields of characteristic $(0,p)$. Then*

$$|\operatorname{Gal}(L/K)_0| = e_{L/K}.$$

Remark 2.0.8. For those whom this means something, $\operatorname{Gal}(L/K)$ is the *inertia group* of L/K.

This suggests an approach to studying ramification in infinite extensions; for an arbitrary Galois extension L/K, we can try to define a filtration on

$$\operatorname{Gal}(L/K) = \varprojlim_{L'/K \text{ finite subextension}} \operatorname{Gal}(L'/K),$$

coming from a filtration on the Galois groups of the finite subextensions. However, we can't quite use this lower numbering filtration to make a definition, because it is not compatible with the quotients in the limit above; if L/K is a finite extension and L'/K is a subextension, then it isn't true that $\operatorname{Gal}(L'/K)_u = \operatorname{Gal}(L/K)_u / \operatorname{Gal}(L/L')_u$. But do not despair; we will fix this by force.

Definition 2.0.9. Let L/K be a finite Galois extension. The Herbrand function $\varphi_{L/K} : [-1,\infty) \to [-1,\infty)$ is the continous, increasing and bijective function defined by

$$\varphi_{L/K(u} = \begin{cases} u, & -1 \leq u \leq 0 \\ \int_0^u [\operatorname{Gal}(L/K)_0 : \operatorname{Gal}(L/K)_t]^{-1} dt, & u > 0. \end{cases}$$

Denote the inverse function of $\varphi_{L/K}$ by $\psi_{L/K}$. The *upper numbering filtration* on $\operatorname{Gal}(L/K)$ is defined by

$$\operatorname{Gal}(L/K)^u = \operatorname{Gal}(L/K)_{\psi_{L/K}(u)}.$$

Remark 2.0.10. Note that $\varphi_{L/K}(0) = 0$, so for any finite extension $\operatorname{Gal}(L/K)^0 = \operatorname{Gal}(L/K)_0$.

By construction, we can check that the upper numbering filtration plays well with quotients (see [Ser79, Chapter IV, §3, Proposition 14]), and so we obtain a filtration on the Galois group for infinite extensions too.

Definition 2.0.11. Let L/K be an arbitrary Galois extension. For $-1 \leq u \leq 0$, We define the *upper numbering filtration* on $\operatorname{Gal}(L/K)$ via the subgroups

$$\operatorname{Gal}(L/K)^u = \varprojlim_{L'/K \text{ finite subextensions}} \operatorname{Gal}(L'/K)^u.$$

With this, we are ready to talk about ramification in infinite extensions.

Definition 2.0.12. Let L/K be a Galois extension. We say L/K is *ramified* if $\operatorname{Gal}(L/K)^0 \neq \{1\}$. Otherwise, we say L/K is *unramified*.

3. An overview of the argument

Now that we have some basic familiarity with ramification theory, let us start describing how we are going to approach our theorem of interest, the computation of

$$H^*(\operatorname{Gal}(\overline{K}/K), \mathcal{O}_C) = H^*\left(\mathcal{O}_C^{h \operatorname{Gal}(\overline{K}/K)}\right).$$

The idea, originally due to Tate, is to consider an ramified intermediate extension K_∞/K with Galois group $\mathbb{Z}_p$; using the fact that fixed points play nicely with quotients, we can then write

$$\mathcal{O}_C^{h\,\mathrm{Gal}(\overline{K}/K)} \simeq \left(\mathcal{O}_C^{h\,\mathrm{Gal}(\overline{K}/K_\infty)}\right)^{h\,\mathrm{Gal}(K_\infty/K)}. \tag{3.0.1}$$

As described in the beginning of [BSSW24, §4.2], we can always construct an extension like this. Given a local field K of characteristic $(0,p)$, let L/K be the extension obtained by adjoining a primitive p^nth root of unity ζ_{p^n} for all $n \geq 1$. This is a ramified extension with Galois group an open subgroup of $\mathbb{Z}_p^\times$, and so it has a maximal quotient isomorphic to $\mathbb{Z}_p$. We can see that K_∞/K corresponding to this quotient is a ramified $\mathbb{Z}_p$-extension of K, called the *cyclotomic $\mathbb{Z}_p$-extension.*

By virtue of having Galois group $\mathbb{Z}_p$, K_∞/K inherits a natural filtration by finite subextensions

$$K \subset K_1 \subset K_2 \subset \cdots \subset K_\infty,$$

where K_n/K corresponds to the subgroup $p^n\mathbb{Z}_p \subset \mathbb{Z}_p$, with Galois group $\mathbb{Z}_p/p^n\mathbb{Z}_p \cong \mathbb{Z}/p^n$. Using these subextensions, Tate obtains a pretty good understanding of the cohomology of our intermediate middle term in Equation 3.0.1. We'll state his result after a brief definition.

