The Life of π

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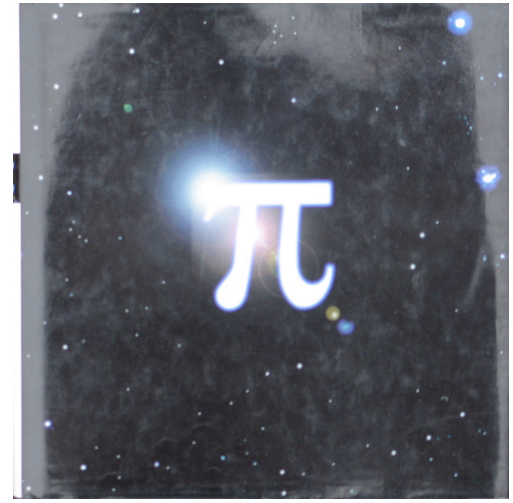
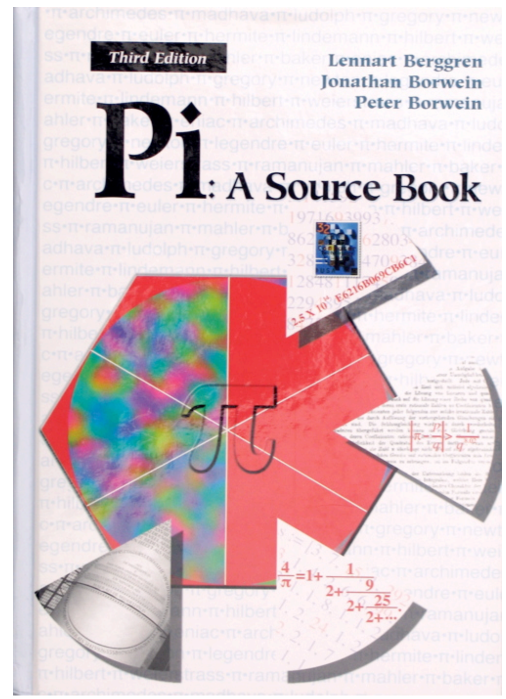
SPAMS, Fall 2005

References

- *“The Joy of Pi”*, Blatner, Penguin 1997
- *“Pi: A Source Book”*, Berggren, Borwein, and Borwein, Springer 2004
- *“A History of Pi”*, Beckmann, Golem 1970
- *“Pi: A Biography of the World’s Most Mysterious Number”*, Posamentier and Lehmann, Prometheus 2004

- http://www.joyofpi.com
- http://www-groups.dcs.st-and.ac.uk/~history/

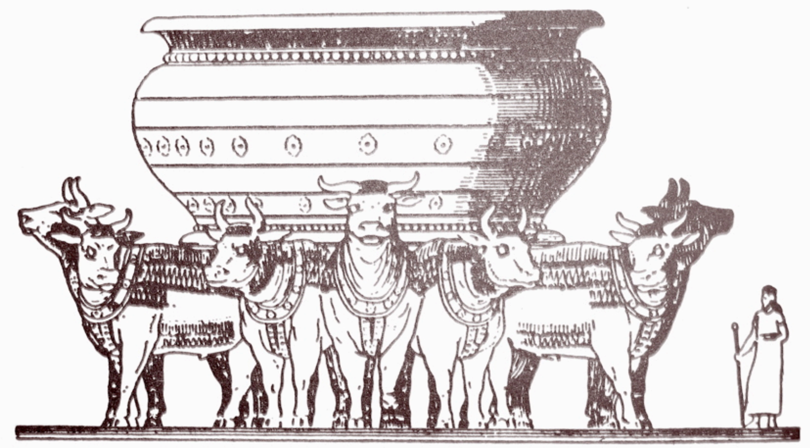




Ancient values for π

- Babylonians: π=3 1/8

- *"And he made the molten sea of ten cubits from brim to brim, round in compass, and the height thereof was five cubits; and a line of thirty cubits did compass it round about" – 1 Kings 7:23*



The molten sea, as reconstructed by Gressmann from the description in 2 Kings vii.[3]

- Implies π=3.

Rhind Papyrus

- Purchased by Henry Rhind in 1858, in Luxor, Egypt
- Scribed in 1650BC, and copied from an earlier work from ~2000BC
- One of the oldest mathematical texts in existence
- Gives a value for π

Problem number 24

"A heap and its 1/7 part become 19. What is the heap?"

$$x + \frac{x}{7} = 19$$

Then 1 heap is 7.

And 1/7 of the heap is 1.

Making a total of 8.

But this is not the right answer, and therefore we must rescale 7 by the proportion of 19/8 to give

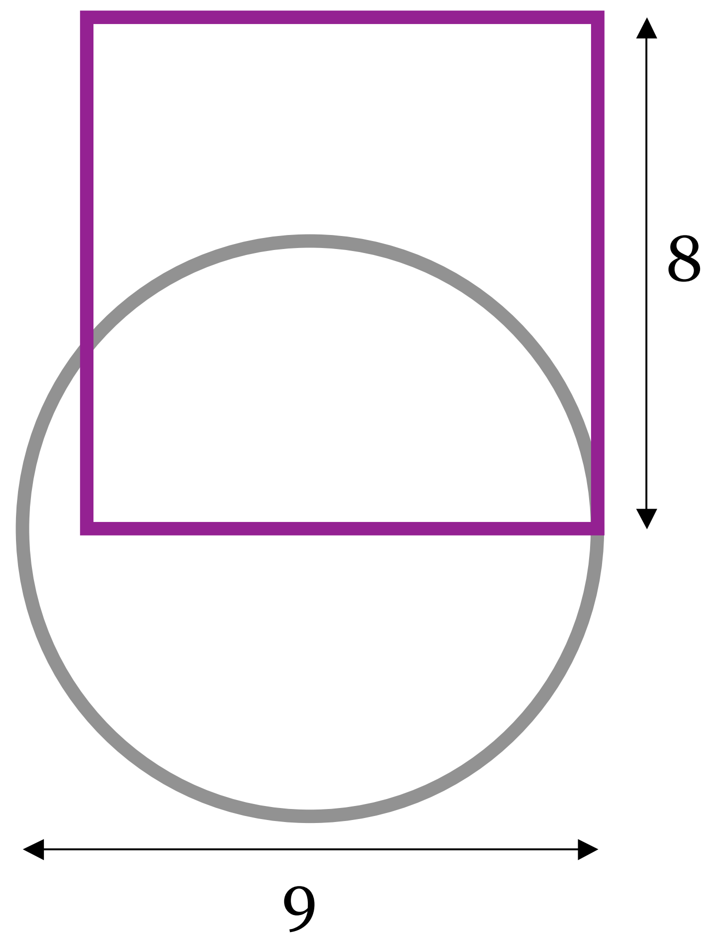
$$7 \times \frac{19}{8} = 16\frac{5}{8}.$$

