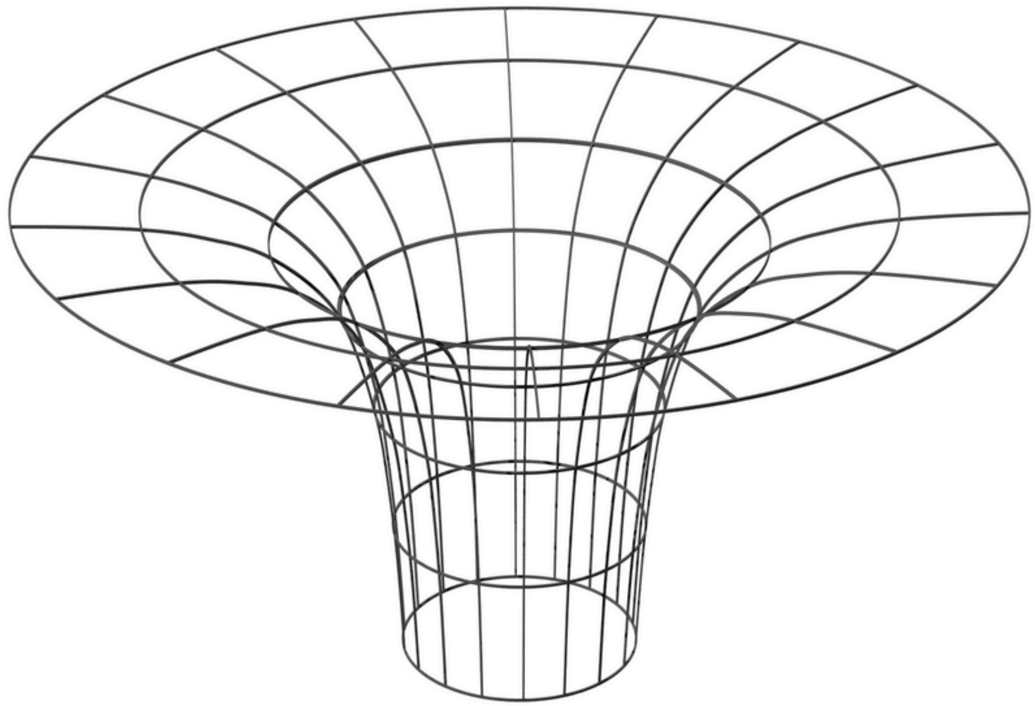


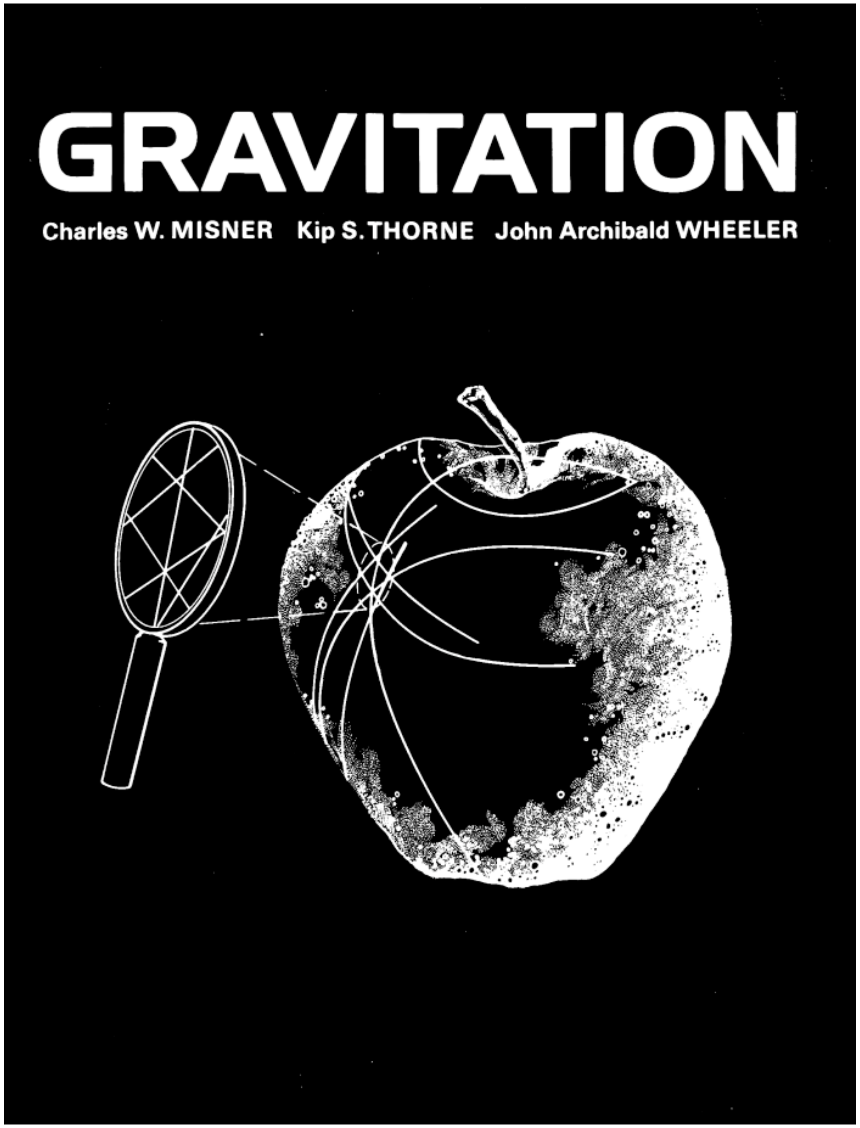
The lighter side of black holes

Chris Rycroft

SPAMS, Spring 2006

Reference

- *"Gravitation"* by Misner, Thorne and Wheeler
- Wheeler coined phrase "black hole" in 1979

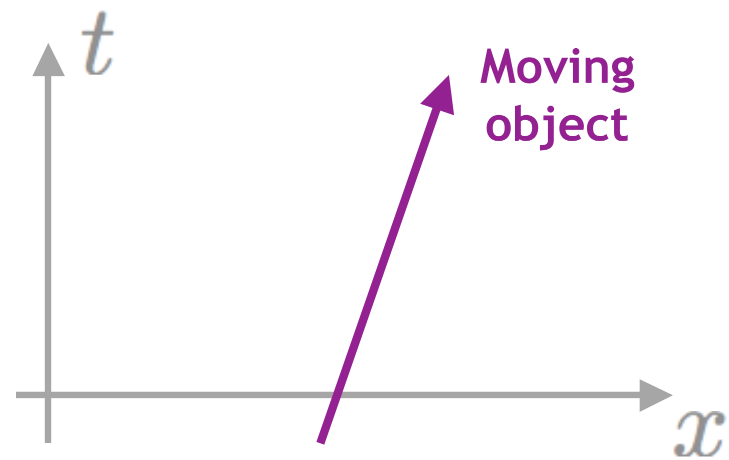


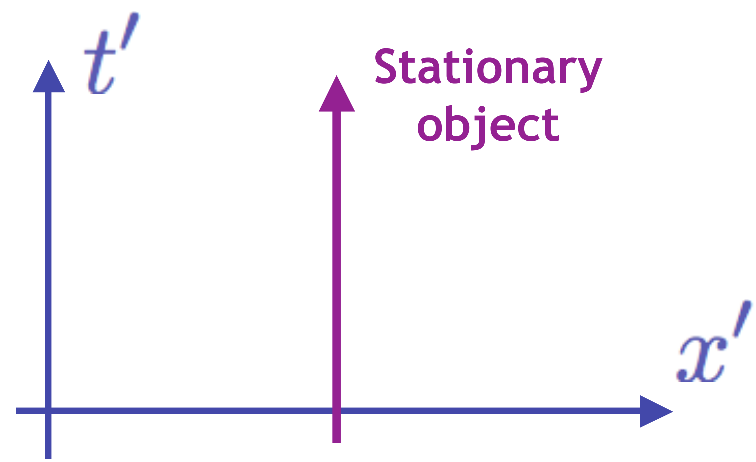
Principle of relativity

- 6th Century BC - Indian word “*sapekshavad*”
- Galilean transformation:

$$
\begin{aligned}
x' &= x + vt \\
t' &= t
\end{aligned}
$$

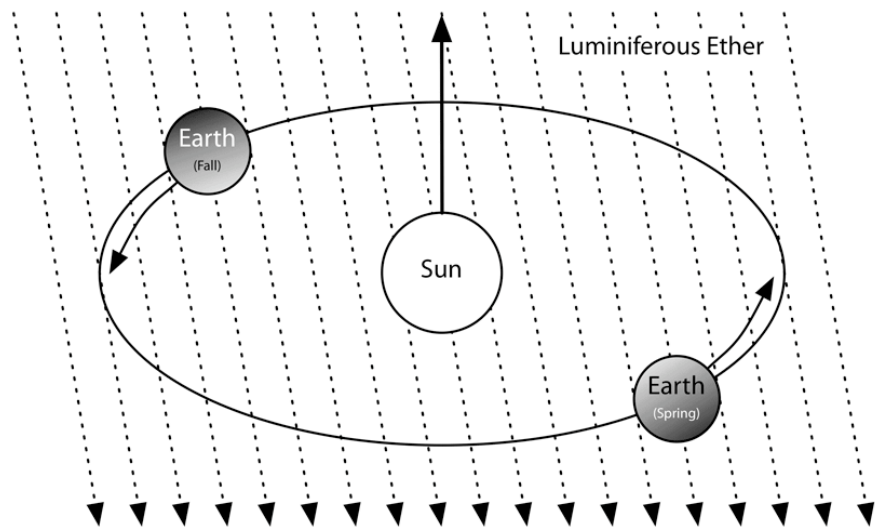
- What about light? Transmitted through “aether”? Speed of light from a star should vary due to Earth’s motion





Experimentation

- Earth moves at 30km/s around sun
- Measure speed of light from a star at different times of year
- Light should move at different speeds
- But light always moves at the same speed!



Einstein - postulates of special relativity

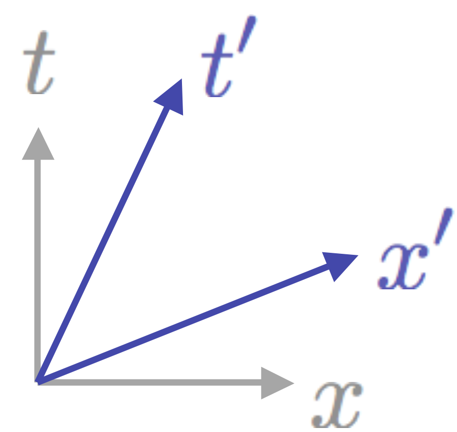
1. The laws of physics are the same in all inertial frames of reference
2. Light is always propagated in empty space with a definite velocity c