Definition 3.0.2. Let M be a module over a valuation ring R. We say M is *almost zero* if it is α-torsion for any $\alpha \in R$ with $v(\alpha) > 0$.

Theorem 3.0.3 ([Tat67], §3, Corollary 1)**.** *We have an isomorphism of $\mathcal{O}_{\hat{K}_\infty}$-modules.*

$$H^i_{\mathrm{cts}}\left(\mathrm{Gal}(\overline{K}/K_\infty), \mathcal{O}_C\right) = \begin{cases} \mathcal{O}_{\hat{K}_\infty}, & i = 0 \\ \textit{almost zero}, & i > 0. \end{cases}$$

Said differently, we can define the complex of solid $\mathcal{O}_{\hat{K}_\infty}$-modules $\mathcal{Y}_0$ to be the cofiber

$$\mathcal{O}_{\hat{K}_\infty} \to \mathcal{O}_C^{h\,\mathrm{Gal}(\overline{K}/K_\infty)} \to \mathcal{Y}_0, \tag{3.0.4}$$

where the first morphism is induced from $\mathcal{O}_{\hat{K}_\infty} \hookrightarrow \mathcal{O}_C$. Then $H^0(\mathcal{Y}_0) = 0$, and for all $i > 0$, $H^i(\mathcal{Y}_0)$ is almost zero in the sense that the action of any $\alpha \in \mathcal{O}_{\hat{K}_\infty}$ with $v(\alpha) > 0$ on it is zero.

We will omit proof of this theorem; for a sketch of the proof, take a look at [BSSW24, Theorem 4.3.2]. Just to say some words that might mean something to people other than me; apparently most of the proof hinges on the observation that $\hat{K}_\infty$ is perfectoid.

Before we can transfer this to study $\mathcal{O}_C^{h\,\mathrm{Gal}(\overline{K}/K)}$, we will need a new way to measure ramification.

Definition 3.0.5. Let L/K be a finite Galois extension of local fields of characteristic $(0,p)$. We have the trace map $\mathrm{tr}_{L/K} : L \to K$, which yields a pairing

$$\begin{aligned} \langle -, - \rangle : L \times L &\to K \\ (x,y) &\mapsto \mathrm{tr}_{L/K}(xy). \end{aligned}$$

The dual of $\mathcal{O}_K$ under this pairing is the set

$$\{\beta \in L : \langle \alpha, \beta \rangle \in \mathcal{O}_K \text{ for all } \alpha \in \mathcal{O}_L\};$$

this is a fractional ideal of L containing $\mathcal{O}_L$. The *different* $\mathcal{D}_{L/K}$ is the ideal of $\mathcal{O}_L$ which is the inverse of this fractional ideal.

The different is an ideal of $\mathcal{O}_L$, so we can write it as (π_L^m) for some m. The valuation of the different $v_L(\mathcal{D}_{L/K})$ will be this m. It turns out that in the case of the finite subextensions K_n of K, we can obtain a bound on this exponent.

Lemma 3.0.6 ([BSSW24], Lemma 4.2.3)**.** *Let K_∞/K be a ramified $\mathbb{Z}_p$-extension. There exists an integer $N \geq 0$ such that for all $n > N$ we have*

$$v_{K_n}(\mathcal{D}_{K_n/K_{n-1}}) \geq p - 1 + e_{K_n}(1 - p^{N-n}). \tag{3.0.7}$$

Definition 3.0.8. Let K_∞/K be a ramified $\mathbb{Z}_p$-extension. We call K_∞/K *sufficiently ramified* if the inequality in Equation 3.0.7 holds for all $n \geq 1$.

Note that if we take N to be the integer appearing in 3.0.6, then K_∞/K_N is sufficiently ramified. Our torsion bounds on $H^*_{\mathrm{cts}}(\mathrm{Gal}(\overline{K}/K), \mathcal{O}_C)$ will come from a careful analysis of these sufficiently ramified extensions. The following proposition will be our workhorse.

Proposition 3.0.9 ([BSSW24], Proposition 4.2.17)**.** *Let K_∞/K be a ramified $\mathbb{Z}_p$-extension. Let $N \geq 0$ be large enough so that K_∞/K_N is sufficiently ramified. Define a p-adically complete abelian group X with continuous $\mathrm{Gal}(K_\infty/K)$-action by the exact sequence*

$$0 \to \mathcal{O}_K \to \mathcal{O}_{\hat{K}_\infty} \to X \to 0.$$

Then $H^i_{\mathrm{cts}}(\mathrm{Gal}(K_\infty/K), X) = 0$ for all $i \neq 1$, and $H^1_{\mathrm{cts}}(\mathrm{Gal}(K_\infty/K), X)$ is p^{N+2}-torsion for $p \neq 1$ and p^{N+3}-torsion for $p \neq 2$.