Problem number 50

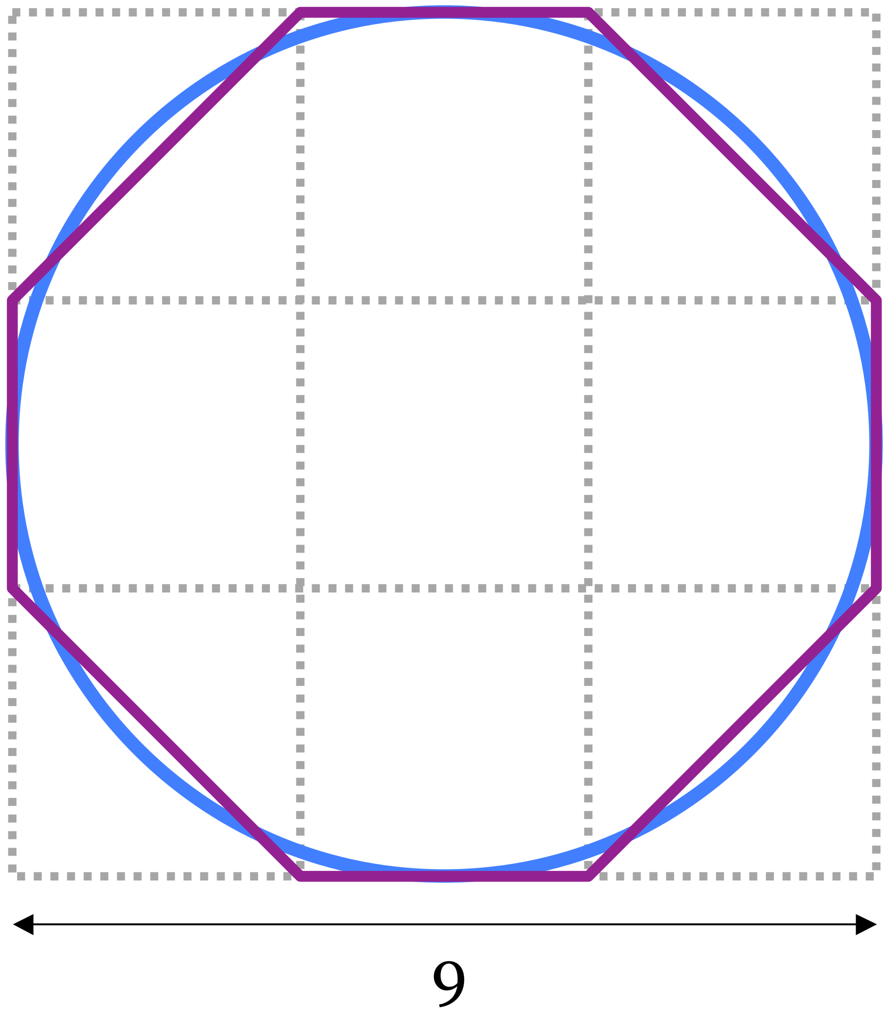
- A circular field with diameter 9 units has the same area as a square with side eight units

$$\pi\left(\frac{9}{2}\right)^2 = 8^2$$

$$\pi = 4\left(\frac{8}{9}\right)^2 = 3.16049\ldots$$



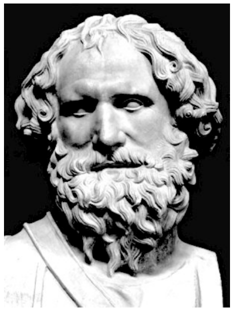
Octagon method?



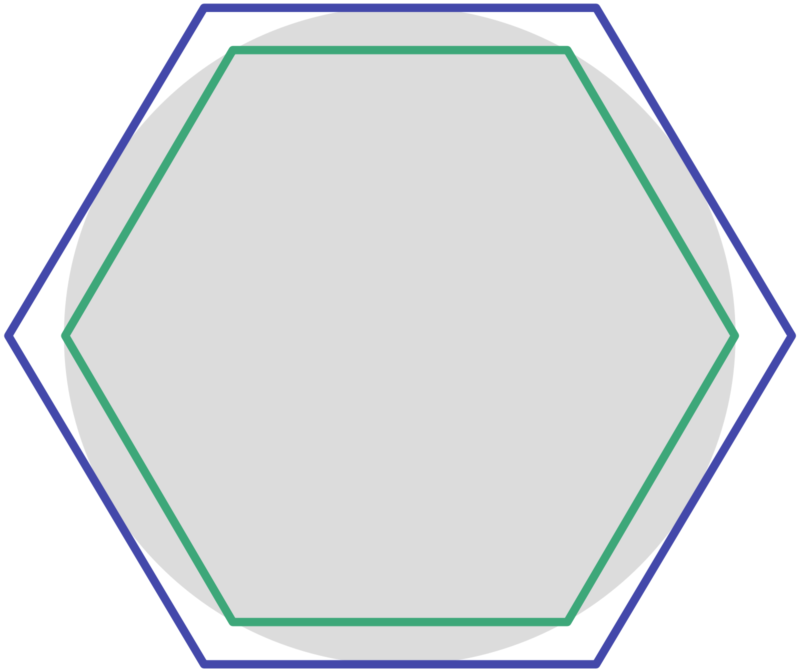
- Egyptians made use of square nets
- Cover circle in 3x3 grid
- Area of octagon is 63, which gives

$$\pi = 4\left(\frac{63}{81}\right) \approx 4\left(\frac{64}{81}\right) = 4\left(\frac{8}{9}\right)^2$$