- Leads to Lorentz transformations

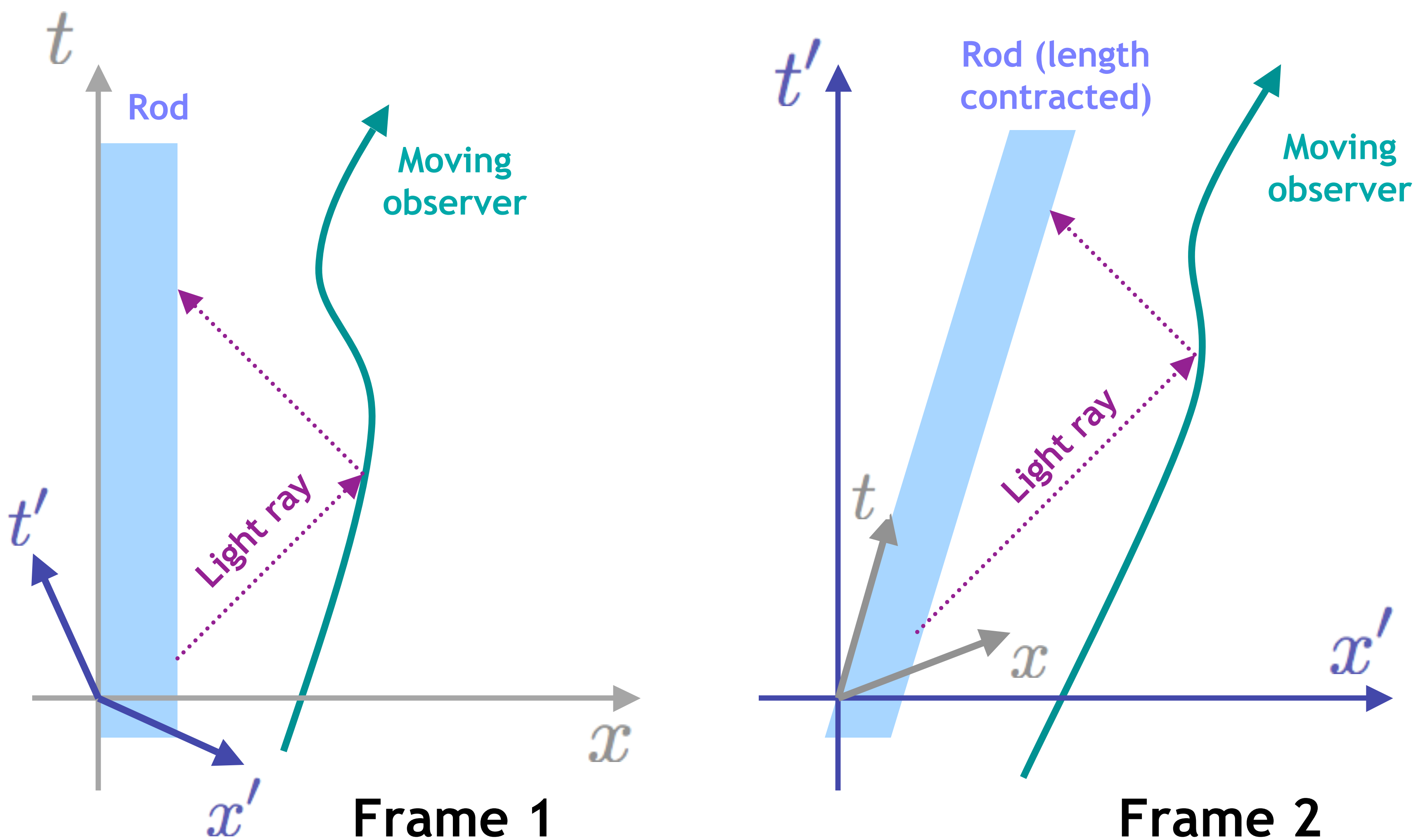
$$\begin{aligned} t' &= \gamma\left(t - \frac{vx}{c^2}\right) \\ x' &= \gamma(x - vt) \end{aligned}$$

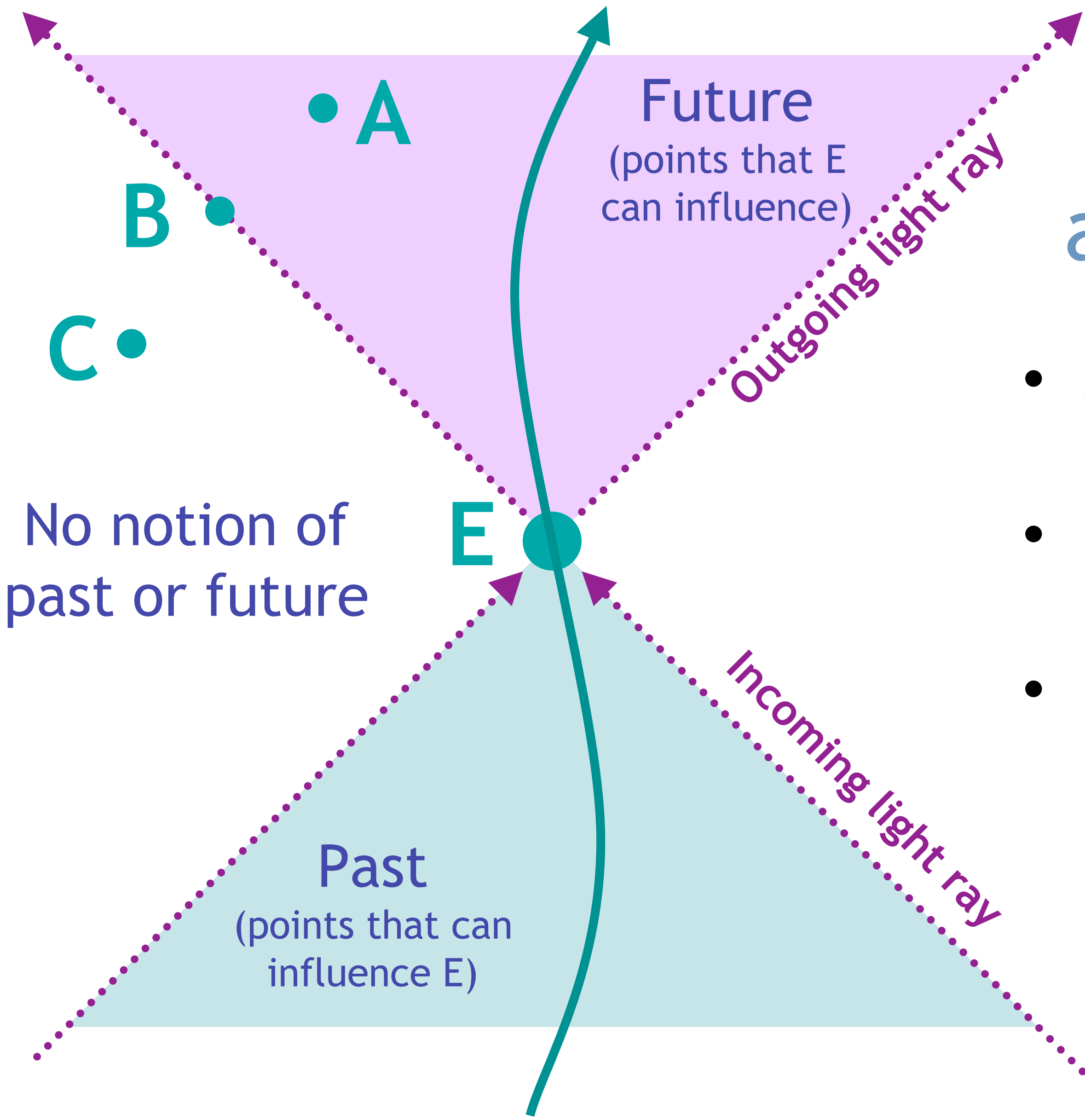
where

$$\gamma = (1 - v^2/c^2)^{-1/2}$$



Spacetime diagrams

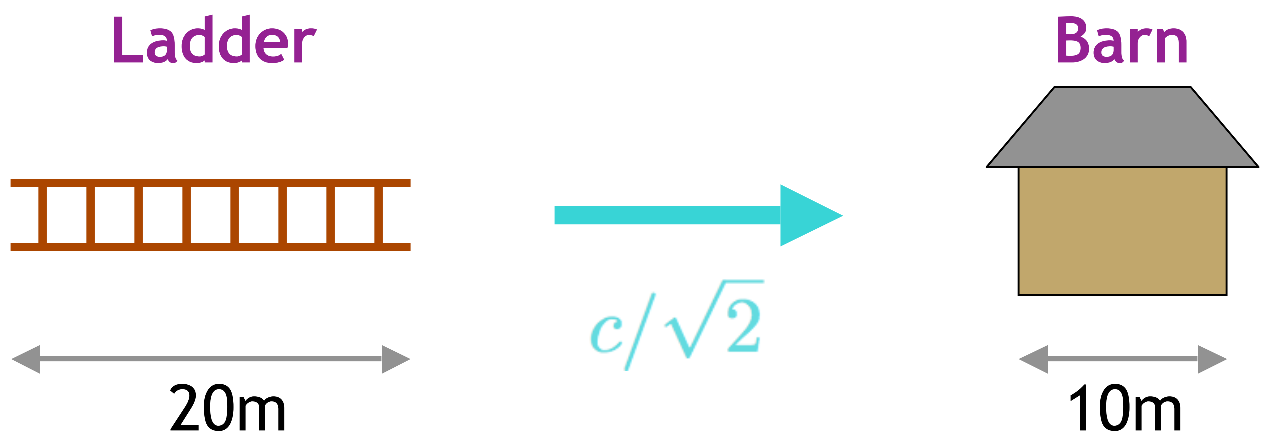




Light cones and causality

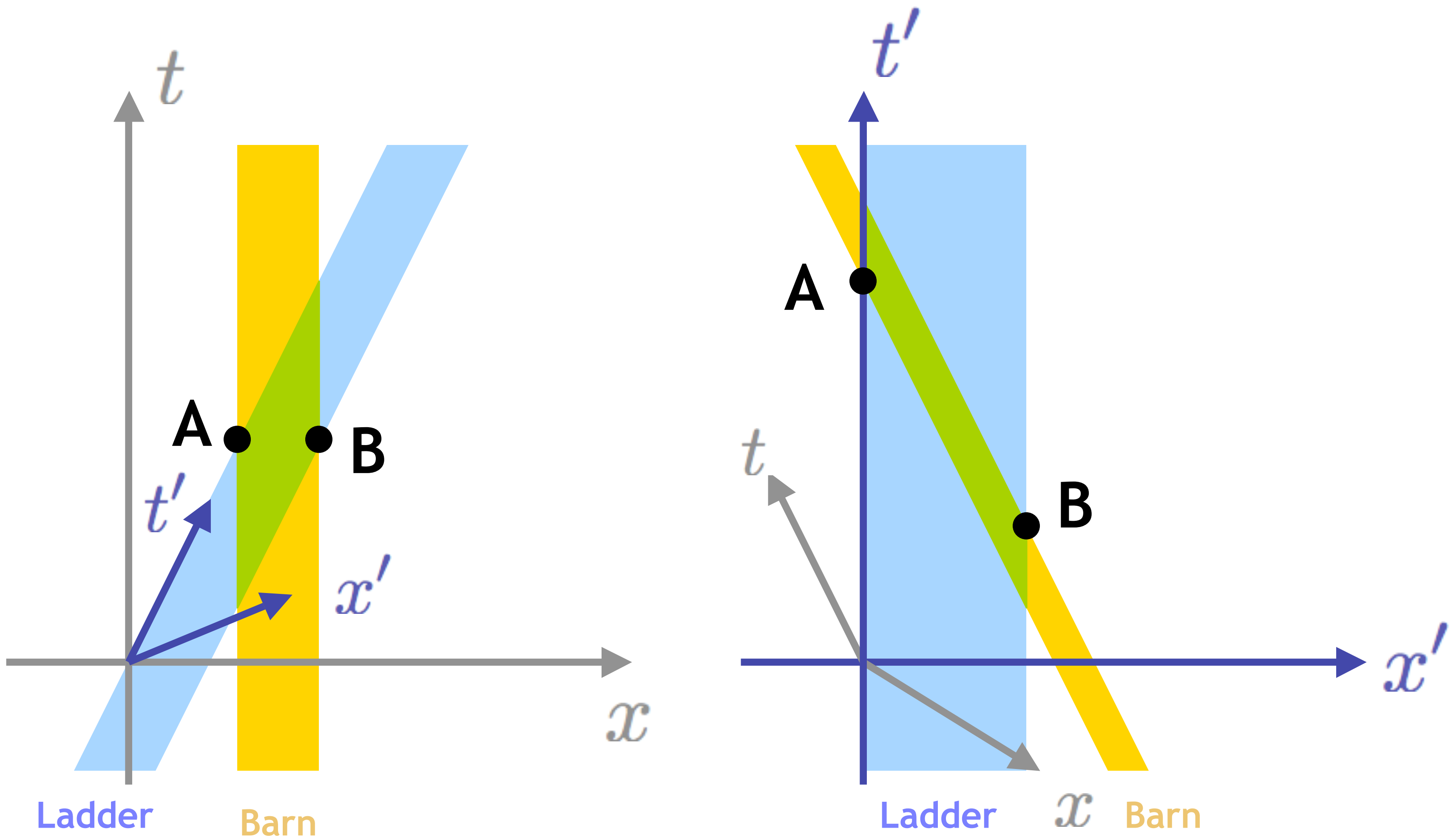
- A and E are **timelike** separated
- B and E are **lightlike** separated
- C and E are **spacelike** separated

Ladder paradox



- Length contraction factor 1/2
- In the rest frame, we see a 10m ladder fit into a 10m barn. Ladder fits into barn.
- In the running frame, we see a 20m ladder fitting into a 5m barn. Ladder does not fit into barn!