Restated in terms of complexes, we have a cofiber sequence

$$\mathcal{O}_K^{h\,\mathrm{Gal}(K_\infty/K)} \to \mathcal{O}_{\hat{K}_\infty}^{h\,\mathrm{Gal}(K_\infty/K)} \to X^{h\,\mathrm{Gal}(K_\infty/K)} =: \mathcal{X}.$$

Then $H^i(\mathcal{X}) = 0$ for $i \neq 1$, and $H^1(\mathcal{X}) = H^1_{\mathrm{cts}}(\mathrm{Gal}(K_\infty/K), X)$ is torsion as claimed above.

We won't prove this today, but next week we might get an idea of how to arrive at this. For now, let's take a look at how we can use this to prove our theorem of interest. First, recall the cofiber sequence from Tate's theorem, appearing in Equation 3.0.4. We can take $\mathrm{Gal}(K_\infty/K)$-fixed points to obtain the cofiber sequence

$$\mathcal{O}_{\hat{K}_\infty}^{h\,\mathrm{Gal}(K_\infty/K)} \to \mathcal{O}_C^{h\,\mathrm{Gal}(\overline{K}/K)} \to \mathcal{Y}, \tag{3.0.10}$$

where $\mathcal{Y} = \mathcal{Y}_0^{h\,\mathrm{Gal}(K_\infty/K)}$. Our first result in this direction is the following theorem.

Theorem 3.0.11 ([BSSW24], Theorem 4.3.10)**.** *Let K_∞/K be a ramified $\mathbb{Z}_p$-extension, and let N be large enough so that K_∞/K_N is sufficiently ramified. Define $\mathcal{Z}$ by the cofiber sequence*

$$\mathcal{O}_K^{h\,\mathrm{Gal}(K_\infty/K)} \to \mathcal{O}_C^{h\,\mathrm{Gal}(\overline{K}/K)} \to \mathcal{Z}, \tag{3.0.12}$$

where the first morphism is the composite $\mathcal{O}_K^{h\,\mathrm{Gal}(K_\infty/K)} \to \mathcal{O}_{\hat{K}_\infty}^{h\,\mathrm{Gal}(K_\infty/K)} \to \mathcal{O}_C^{h\,\mathrm{Gal}(\overline{K}/K)}$. Then

$$H^i(\mathcal{Z}) = \begin{cases} 0, & i = 0 \\ p^{N+3}\text{-torsion (resp., } p^{N+5}\text{-torsion)}, & i = 1 \\ p\text{-torsion (resp., } p^2\text{-torsion)}, & i \geq 2 \end{cases}$$

as p is odd or even, respectively.

Proof. We have three cofiber sequences

$$\mathcal{O}_K^{h\,\mathrm{Gal}(K_\infty/K)} \to \mathcal{O}_{\hat{K}_\infty}^{h\,\mathrm{Gal}(K_\infty/K)} \to \mathcal{X}$$
$$\mathcal{O}_{\hat{K}_\infty}^{h\,\mathrm{Gal}(K_\infty/K)} \to \mathcal{O}_C^{h\,\mathrm{Gal}(\overline{K}/K)} \to \mathcal{Y}$$
$$\mathcal{O}_K^{h\,\mathrm{Gal}(K_\infty/K)} \to \mathcal{O}_C^{h\,\mathrm{Gal}(\overline{K}/K)} \to \mathcal{Z},$$

from Proposition 3.0.9, Equation 3.0.10, and the statement of the theorem, respectively. The octahedral axiom then tells us that we have a cofiber sequence

$$\mathcal{X} \to \mathcal{Z} \to \mathcal{Y}.$$

Our workhorse Proposition 3.0.9 gives us control of the torsion in $\mathcal{X}$, and the following lemma gives us control of the torsion in $\mathcal{Y}$.

Lemma 3.0.13 ([BSSW24], Lemma 4.3.7)**.** *$H^0(\mathcal{Y}) = 0$, and for all $i \geq 1$, $H^i(\mathcal{Y})$ is p-torsion if p is odd, and p^2-torsion if $p = 2$.*

I'll just give one comment on this lemma; if you are okay with just having that it is p^2-torsion for all primes, this follows from considering the homotopy fixed points spectral sequence for $\mathcal{Y}$. To get that it is p-torsion for odd p requires some extra thought.