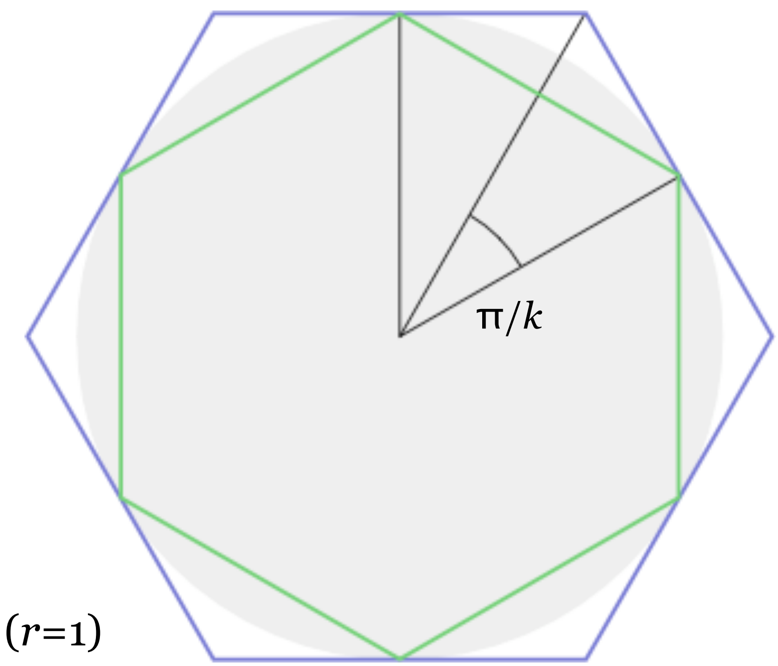
Archimedes (287BC-212BC)



- Brilliant physicist, engineer, mathematician
- Links circumference relation and area relation; shows $\pi = \pi'$ in $C = 2\pi' r$ and $A = \pi r^2$
- Sandwiches circle between inscribed and superscribed polygons



Archimedes' Method



- Let a_n and b_n be the circumferences of inscribed and superscribed polygons of 3.2^{n-1} sides
- For k sides,

$$a_n = k\tan\frac{\pi}{k}, b_n = k\sin\frac{\pi}{k}$$

$$\Rightarrow \frac{1}{a_n} + \frac{1}{b_n} = \frac{2}{a_{n+1}}, a_{n+1}b_n = b_{n+1}^2$$

Archimedes uses n=6 (96 sides) to find $3\frac{10}{71} < \pi < 3\frac{1}{7}$!

The Dark Ages

- Religious persecution of science brings study of π to a halt in Europe
- Most developments in Asia
- Decimal notation
- Zu Chongzhi (429-500) uses a polygon with 3×2^{14} sides; obtains π=3.1415926…
- Also finds ratio π≈355/113
- Best result for a millennium!

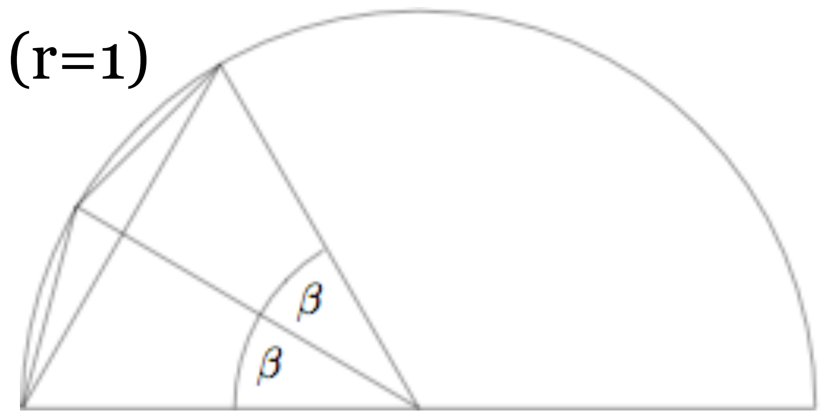
(Zu Chongzhi postage stamp)

European Middle Ages (1000-1500)

- Figures of 3 1/8, 22/7 and $(16/9)^2$ still in use
- Decimal notation gradually introduced
- Leonardo of Pisa (Fibonacci) uses Archimedes' method, but more accurately; obtains

$$\pi = \frac{864}{275} = 3.141818\ldots$$

François Viète (1540-1603)



- Finds the first infinite sequence
- Let *A(n)* be the area of the *n*-sided inscribed polygon

For the diagram, $A(2n)/A(n) = \cos\beta$ so

$$A(2^k n) = A(n) \times \frac{A(2n)}{A(n)} \times \frac{A(4n)}{A(2n)} \times \ldots \times \frac{A(2^k n)}{A(2^{k-1}n)}$$
$$= A(n) \cos\beta \cos\beta/2 \ldots \cos\beta/2^{k-1}$$

Using $n = 4, k \to \infty$ and $\cos\theta/2 = \sqrt{(1+\cos\theta)/2}$,

$$\frac{2}{\pi} = \sqrt{\frac{1}{2}}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}}}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}}}}\ldots$$

James Gregory (1638-1675)

- Discovers the arctangent series

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots$$

- Putting $x = 1$ gives

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots$$

- Extremely slow convergence

Isaac Newton (1642-1727)

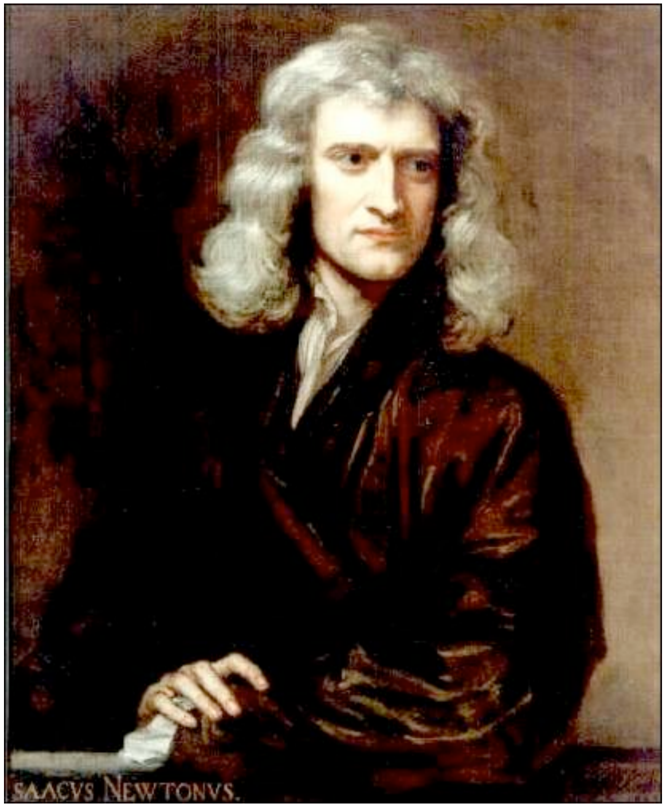
- Discovers related series

$$\sin^{-1} x = x + \frac{1}{2}\frac{x^3}{3} + \frac{1 \times 3}{2 \times 4}\frac{x^5}{5} + \ldots$$