The ladder paradox explained

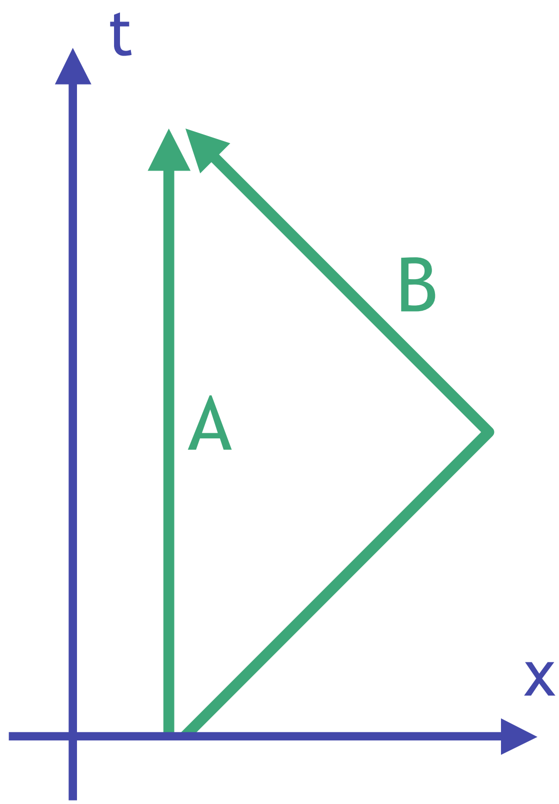


Time dilation and the twin paradox

- Proper time: time measured by a clock traveling with an observer
- For any path

$$d\tau^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2$$

Twins A and B are aged 20. Twin A stays on Earth. Twin B travels at 0.9c for ten years away, and then 0.9c for ten years back. When they meet, A is 40 and B is 28.



A metric and four-vectors

- Events can be labeled by four vectors

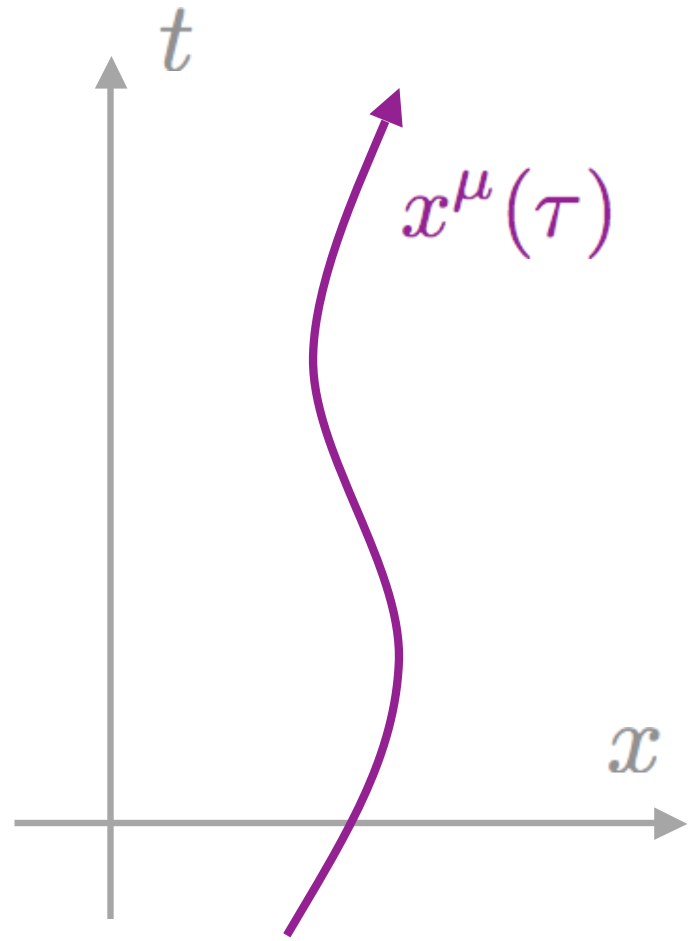
$$x^\mu = (t, x, y, z)$$

- World line of an observer given by $x^\mu(\tau)$
- Metric

$$\eta_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

- Contravariant vector

$$x_\nu = \eta_{\mu\nu} x^\mu = (ct, -x, -y, -z)$$



Momentum and energy

- Four-velocity

$$v^\mu = \frac{dx^\mu}{d\tau} \qquad\qquad v^\mu v_\mu = \eta_{\mu\nu} x^\mu x^\nu = 1$$

- Four-momentum

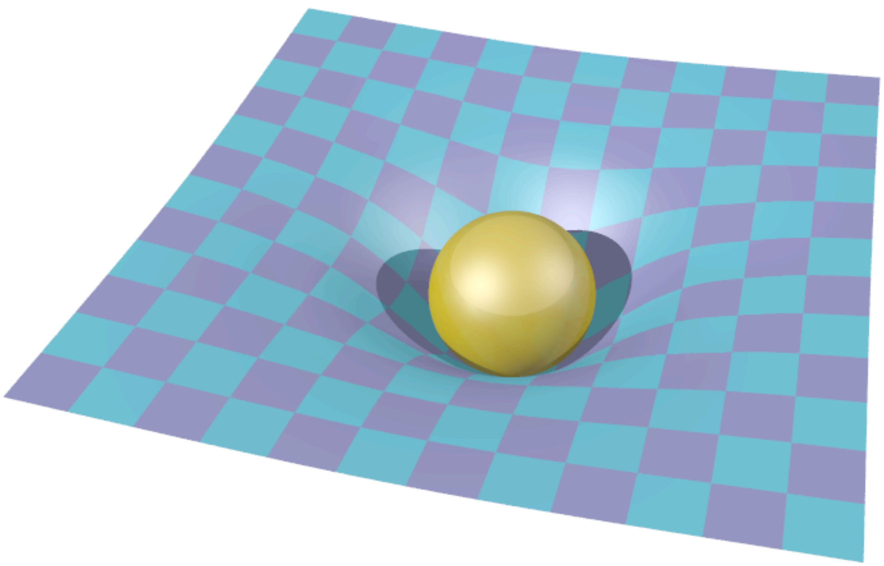
$$p^\mu = m_0 v^\mu = (E, cp_x, cp_y, cp_z)$$
$$p^\mu p_\mu = E^2 - (pc)^2$$

- Since this is invariant, consider it in the frame where the velocity is zero. Then

$$v^\mu = (c^2, 0, 0, 0)$$
$$E = m_0 c^2$$

But what about gravity?

- Special relativity works in flat space
- Particles travel along straight lines (geodesics)
- The presence of mass and energy bends space
- Can be encapsulated by replacing $\eta_{\mu\nu}$ with a general metric $g_{\mu\nu}$
- Locally $g_{\mu\nu}$ becomes $\eta_{\mu\nu}$



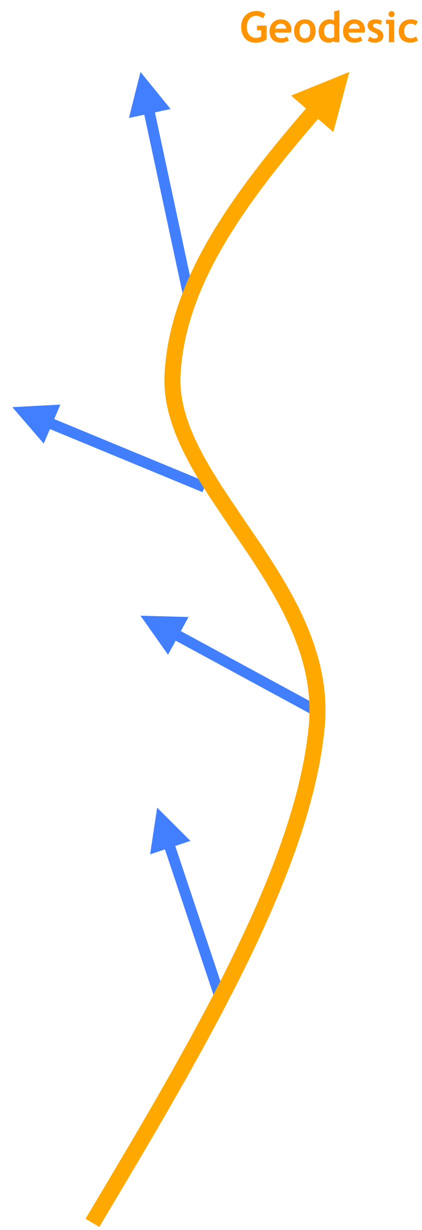
Covariant derivative

- Usual derivative ∂ is not a tensor
- Parallel transport - move a vector along a geodesic preserving angles/length
- Mathematically

$$\nabla_\beta v^\alpha = \partial_\beta v^\alpha + \Gamma^\alpha_{\gamma\beta} v^\gamma$$

- Christoffel symbols

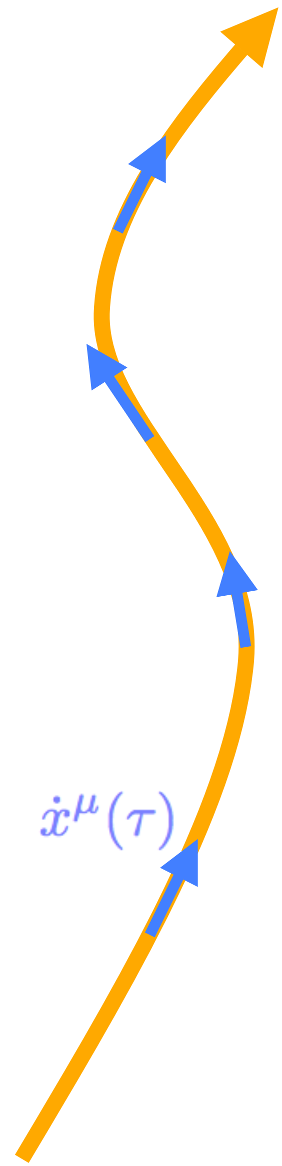
$$\Gamma^\alpha_{\gamma\beta} = \frac{1}{2} g^{\alpha\delta} (\partial_\beta g_{\delta\gamma} + \partial_\gamma g_{\delta\beta} - \partial_\delta g_{\gamma\beta})$$



Geodesic equation

- Consider a line $x^\mu(\tau)$
- For a geodesic, $\dot{x}^\mu(\tau)$ is parallely transported along the line
- Thus we get the geodesic equation

$$\begin{aligned} 0 &= (\dot{x}^\mu \nabla_\mu)\dot{x}^\nu \\ &= \dot{x}^\mu \partial_\mu \dot{x}^\nu + \dot{x}^\mu \Gamma^\nu_{\alpha\mu} \dot{x}^\alpha \\ &= \ddot{x}^\nu + \Gamma^\nu_{\alpha\mu} \dot{x}^\alpha \dot{x}^\mu \end{aligned}$$