In any case, our theorem then follows from this lemma and Proposition 3.0.9, by considering the long exact sequence in cohomology associated to $\mathcal{X} \to \mathcal{Z} \to \mathcal{Y}$. ☐

We are almost at our goal! Let's see how we can use this theorem to finally prove Theorem 1.0.3, which we will restate a little more precisely below.

Theorem 3.0.14. *Let K be a local field of characteristic $(0, p)$. Let C be the completion of an algebraic closure $\overline{K}/K$. Then*

(1) $H^0(\mathrm{Gal}(\overline{K}/K), \mathcal{O}_C) = \mathcal{O}_K$
(2) There exists an isomorphism of $\mathcal{O}_K$-modules
$$H^1_{\text{cts}}(\mathrm{Gal}(\overline{K}/K), \mathcal{O}_C) \cong \mathcal{O}_K \oplus T$$
where T is p^4-torsion (resp., p^6-torsion) as p is odd or even, respectively.
(3) For $i > 1$, $H^i_{\text{cts}}(\mathrm{Gal}(\overline{K}/K), \mathcal{O}_C)$ is p-torsion (resp., p^2-torsion) as p is odd or even, respectively.

Remark 3.0.15. If K has *tame ramification*; that is, if $p \nmid e_K$, or if it is a cyclotomic extension of a tamely ramified field, then the bound in (2) can be improved to p^3 for p odd, and p^5 for p even.

Proof. Let K_∞/K be a ramified $\mathbb{Z}_p$-extension. Take a look at the long exact sequence associated to sequence in Equation 3.0.12,

$$\mathcal{O}_K^{h\,\mathrm{Gal}(K_\infty/K)} \to \mathcal{O}_C^{h\,\mathrm{Gal}(\overline{K}/K)} \to \mathcal{Z}.$$

The action of $\mathrm{Gal}(K_\infty/K)$ on $\mathcal{O}_K$ is trivial, so we obtain $H^0(\mathrm{Gal}(\overline{K}/K), \mathcal{O}_C) = \mathcal{O}_K$, proving (1). We further retrieve isomorphisms $H^i_{\text{cts}}(\mathrm{Gal}(\overline{K}/K), \mathcal{O}_C) \cong H^i(\mathcal{Z})$ for $i > 1$; Theorem 3.0.11 then gives us (3).

Towards proving (2), we will take a look at last part of this long exact sequence, with the stuff in degree 1:

$$0 \to \mathcal{O}_K \to H^1_{\text{cts}}(\mathrm{Gal}(\overline{K}/K), \mathcal{O}_C) \to H^1(\mathcal{Z}) \to 0.$$

For starters, assume that K_∞/K is sufficiently ramified. Then Theorem 3.0.11 gives us that $H^1(\mathcal{Z})$ is p^3- or p^5-torsion as p is odd or even. Further, we have the following lemma:

Lemma 3.0.16. *Given an exact sequence of $\mathcal{O}_K$-modules*

$$0 \to M_1 \to M \to M_2 \to 0,$$

with M_1 finite free of rank r and M_2 killed by p^n, there exists an isomorphism $M \simeq \mathcal{O}_K^{\oplus r} \oplus T$, where T is p^n-torsion.

This allows us to conclude that $H^1_{\text{cts}}(\mathrm{Gal}(\overline{K}/K), \mathcal{O}_C) \cong \mathcal{O}_K \oplus T$, where T is p^3/p^5-torsion.

In the general case, this line of argument shows us that $H^1_{\text{cts}}(\mathrm{Gal}(\overline{K}/K), \mathcal{O}_C) \cong \mathcal{O}_K \oplus T$, where T is p^r-torsion for some r; we need to revise this down using the sufficiently ramified case. If we let N be large enough that K_∞/K_N is sufficiently ramified, then considering the short exact sequence above shows us that we have an injection $\mathcal{O}_{K_N} \to H^1_{\text{cts}}(\mathrm{Gal}(\overline{K}/K), \mathcal{O}_C)$ whose cokernel is killed by p^3/p^5. We then can look at the *inflation-restriction sequence* for the tower $K_\infty/K_n/K$; in part, this reads

$$H^1(\mathrm{Gal}(K_N/K), \mathcal{O}_{K_N}) \to H^1_{\text{cts}}(\mathrm{Gal}(\overline{K}/K), \mathcal{O}_C) \to H^1_{\text{cts}}(\mathrm{Gal}(\overline{K}/K_N), \mathcal{O}_C)^{\mathrm{Gal}(K_N/K)}.$$

By a theorem of Sen ([BSSW24, Theorem 4.0.2]), the term on the left of this sequence is p-torsion, and so we can see that p^4/p^6 kills all torsion in $H^1(\mathrm{Gal}(\overline{K}/K), \mathcal{O}_C)$, proving (2). ☐

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