- Putting x=1/2 gives

$$\frac{\pi}{6} = \frac{1}{2} + \frac{1}{2}\frac{1}{3.2^3} + \frac{1.3}{2.4}\frac{1}{5.2^5} + \ldots$$

- Converges much faster
- Newton calculates 15 digits, but is "embarrassed"



John Machin (1680-1752)

- If $\tan\alpha = 1/5$

$$\tan 2\alpha = \frac{2\tan\alpha}{1-\tan^2\alpha} = \frac{5}{12} \quad \Rightarrow \quad \tan 4\alpha = \frac{2\tan 2\alpha}{1-\tan^2 2\alpha} = \frac{120}{119}$$

- This is very close to one, and we see

$$\tan(4\alpha - \pi/4) = \frac{\tan 4\alpha - 1}{1+\tan 4\alpha} = \frac{1}{239}$$

- From this, we obtain

$$\frac{\pi}{4} = 4\tan^{-1}\frac{1}{5} - \tan^{-1}\frac{1}{239}$$

Alternative derivation

- Many arctangent formulae can be derived
- Can also be found using complex numbers – consider

$$(3+i)(3+i)(7+i) = 50 + 50i$$

- Since the arguments of complex numbers add, we know that

$$2\arg(3+i) + \arg(7+i) = \arg(50+50i)$$

$$2\tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{7} = \frac{\pi}{4}$$

William Jones (1675-1749): the first use of "π"

Taking a as an arc of 30°, aud t as a tangent in a figure given, he states (p. 243):

$$6a, \text{ or } 6 \times t - \frac{1}{3}t^2 + \frac{1}{5}t^5, \text{ \&c.} = \frac{1}{2} \text{ Periphery } (\pi) \ldots$$

Let

$$\alpha = 2\sqrt{3},\ \beta = \frac{1}{3}\alpha,\ \gamma = \frac{1}{3}\beta,\ \delta = \frac{1}{3}\gamma, \text{ \&c.}$$

Then

$$\alpha - \frac{1}{3}\beta + \frac{1}{5}\gamma - \frac{1}{7}\delta + \frac{1}{9}\epsilon, \text{ \&c.} = \frac{1}{2}\pi,$$

or

$$\alpha - \frac{1}{3}\frac{3\alpha}{9} + \frac{1}{5}\frac{\alpha}{9} - \frac{1}{7}\frac{3\alpha}{9^2} + \frac{1}{9}\frac{\alpha}{9^2} - \frac{1}{11}\frac{3\alpha}{9^3} + \frac{1}{13}\frac{\alpha}{9^3}, \text{ \&c.}$$

Theref. the (Radius is to ½ Periphery, or) Diameter is to the Periphery, as 1,000, &c to 3.141592653 . 5897932384 . 6264338327 . 9502884197 . 1693993751 . 0582097494 . 4592307816 . 4062862089 . 9862803482 . 5342117067. 9+ True to above a 100 Places; as Computed by the accurate and Ready Pen of the Truly Ingenious Mr. *John Machin*.

William Jones (1675-1749): the first use of “π”

There are various other ways of finding the *Lengths*, or *Areas* of particular *Curve Lines*, or *Planes*, which may very much facilitate the Practice; as for Instance, in the *Circle*, the *Diameter* is to *Circumference* as 1 to

$$\frac{16}{3} - \frac{4}{239} - \frac{1}{3}\frac{16}{5^3} - \frac{4}{239^3} + \frac{1}{5}\frac{16}{5^5} - \frac{4}{239^5} -, \&c. =$$

$$3.14159, \&c. = \pi \ldots$$

Whence in the *Circle*, any one of these three, α, c, d, being given, the other two are found, as, $d = c \div \pi = \overline{\alpha \div \frac{1}{4}\pi}|^{\frac{1}{2}}$, $c = d \times \pi = \overline{\alpha \times 4\pi}|^{\frac{1}{2}}$, $\alpha = \frac{1}{4}\pi d^2 = c^2 \div 4\pi$.

Euler (1707-1783)

"He calculated just as men breathe, as eagles sustain themselves in the air" – François Arago

- Adopted π symbol
- Derived many series, such as

$$\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots$$

- Calculated 20 digits in an hour using

$$\frac{\pi}{4} = 5\tan^{-1}\frac{1}{7} + 2\tan^{-1}\frac{3}{79}$$

The arctangent digit hunters

- **1706:** John Machin, *100 digits*
- **1719:** Thomas de Lagny, *112 digits*
- **1739:** Matsunaga Ryohitsu, *50 digits*
- **1794:** Georg von Vega, *140 digits*
- **1844:** Zacharias Dase, *200 digits*
- **1847:** Thomas Clausen, *248 digits*
- **1853:** William Rutherford, *440 digits*
- **1876:** William Shanks, *~~707~~ digits*

(incorrect after 527 digits!)

A short poem to Shanks

Seven hundred seven
Shanks did state
Digits of π he would calculate
And none can deny
It was a good try
But he erred in five twenty eight!

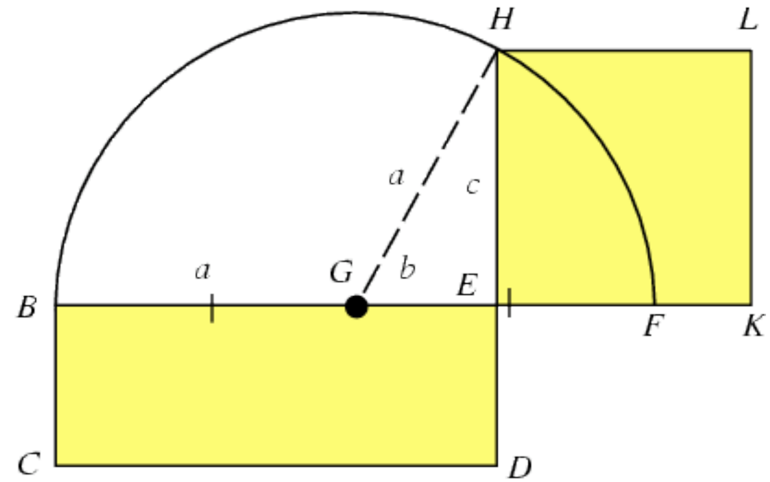
Transcendence

- 1767: Lambert proves π is irrational
- 1794: Legendre proves π and π^2 irrational: they can't be written as p/q
- 1882: Lindemann proves π is transcendental – π can't be expressed as the solution to an algebraic equation

$$a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0 = 0$$

You can’t square a circle

- Given a rectangle, you can construct a square of equal area with just geometry
- What about for a circle? If you could do it, you could geometrically find π
- But geometry will never give you a transcendental number, so it’s impossible
- Lots of people tried anyway