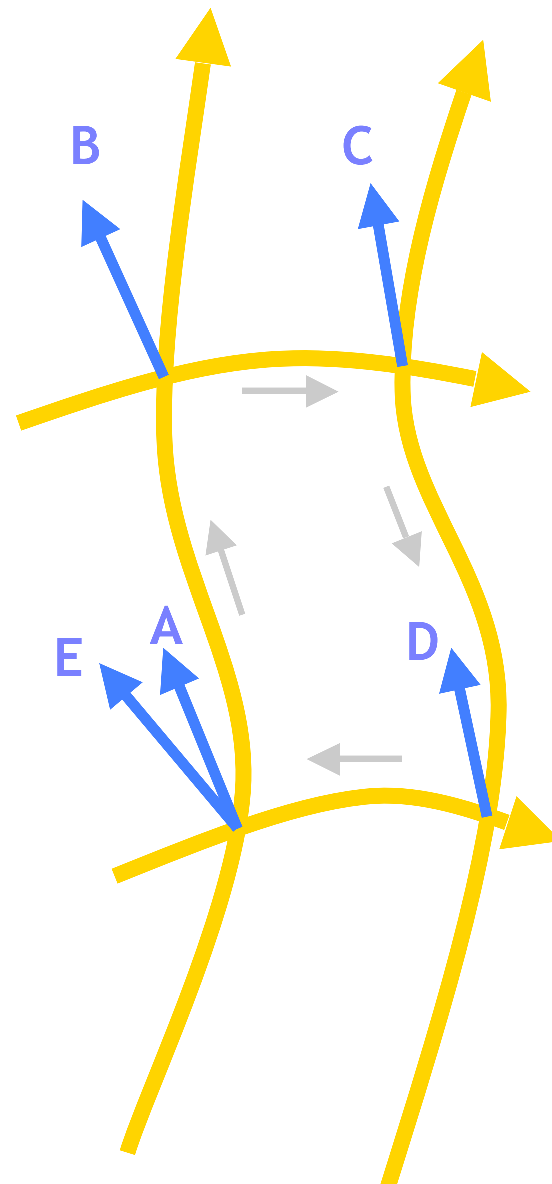


Riemann Curvature Tensor

- Captures the amount of deviation by parallely transporting a vector around a square of geodesics

$$R^{\alpha}{}_{\beta\gamma\delta} = \partial_\gamma \Gamma^{\alpha}_{\beta\delta} - \partial_\delta \Gamma^{\alpha}_{\beta\gamma} + \Gamma^{\alpha}_{\mu\gamma}\Gamma^{\mu}_{\beta\delta} - \Gamma^{\alpha}_{\mu\delta}\Gamma^{\mu}_{\beta\gamma}$$

- Derived quantities
  - Ricci tensor $R_{\mu\nu} = R^{\alpha}{}_{\mu\alpha\nu}$
  - Ricci scalar $R = g^{\mu\nu} R_{\mu\nu}$
  - Einstein tensor $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$



Einstein Field Equation

$$G_{\mu\nu} = 8\pi T_{\mu\nu}$$

Curvature

(Einstein tensor)

Mass

(Stress-Energy tensor)

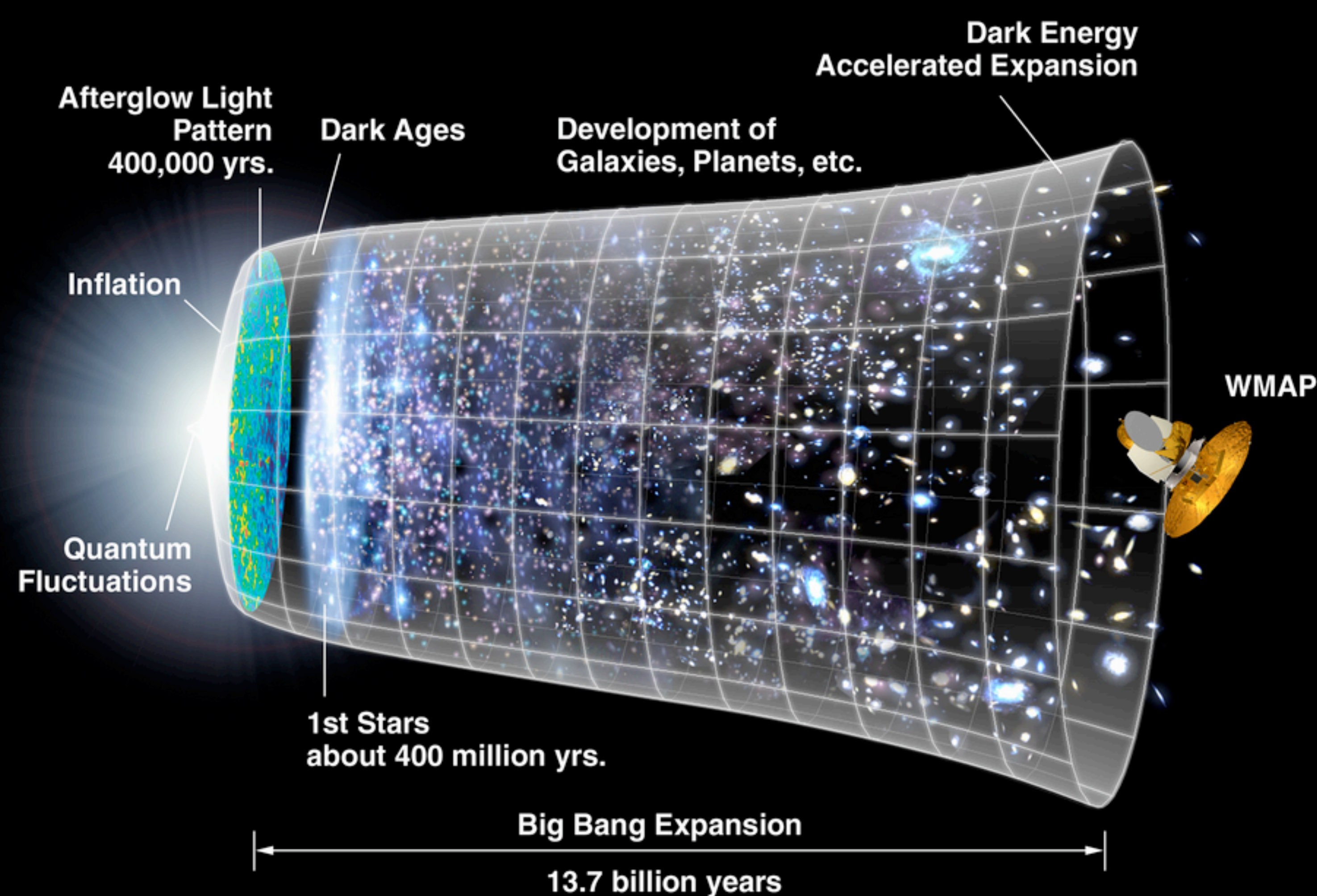
"Spacetime grips mass, telling it how to move and mass grips spacetime, telling it how to curve" – J. Wheeler

Einstein Field Equation

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi T_{\mu\nu}$$

Cosmological constant

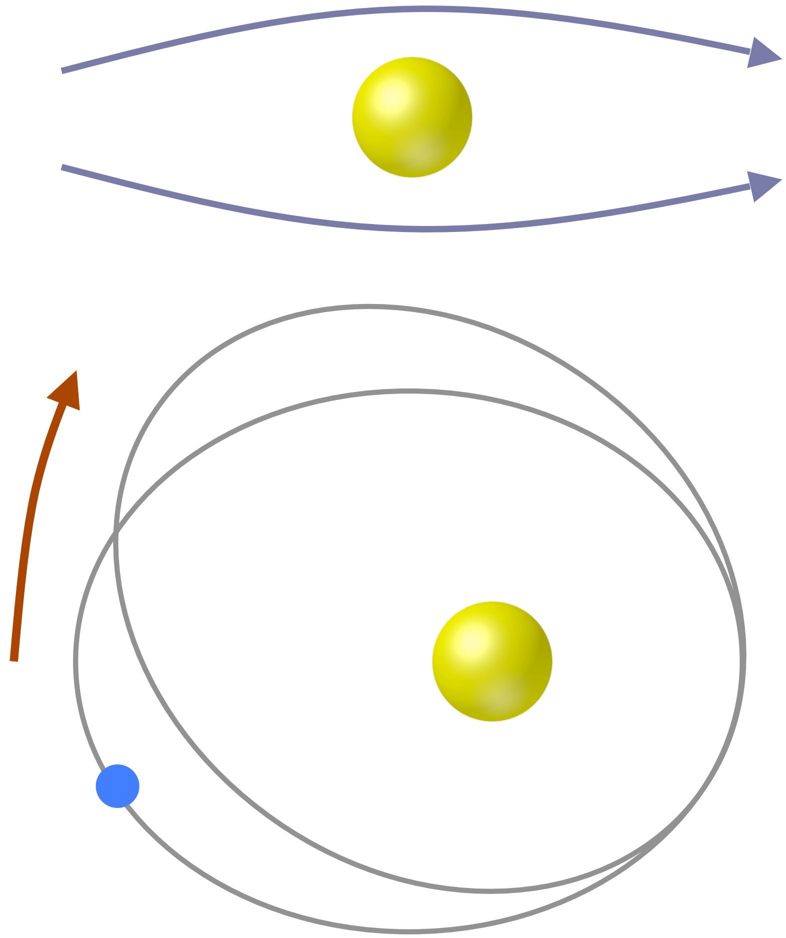
- Consistent with equations and proposed by Einstein to create steady universe
- Big bang theories suggest it is zero. Einstein called it “his biggest blunder”
- Modern theories suggest it might actually be true



Astronomy Picture of the Day, March 23rd 2006
http://antwrp.gsfc.nasa.gov/apod/astropix.html

NASA/WMAP Science

Measurable implications



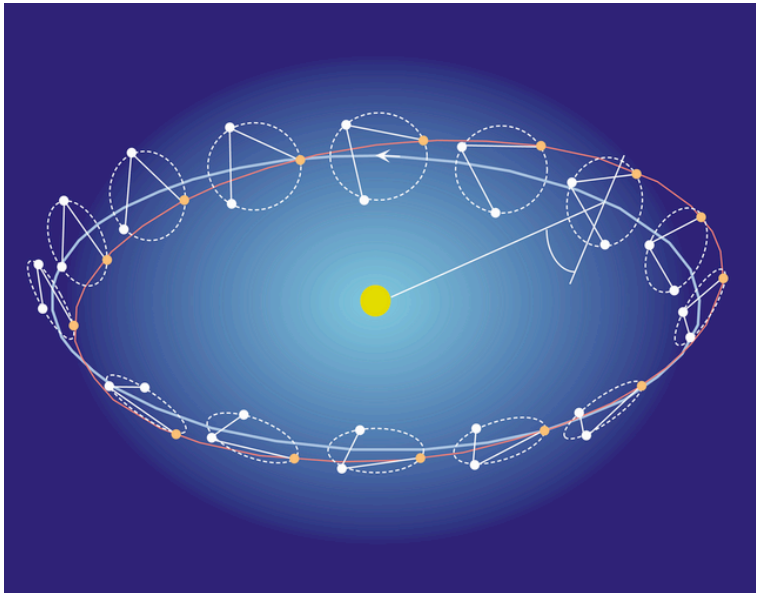
- Bending of starlight around the Sun
- Corrections to planetary orbits
- Gravity waves

$$g_{\mu\nu} = \eta_{\mu\nu} + \epsilon h_{\mu\nu}$$

- The perturbations obey a wave equation

Detecting gravity waves

- LIGO - Laser Interferometer Gravitational-Wave Observatory
- Detect length changes of one part in 10^{21}
- LISA - Laser Interferometer Space Antenna