How to square a rectangle

The circle-squarers

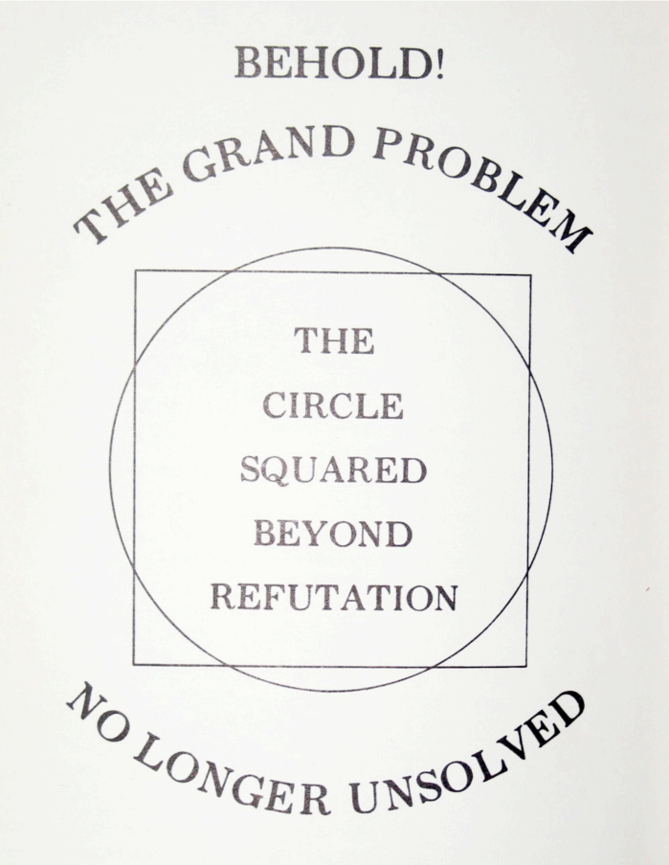
- *“With the straight ruler I set to make the circle four-cornered.” – Aristophanes, The Birds, 414BC*

- *“I have found, by the operation of figures, that this proportion is as 6 to 19. I am asked what evidence I have to prove that the proportion the diameter of a circle has to its circumference is as 6 to 19? I answer, there is no other way to prove that an apple is sour, and why it is so, than by common consent.” – John Davis, The Measure of the Circle, 1854*

- *“It is utterly impossible for one to accomplish the work in a physical way; it must be done metaphysically and geometrically, not mathematically.” – A. S. Raleigh, Occult Geometry, 1932*

Circle-squaromania: Carl Theodore Heisel

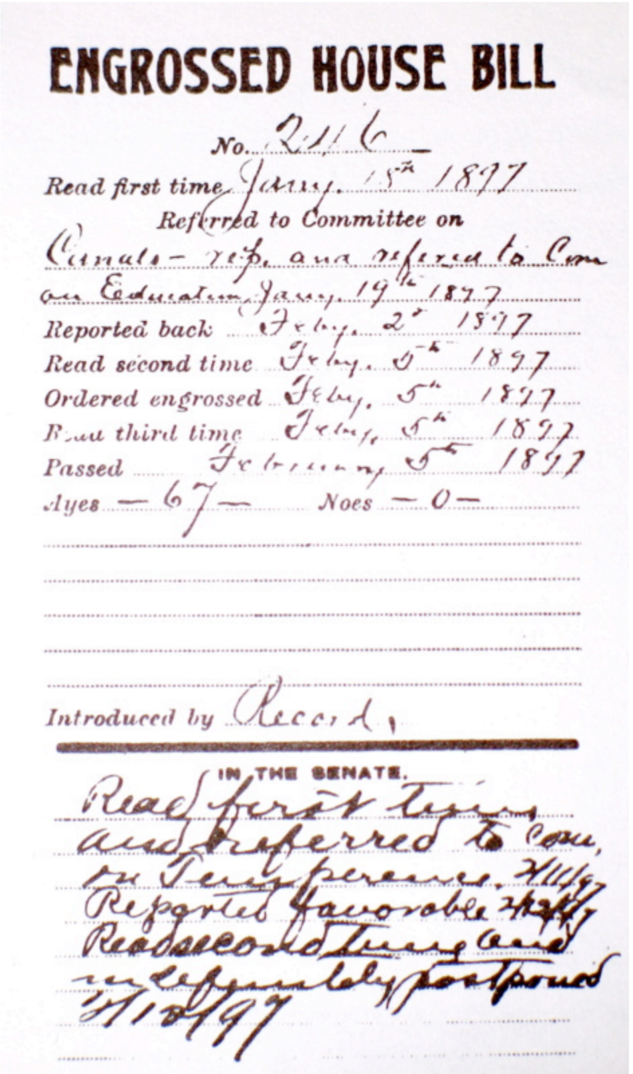
- In his 1931 book, he squares the circle, rejects decimal notation, and disproves the Pythagorean theorem
- Finds π=256/81, and verifies it by checking for circles with radius 1,2,…,9 "thereby furnishing incontrovertible evidence of the exact truth"
- There's a copy in the Harvard library



Title page from Heisel's book

Edwin J. Goodwin, Indiana

- "Squares the circle" in 1888; introduced to Indiana house in 1897
- *"A bill for an act introducing a new mathematical truth and offered as a contribution to education to be used only by the state of Indiana free of cost by paying any royalties whatever on the same, provided it is accepted and adopted by the official action of the legislature of 1897"*
- Passed unanimously by the house 67-0, without fully understanding the content of the bill



ENGROSSED HOUSE BILL

No. 246

Read first time Jany. 18th 1897

Referred to Committee on

Canals – rep. and referred to Com on Education Jany. 19th 1897

Reported back Feby. 2d 1897

Read second time Feby. 5th 1897

Ordered engrossed Feby. 5th 1897

Read third time Feby. 5th 1897

Passed February 5th 1897

Ayes —67— Noes —0—

Introduced by Record,

IN THE SENATE.

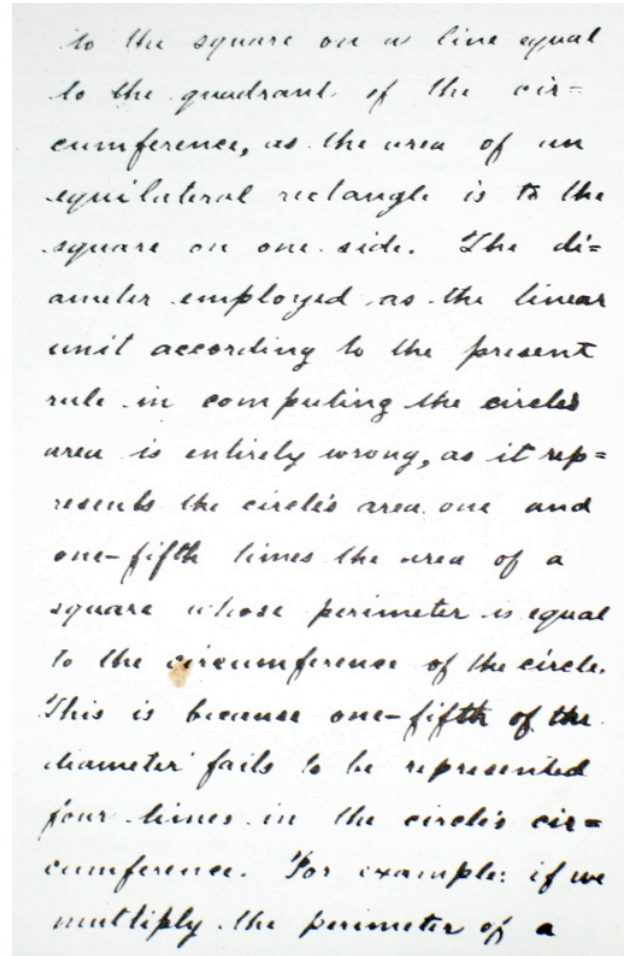
Read first time and referred to Com. on Temperance. 2/11/97

Reported favorable 2/12/97

Read second time and indefinitely postponed 2/12/97

Edwin J. Goodwin, Indiana

- *"It has been found that a circular area is to the square on a line equal to the quadrant of the circumference, as the area of an equilateral rectangle is to the square on one side"*
- Six different values of π!
- Prof C. A. Waldo (Purdue) visiting at the time, and is shocked that the billed passed
- Persuades Senate to indefinitely postpone action on it

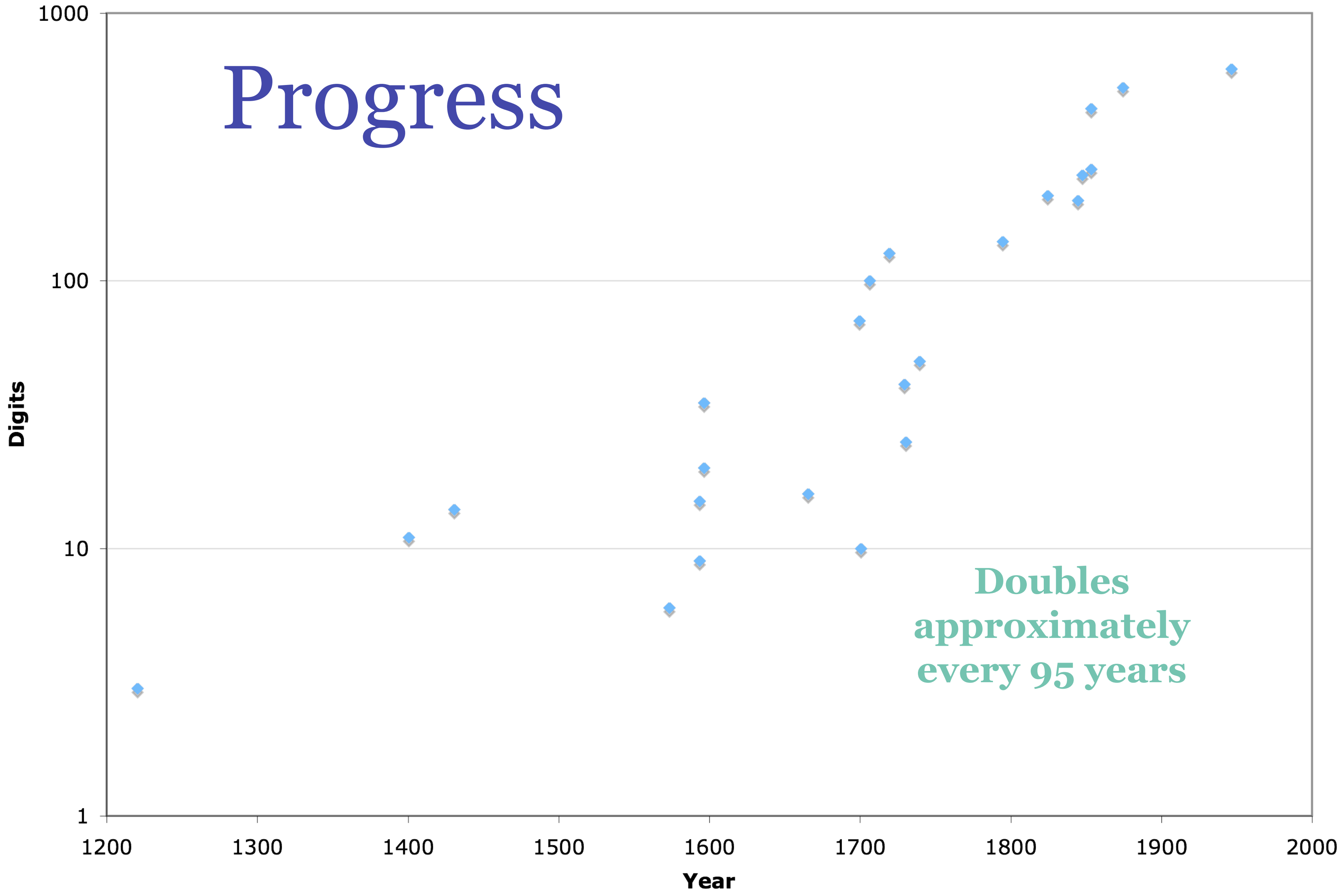


to the square on a line equal
to the quadrant of the cir-
cumference, as the area of an
equilateral rectangle is to the
square on one side. The di-
ameter employed as the linear
unit according to the present
rule in computing the circles
area is entirely wrong, as it rep-
resents the circle's area one and
one-fifth times the area of a
square whose perimeter is equal
to the circumference of the circle.
This is because one-fifth of the
diameter fails to be represented
four times in the circle's cir-
cumference. For example: if we
multiply the perimeter of a

Ramanujan (1887-1920)