Penrose diagrams: a flat universe

- Change variables

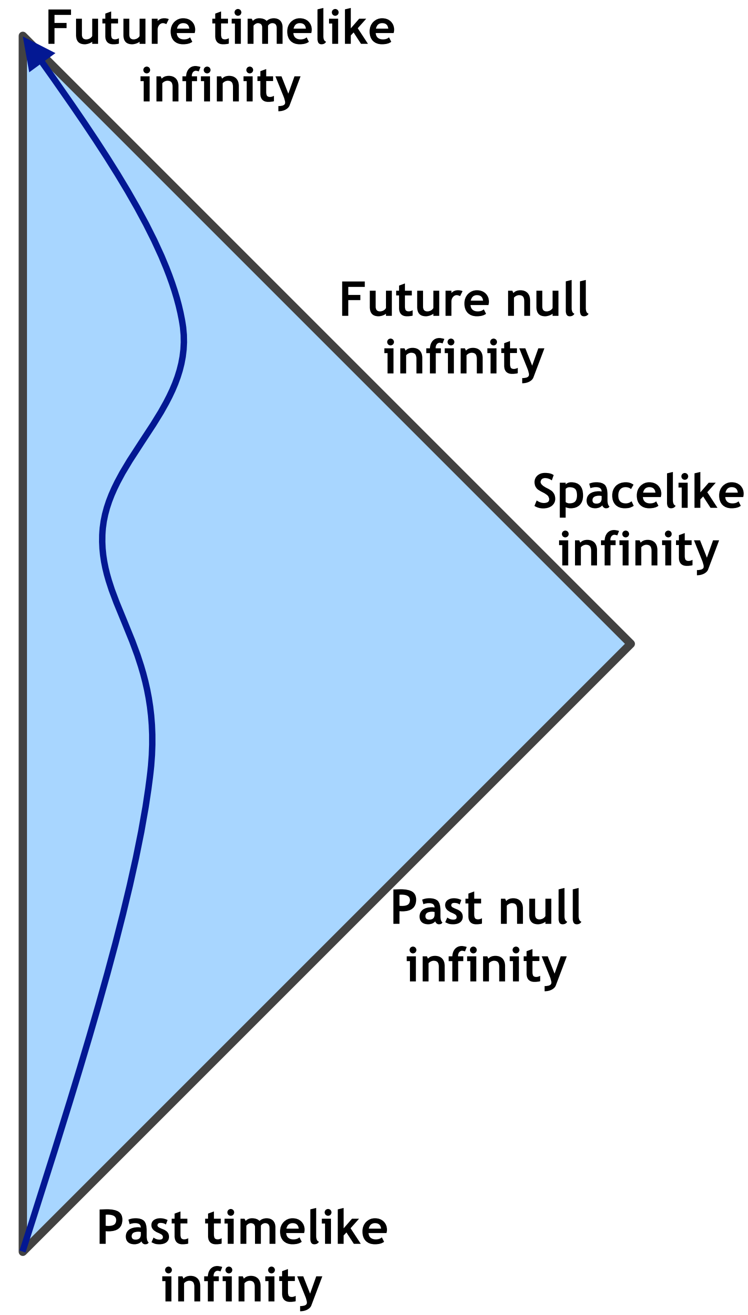
$$2ct = \tan\frac{t'+r'}{2} + \tan\frac{t'-r'}{2}$$
$$2r = \tan\frac{t'+r'}{2} - \tan\frac{t'-r'}{2}$$

- Coordinates have ranges

$$r' > 0, |r' \pm t'| < \pi$$

- Metric becomes

$$ds^2 = \frac{dt'^2 - dr'^2}{4\cos^2\frac{t'+r'}{2}\cos^2\frac{t'-r'}{2}}$$

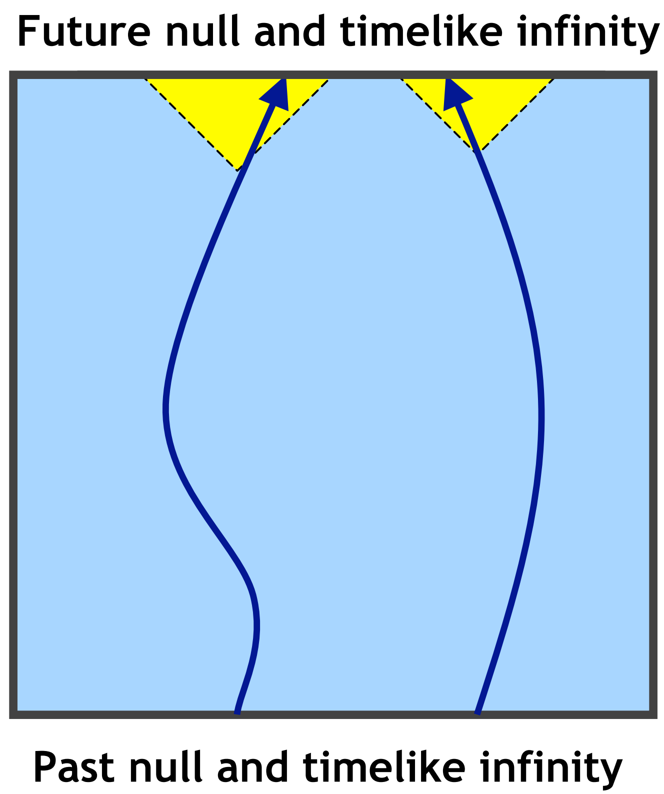


de Sitter space

- Empty space with a positive cosmological constant

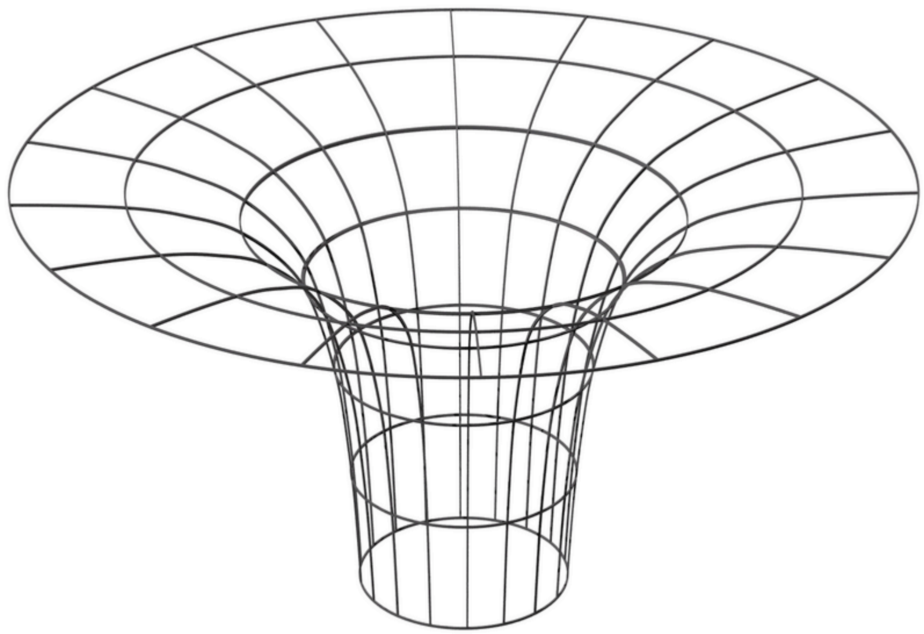
$$ds^2 = dt^2 - \alpha^2 \cosh^2(t/\alpha) d\Omega_{n-1}^2$$

- Distances between observers increase
- In two dimensions, Penrose diagram is a square
- Areas can no longer be reached

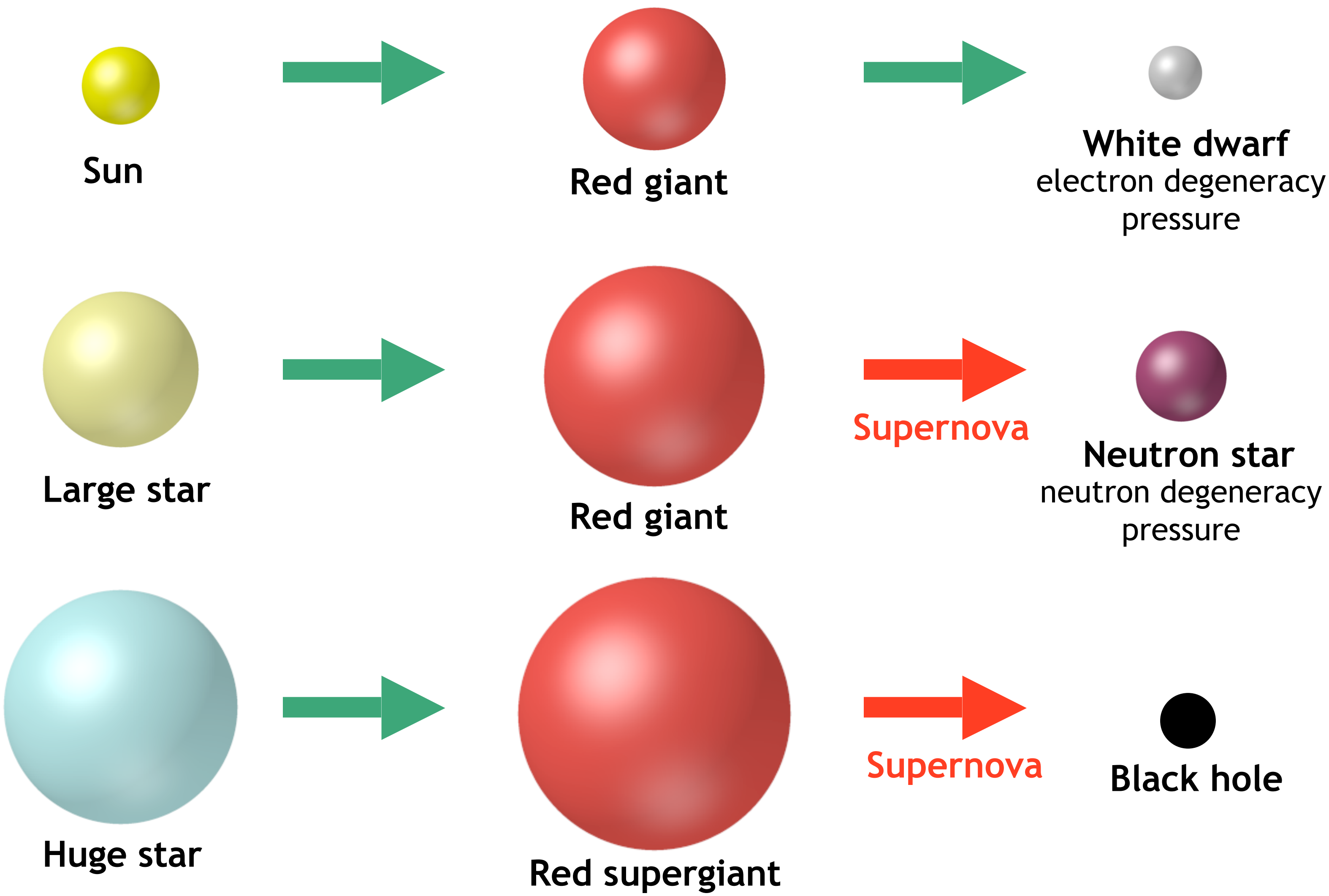


Black holes

- John Michell (1783): gravitational escape velocity for a sufficiently heavy body would exceed the speed of light
- Allowed by general relativity
- Schwarzschild radius is given by $r_s = 2GM/c^2$
  - 9mm for Earth
  - 3km for Sun

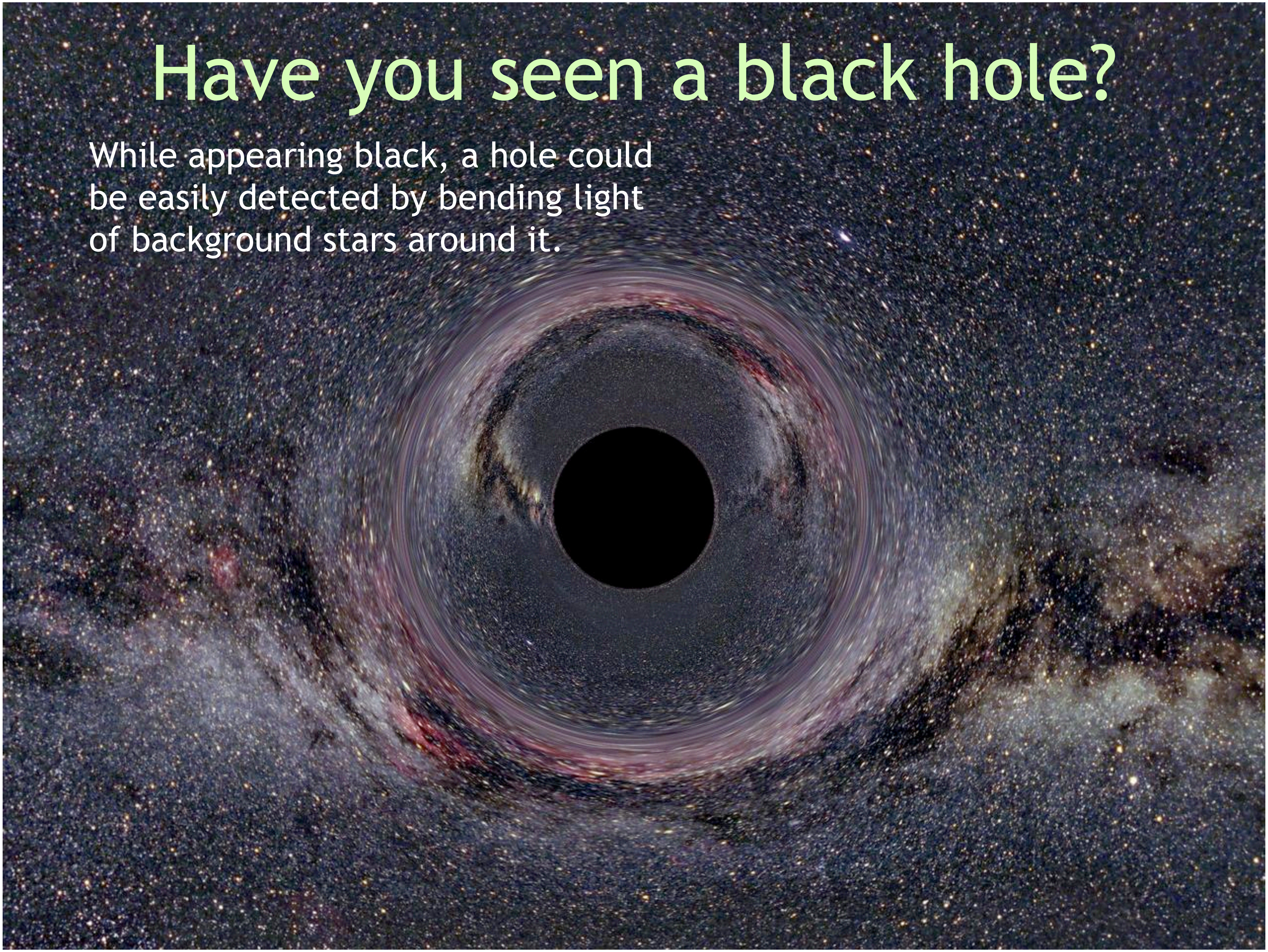


Stellar evolution

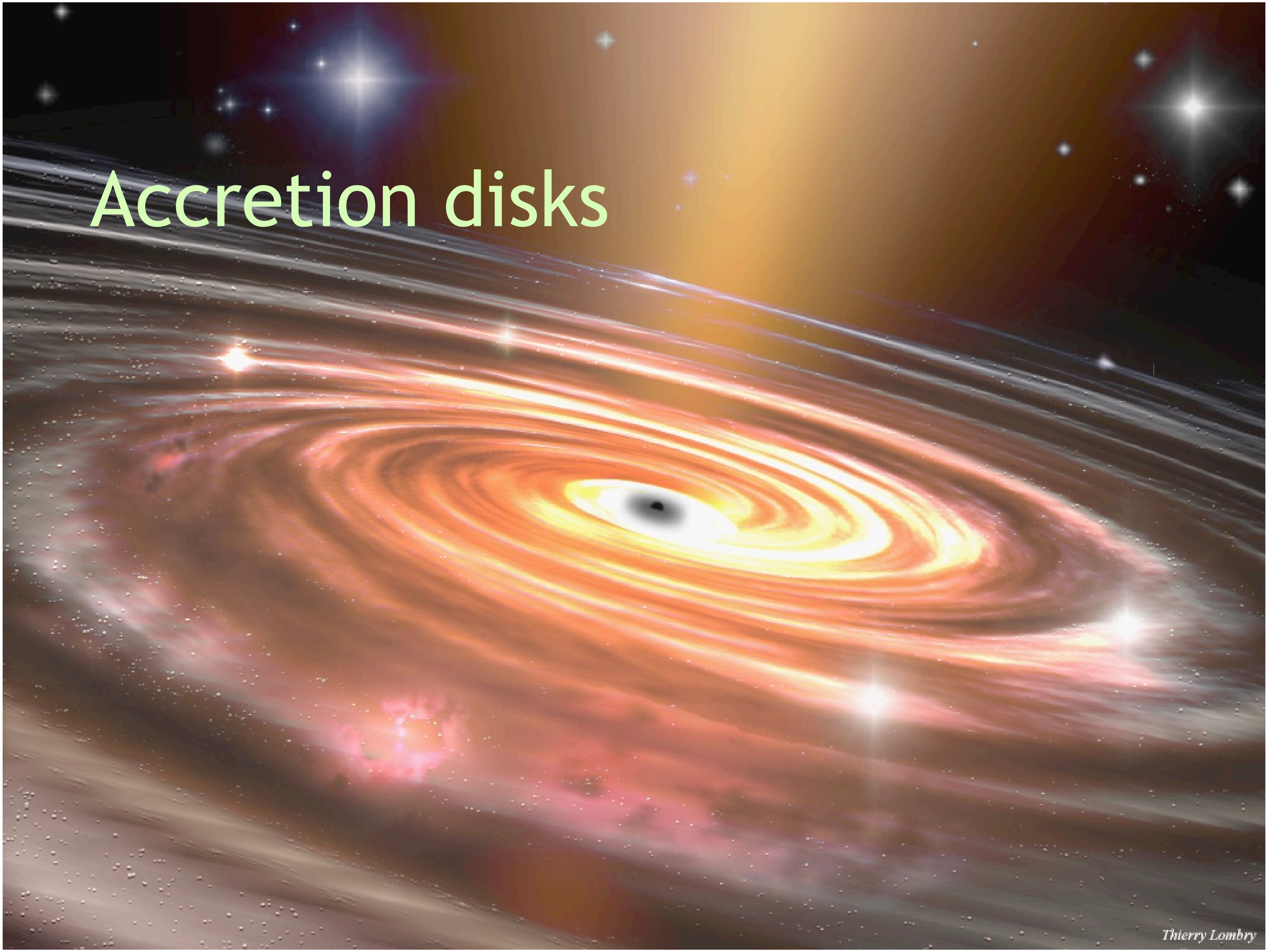


Have you seen a black hole?

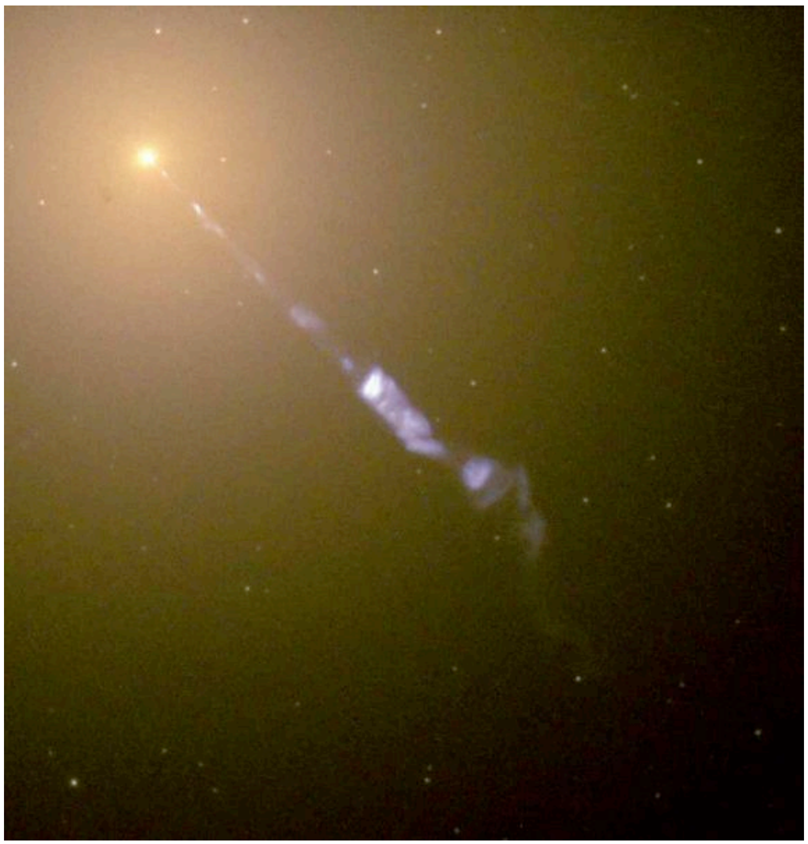
While appearing black, a hole could be easily detected by bending light of background stars around it.



Accretion disks



Supermassive black holes

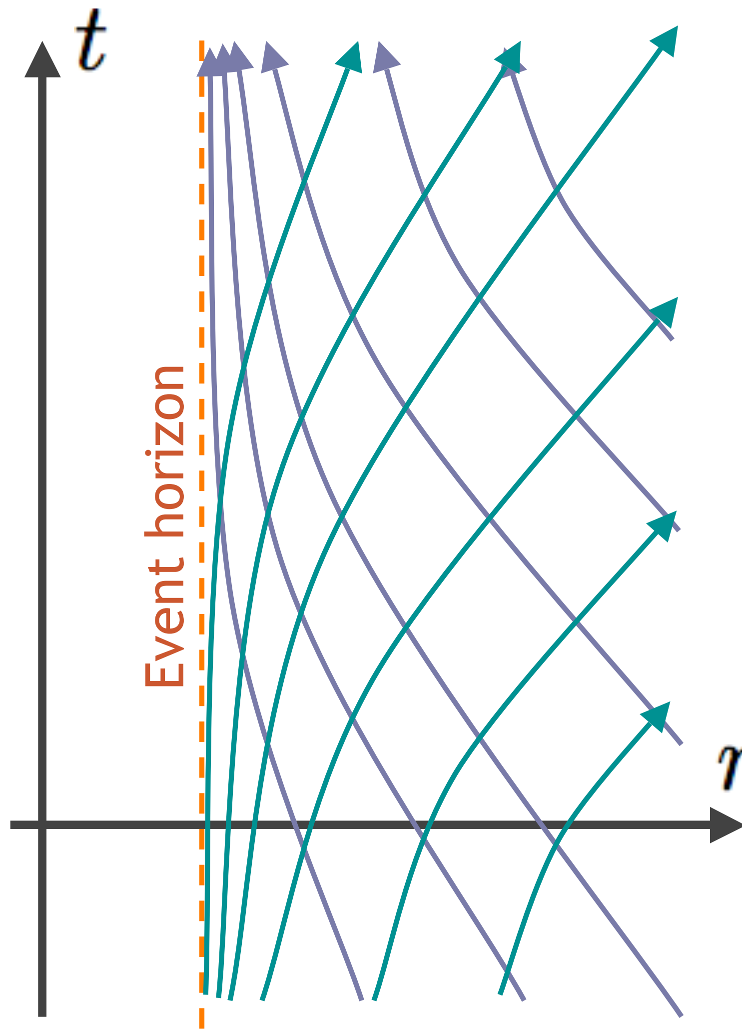


- Mounting evidence of supermassive black holes at the centers of galaxies
- Hard to see in our own galaxy, since obscured by dust
- Nearby galaxy M87 projects enormous filament of subatomic particles from a black hole, perhaps with two billion times the Sun’s mass