- Born to a family without wealth in Southern India
- Math prodigy, but failed college entrance exams
- Went to Cambridge to work with G. H. Hardy in 1913
- Work formed the basis of many modern π formulae
- Became chronically ill during WWI
- Returned to India in 1919; died a year later

$$\frac{1}{\pi} = \frac{2\sqrt{2}}{9801} \sum_{n=0}^{\infty} \frac{(4n)!}{(n!)^4} \times \frac{1103 + 26390n}{396^n}$$



Progress
Doubles approximately every 95 years
1000
100
10
1
Digits
1200
1300
1400
1500
1600
1700
1800
1900
2000
Year

Machine calculators

- 1947: Ferguson uses mechanical calculator to compute 710 digits
- 1949: ENIAC calculates 2037 digits in 70 hours – the first automatic calculation of π
- Used the Machin formula
- Built robustness into code

ENIAC statistics: 10 feet tall, 1800ft² floor area, 30 tons, 18000 vacuum tubes, 10000 capacitors, 5000 operations a second

Shanks and Wrench: 100,000 decimals (1961)

"I feel the need, the need for speed!"
– Maverick and Goose

(IBM 7090)

- Used the arctangent formula

$$\frac{\pi}{4} = 6\tan^{-1}\frac{1}{8} + 2\tan^{-1}\frac{1}{37} + \tan^{-1}\frac{1}{239}$$

- Factor of 1/8 becomes a shift in binary
- Calculate two terms at a time to halve the number of divisions

$$\tan^{-1}\frac{1}{m} = \sum_{k=0}^{\infty}\frac{m[(4k+3)m^2-(4k+1)]}{(16k^2+16k+3)m^{4(k+1)}}$$

- Calculated in 8h43m; verified with a second arctangent formula

Faster algorithms (1)

- Arctangent formulae give a fixed number of digits per iteration
- 1976: Brent and Salamin find *quadratically* convergent algorithm
- 1985: Borwein and Borwein find *quartically* convergent algorithm:

$$a_0 = 6 - 4\sqrt{2}, y_0 = \sqrt{2} - 1$$

$$y_{k+1} = \frac{1 - (1 - y_k^4)^{1/4}}{1 + (1 - y_k^4)^{1/4}}$$

$$a_{k+1} = a_k(1 + y_{k+1})^4 - 2^{2k+3} y_{k+1}(1 + y_{k+1} + y_{k+1}^2)$$

$$a_k \to 1/\pi$$

Faster algorithms (2)

- Rapid convergence comes with a drawback
- Methods require high-precision division, and high-precision square root both computationally expensive
- Multiplication rapid using FFT technique
- To divide x by a, use the quadratically convergent scheme

$$x_{k+1} = x_k(2 - ax_k)$$

Digit extraction formulae

- Bailey, Borwein, and Plouffe (1996)

$$\pi = \sum_{i=0}^{\infty} \frac{1}{16^i} \left(\frac{4}{8i+1} - \frac{2}{8i+4} - \frac{1}{8i+5} - \frac{1}{8i+6} \right)$$

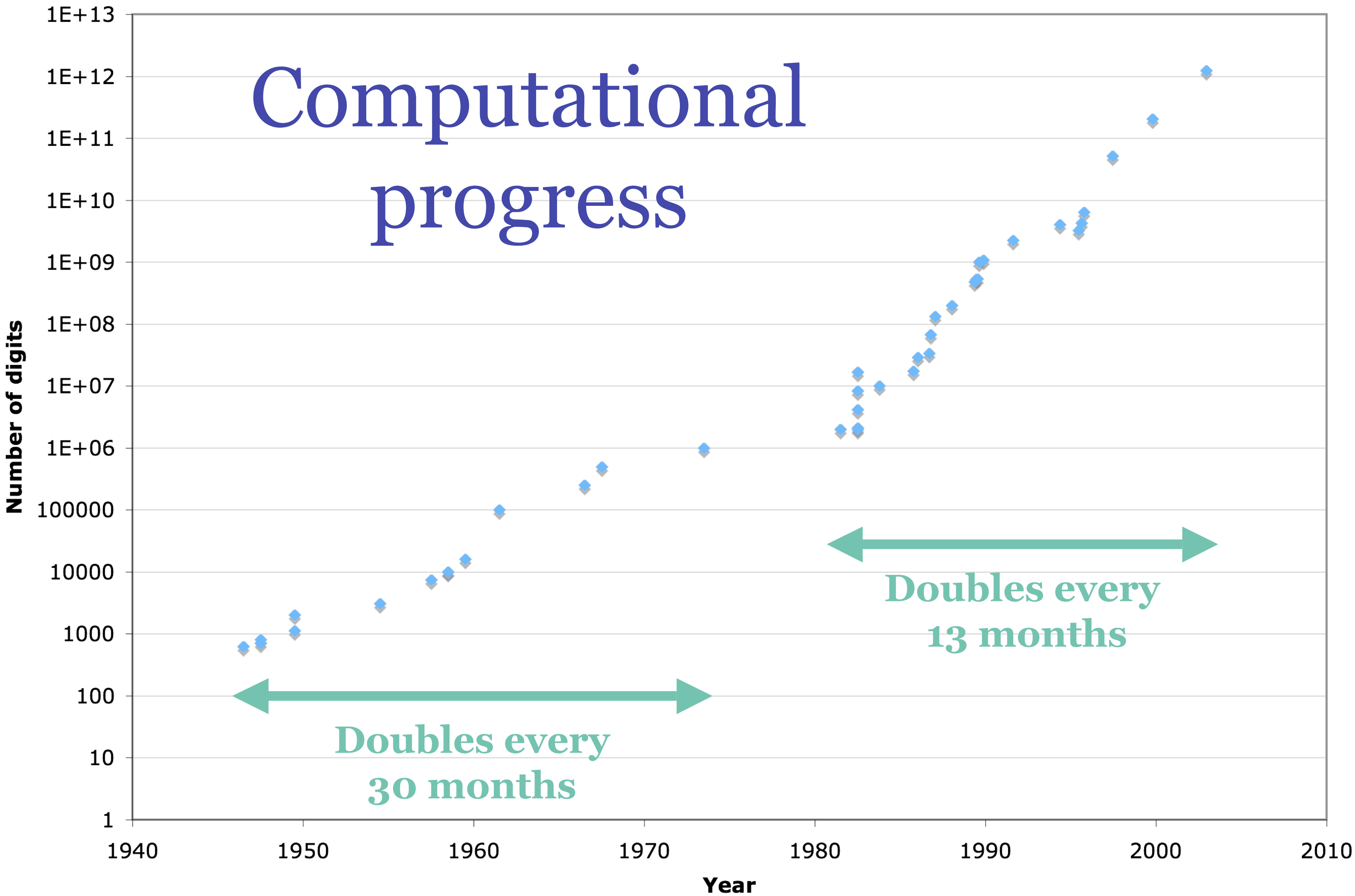
- Can extract a hexadecimal digit without computing the previous ones in O(n) time and O(log(n)) space
- No such expression for a decimal base