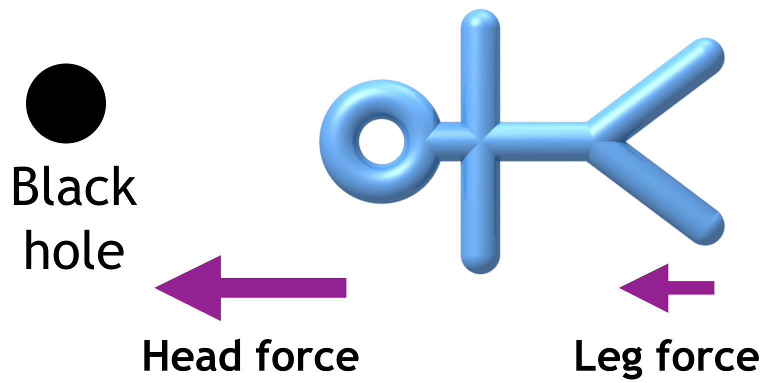
Schwarzschild metric

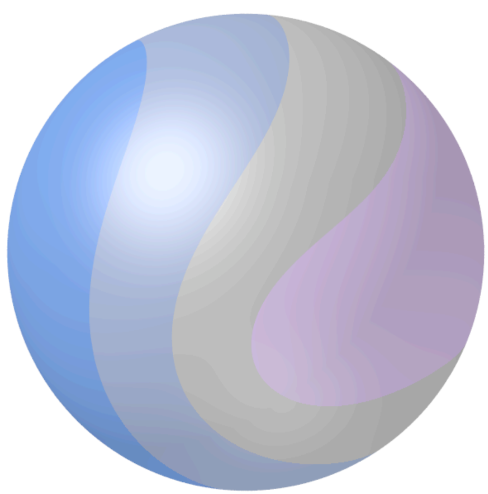
$$ds^2 = c^2\left(1-\frac{2GM}{c^2r}\right)dt^2 - \left(1-\frac{2GM}{c^2r}\right)^{-1}dr^2$$
$$-r^2(d\theta^2+\sin^2\theta\,d\phi^2)$$

- Becomes the standard metric for large r
- Event horizon at the Schwarzschild radius
- Infalling particles have $t \to \infty$ as they approach the event horizon



What is inside?





- Tidal forces make it hard (but not impossible) to get inside
- An infalling observer reaches the event horizon in finite proper time but infinite t
- Extend our coordinate system to examine the interior

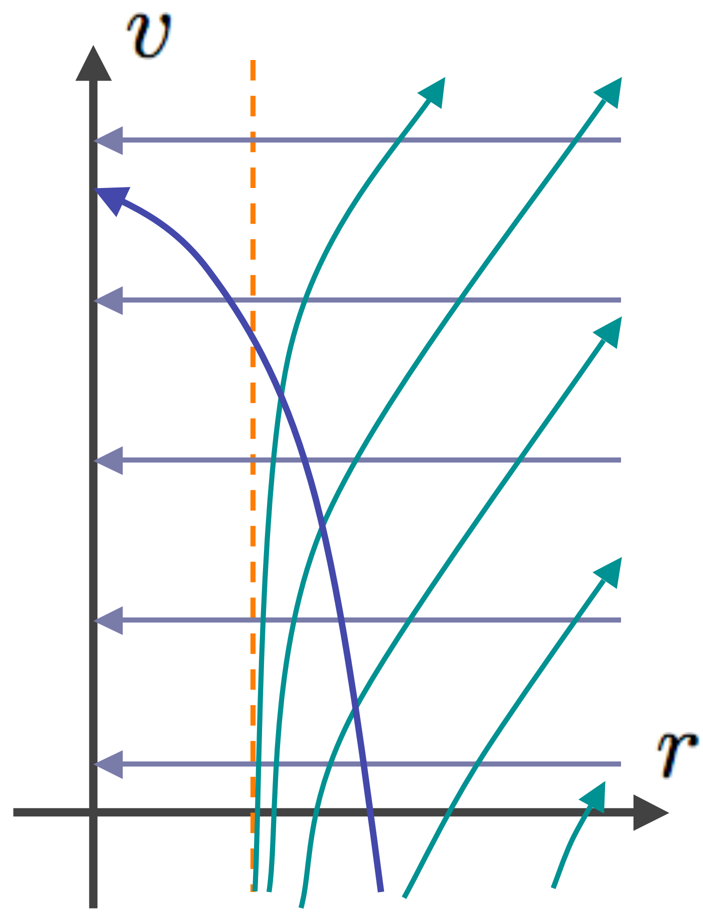
Eddington-Finkelstein coordinates

- Introduce new coordinate along radially infalling null geodesics

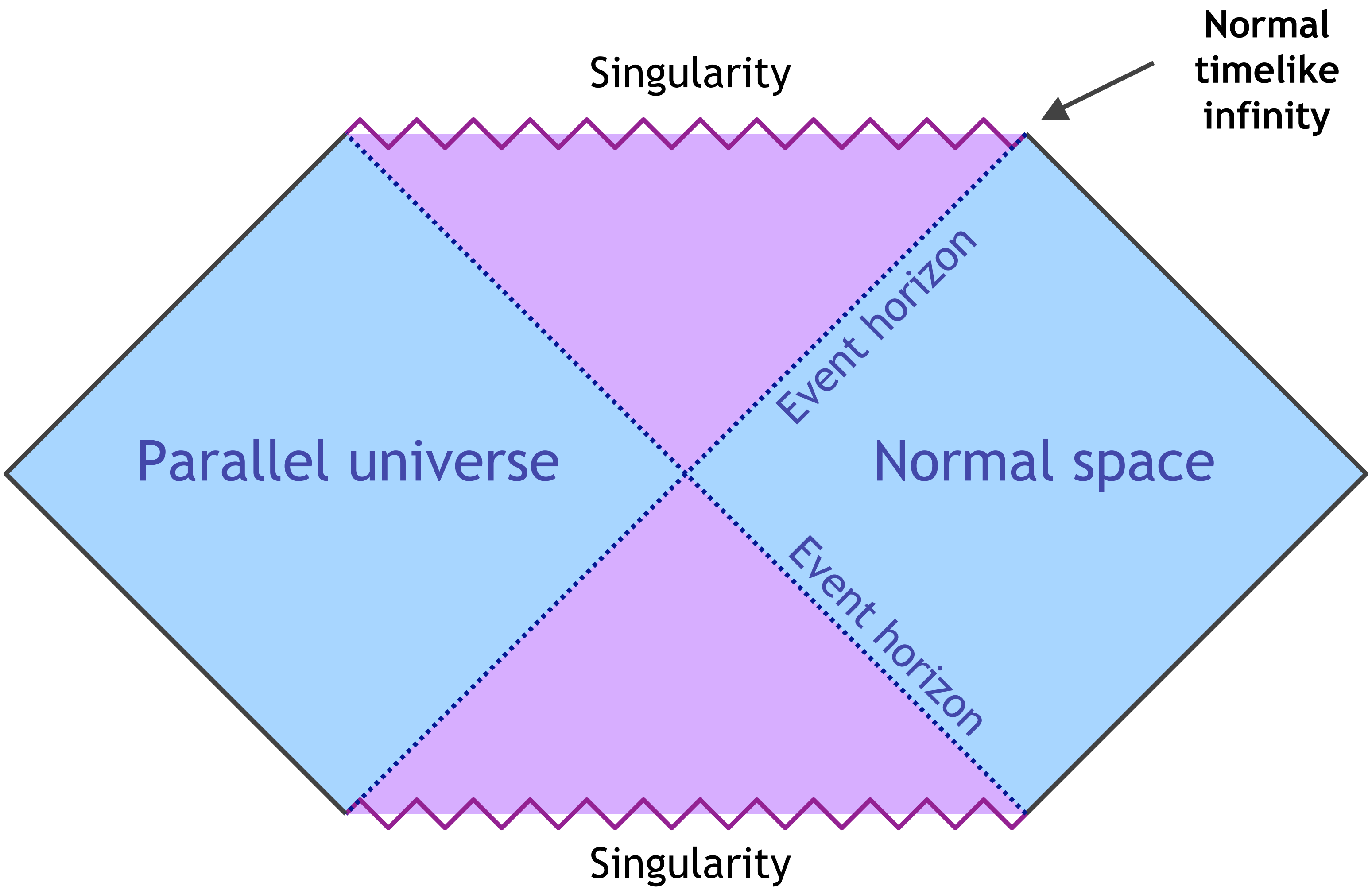
$$v = ct + r + \frac{GM}{c^2}\log\left|\frac{rc^2}{2GM} - 1\right|$$

- Metric becomes

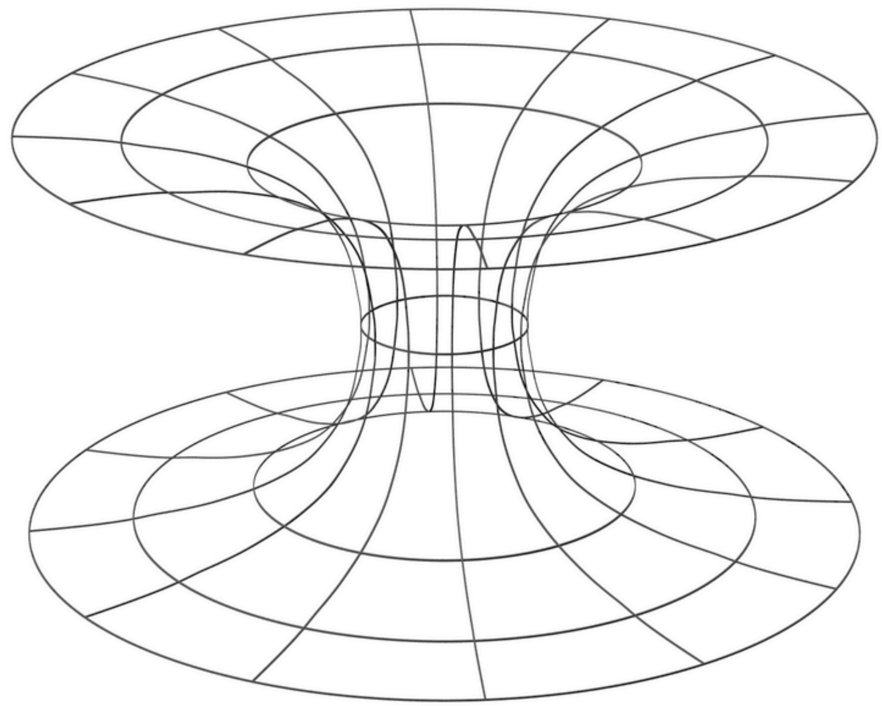
$$ds^2 = \left(1 - \frac{2GM}{r^2 c}\right)dv^2 - 2dv\,dr - r^2 d\Omega^2$$



Penrose diagram



Spacelike wormhole



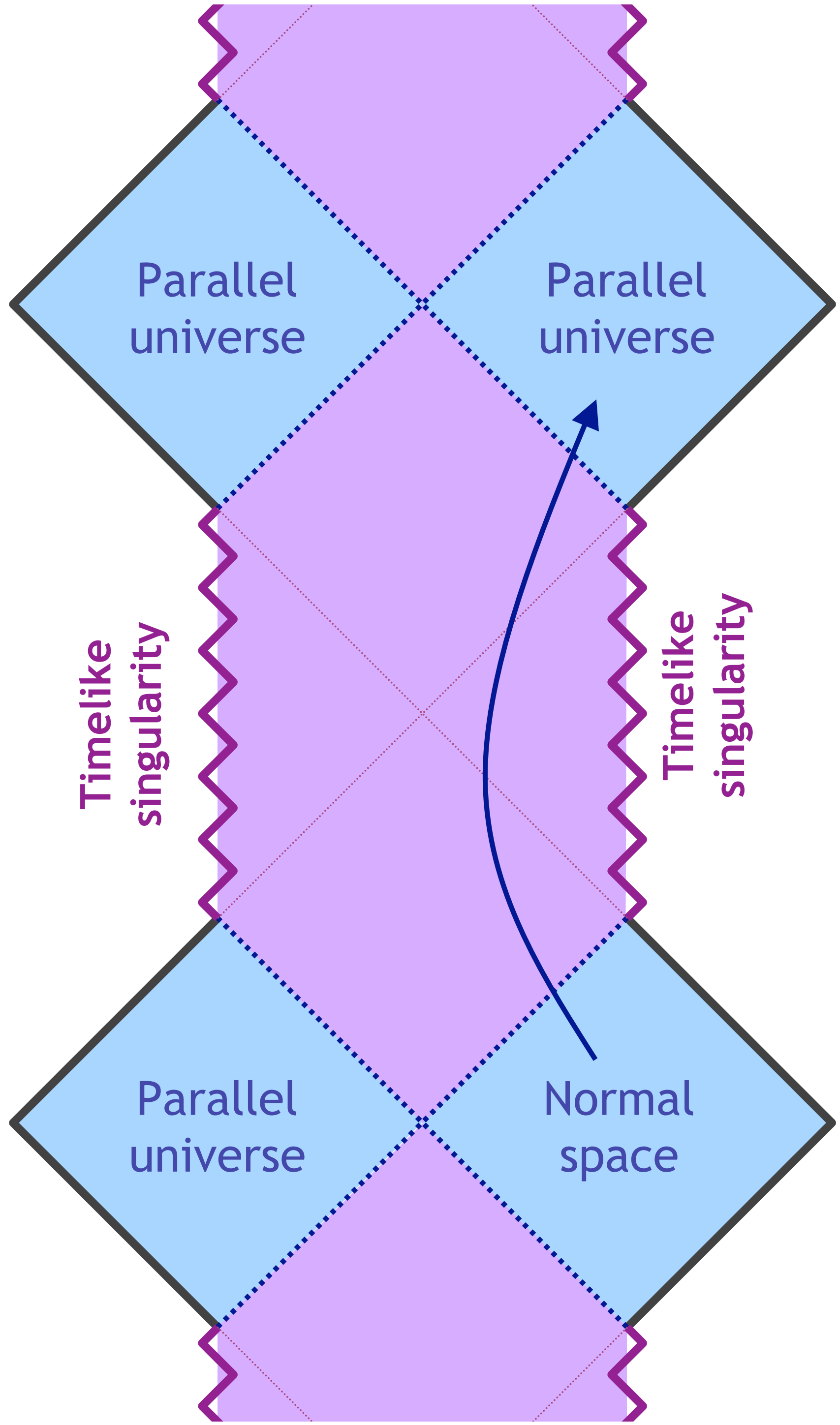
- Connects two areas of space
- Only spacelike trajectories between the two sides
- Is it possible to create wormholes for timelike paths?

Charged black hole

- Charge is conserved: throwing charge into a black hole makes it charged
- Reissner-Nordstrom metric

$$\begin{aligned} ds^2 &= \left(1-\frac{2M}{r}+\frac{Q^2}{r^2}\right)dt^2 \\ &\quad -\left(1-\frac{2M}{r}+\frac{Q^2}{r^2}\right)dr^2 \\ &\quad -r^2(d\theta^2+\sin^2\theta d\phi^2) \end{aligned}$$

- Singularity becomes timelike



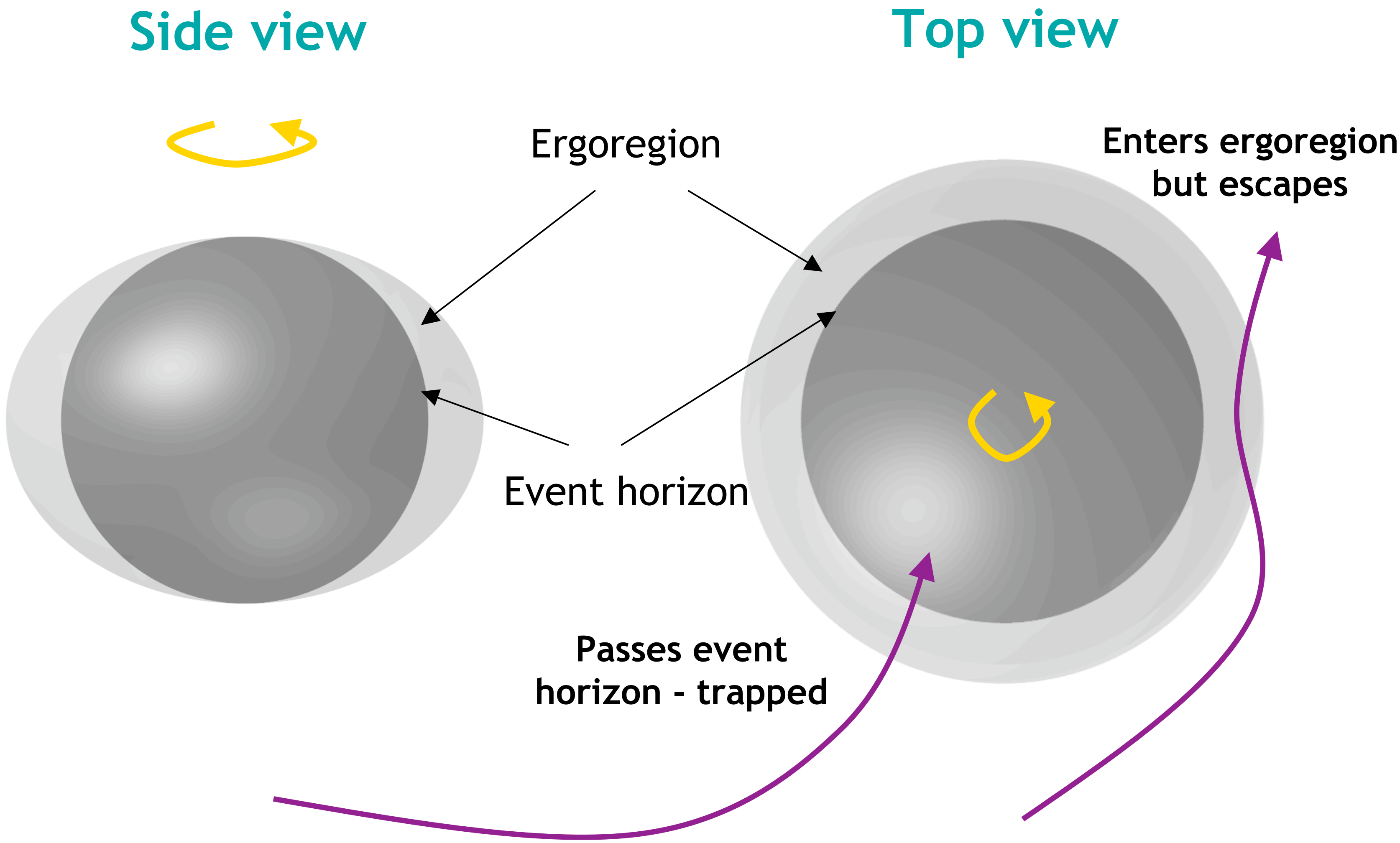
A rotating hole - the Kerr metric

- Exact solution for rotating black hole, discovered in 1963

$$\begin{aligned} ds^2 \quad = \quad & dt^2 \left(1 - \frac{2Mr}{r^2 + a^2\cos^2\theta}\right) + \frac{4aMr\sin^2\theta}{r^2 + a^2\cos^2\theta} dt\, d\phi - \\ & \frac{r^2 + a^2\cos^2\theta}{r^2 - 2Mr + a^2} dr^2 - (r^2 + a^2\cos^2\theta) d\theta^2 \\ & - \left(r^2 + a^2 + \frac{2a^2Mr\sin^2\theta}{r^2 + a^2\cos^2\theta}\right) \sin^2\theta\, d\phi^2 \end{aligned}$$

- Becomes the standard metric for large r
- The $dr\, d\phi$ term corresponds to a “twist” of spacetime, creating *frame-dragging*

Rotating black hole geometry

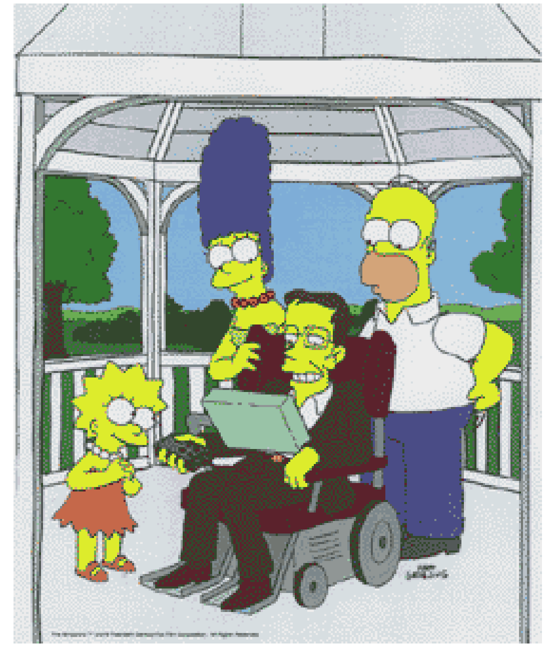


Further results

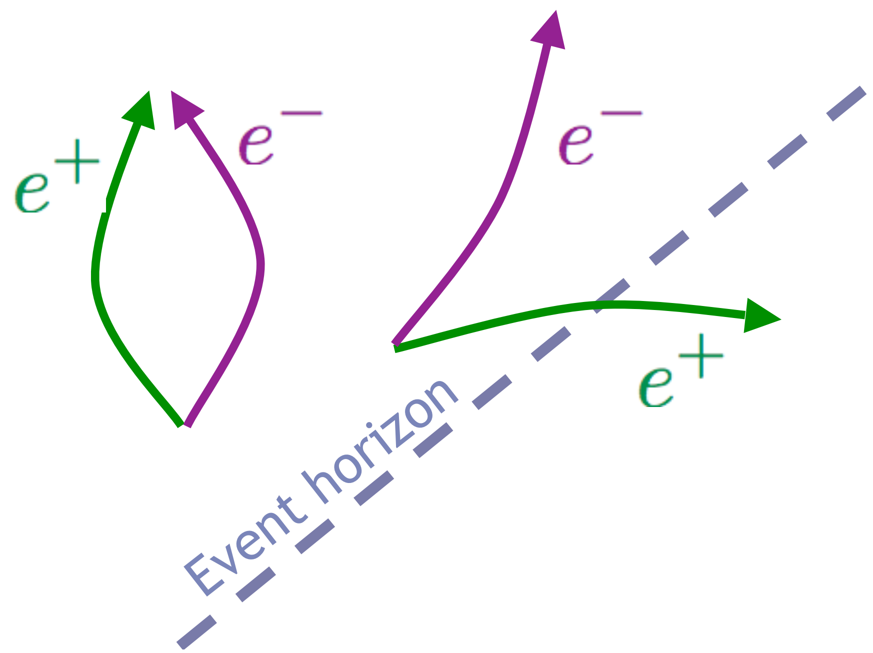
- Black holes can merge but can’t bifurcate
- Black holes have no hair: uniquely characterized by mass, charge, and angular momentum
- Thermodynamic analogues
- Quantum effects…

Black holes ain't so black

"Your theory of a doughnut-shaped universe is interesting ... I may have to steal it." – Stephen Hawking



- Quantum effects near the event horizon cause the black hole to slowly emit a small amount of radiation
- Black holes may "evaporate"
- ***Really*** slow - a black hole with the Sun's mass takes 10^{67} years to evaporate



Conclusions