Hardcore digit hunters

- Chudnovsky brothers (USA) and Kanada (Japan) swap digit record throughout 1980's
- Chudnovsky formula

$$\frac{1}{\pi} = 12 \sum_{n=0}^{\infty} (-1)^n \frac{(6n)!}{(n!)^3 (3n)!} \frac{13591409 + n545140134}{(640320^3)^{n+1/2}}$$

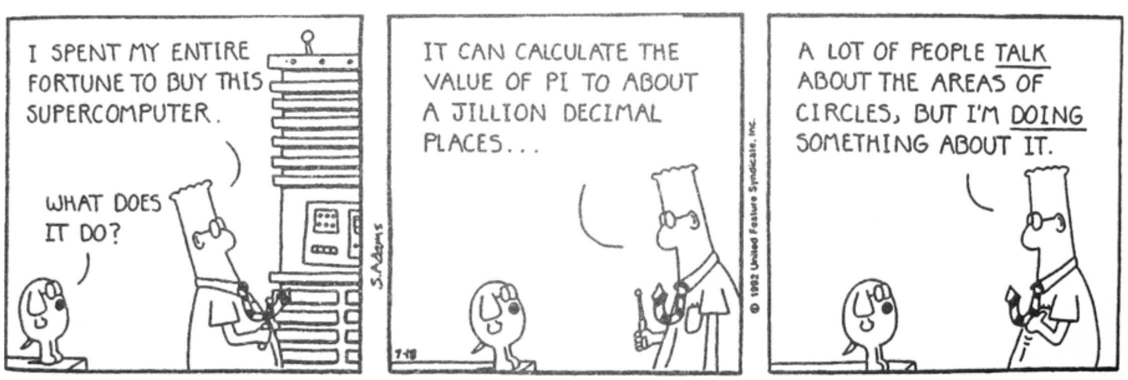
- 15 digits per term



Computational progress
Doubles every 13 months
Doubles every 30 months
1E+13
1E+12
1E+11
1E+10
1E+09
1E+08
1E+07
1E+06
100000
10000
1000
100
10
1
Number of digits
1940
1950
1960
1970
1980
1990
2000
2010
Year

Current record – 1,241,100,000,000 digits

- Computed in December 2002 by Kanada *et al.*
- Hitachi SR8000: 64 nodes, 14.4GFlops/node
- Used two different arctangent formulae
- Takes 601 hours, including the time to move 400Tb of data from memory to disk



Last 500 digits

(3) 500 digits ending 1,241,100,000,000-th
(1,241,099,999,501 - 1,241,100,000,000)

3716787169 6567692125 2797286901 8503557537 6530193499
3533850167 1616469990 5984454421 7623131551 5483436562
7806800557 0748706663 5108659327 6579461496 7987525534
7689068277 7037671632 7753867760 1776471900 9279382597
6527339324 6948904759 2872702485 4618972965 3547547082
4504016840 2350653293 6254205392 4502959326 3809170954
8310279798 7965959470 8455199922 4435552002 5054585883
0997016164 9607402417 5296690907 5622217705 1785600450
0707455198 1744551596 6313820124 4825046054 2311034186
5591198918 2262704528 2696896699 2856706487 3410311045

(passes all tests for randomness)

Frequency analysis for the first 1.2×10^{12} digits

0	:	119999636735	1	:	120000035569
2	:	120000620567	3	:	119999716885
4	:	120000114112	5	:	119999710206
6	:	119999941333	7	:	119999740505
8	:	120000830484	9	:	119999653604

(Acceptable χ^2 value)

Interesting sequences

```
012345678910  : from 1,198,842,766,717-th of pi
 432109876543 : from   149,589,314,822-th of pi
 543210987654 : from   197,954,994,289-th of pi
7654321098765 : from   403,076,867,519-th of pi
 567890123456 : from 1,046,043,923,886-th of pi
4567890123456 : from 1,156,515,220,577-th of pi
777777777777  : from   368,299,898,266-th of pi
999999999999  : from   897,831,316,556-th of pi
111111111111  : from 1,041,032,609,981-th of pi
888888888888  : from 1,141,385,905,180-th of pi
666666666666  : from 1,221,587,715,177-th of pi
271828182845  : from 1,016,065,419,627-th of pi
314159265358  : from 1,142,905,318,634-th of pi
```

π curiosities

$$\pi \approx \sqrt{2} + \sqrt{3} = 3.14626\ldots$$

$$\pi \approx \frac{47^3 + 20^3}{30^3} - 1 = 3.141592593\ldots$$

$$e^\pi - \pi = 19.999099979\ldots$$

$$\pi \approx \left(\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{7}}}}}}}}}}\right)^{\sqrt{9! - \sqrt{\sqrt{4!}}}} = 3.141592624$$

Coincidences

$$4\sum_{k=1}^{500000}\frac{(-1)^{k-1}}{2k-1} = 3.14159\mathbf{0}6535897932\mathbf{40}46264338327\mathbf{6}9502884197\ldots$$

This agrees with π to forty decimal places, *except* for the four places shown

- The sequence 999999 occurs in the first 1000 digits
- The probability any ten digit block contains one of each number is ~1/40000. Interestingly, this happens in the seventh block (digits 61-70).

π Memorizing

- A very cool party trick
- 1970's: world record held by Simon Plouffe (4096 digits)
- 1983: Rajan Mahadevan sets record with 31811 digits
- 1995: Hiroyuki Goto sets record with 42000 digits!

(Simon Plouffe)

Pi Mnemonics

- *“See, I have a rhyme assisting my feeble brain, it’s tasks oft-times resisting.”*
- *“How I wish I could remember pi.”*
- *“How I want a drink, alcoholic of course, after the heavy lectures involving quantum mechanics.”*

Pi Mnemonics

Sir, I bear a rhyme excelling
In mystic force and magic spelling
Celestial sprites elucidate
All my own striving can't relate
Or locate they who can cogitate
And so finally terminate. Finis.

π=3.14159265358979323846264338327 95...

π problem

- Show that there exists exactly one solution for $a, b, c \in \mathbb{N}$ such that

$$\frac{\pi}{4} = \tan^{-1}\frac{1}{a} + \tan^{-1}\frac{1}{b} + \tan^{-1}\frac{1}{c}$$