Quantum D-modules and the elliptic braid group

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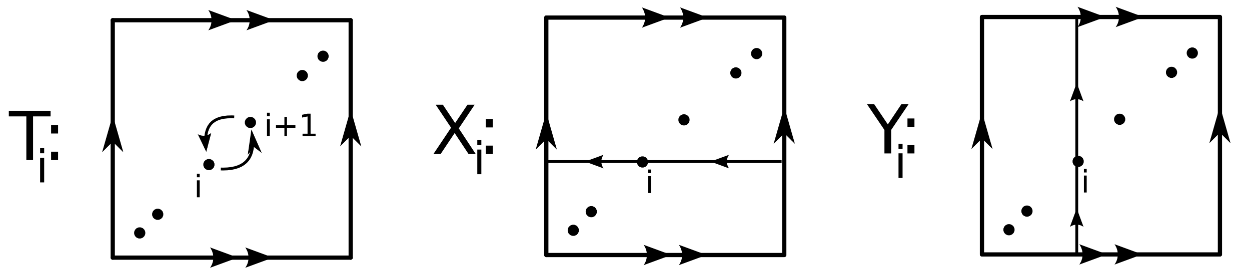
Definition 1 (Birman, Scott) *The elliptic braid group, B_n^{Ell}, is the fundamental group of the configuration space of n points on the torus. It is generated by*

- *the commuting elements $X_1, \ldots, X_n$*
- *the commuting elements $Y_1, \ldots, Y_n$*
- *and the braid group of the plane,*

$$B_n = \left\langle T_1, \ldots, T_{n-1} \quad \middle| \quad \begin{array}{l} T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1} \forall i, \\ T_i T_j = T_j T_i, |i-j| \geq 2 \end{array} \right\rangle .$$



The cross relations are:

- $T_i X_i T_i = X_{i+1}$,
- $T_i Y_i T_i = Y_{i+1}$,
- $X_1 Y_2 = Y_2 X_1 T_1^2$
- $\tilde{Y} X_i = X_i \tilde{Y}$, *where* $\tilde{Y} = \prod_j Y_j$.

Definition 2 *Cherednik's double affine Hecke algebra (DAHA) $\mathcal{H}(q,t)$ is the quotient of the group algebra $\mathbb{C}B_n^{Ell}$ of the elliptic braid group by the additional relations*

$$(T_i - q^{-1}t)(T_i + q^{-1}t^{-1}) = 0,$$

where q, t are complex parameters.

Remark 1 *Note that when $q, t = 1$, we have an isomorphism $\mathcal{H}(1,1) \cong \mathbb{C}[S_n \ltimes \mathbb{Z}^{2n}]$. If we are more careful, and remember first derivatives of Y_i, we get the so-called trigonometric Cherednik algebra $\mathcal{H}_n^{tr}(k)$.*

Two nice theorems

Theorem 3 (Calaque, Enriquez, Etingof) *Let $G = SL(N)$, let M be a $D(G)$-module, and let $V = \mathbb{C}^N$ be the defining representation for $\mathfrak{g} = sl_N$. Consider the space*

$$W = (\underbrace{V \otimes \cdots \otimes V}_{n} \otimes M)^{inv},$$

of invariants with respect to the adjoint action of vector fields on M. Then W carries an action of the trigonometric Cherednik algebra $H_n^{tr}(k)$, where $k = N/n$.

- s_{ij} were defined by the usual symmetric group action on $V^{\otimes n}$
- X_i were defined by:

$$X_i = \sum_{r,s} (E_s^r \otimes A_r^s)_{i,0}$$

- y_i were defined by:

$$y_j = k(\sum_p (b_p \otimes L_{b_p})_{j,0} + \sum_{i<j} s_{ij})$$

Theorem 4 (Lyubashenko, Majid) *Let $\mathcal{C}$ be a ribbon tensor category, let $V \in \mathcal{C}$, and let A be the "CoEnd" object, if it exists (e.g. if $\mathcal{C}$ is semi-simple then $A = \oplus W^* \otimes W$). Consider the space*

$$W = (\underbrace{V \otimes \cdots \otimes V}_{n} \otimes A)^{inv} := \mathit{Hom}(1, \underbrace{V \otimes \cdots \otimes V}_{n} \otimes A)$$

of invariants. W carries an action of the elliptic braid group on n strands.

These theorems overlap by applying Theorem 4 to $U = U_t(sl_N)$, and taking a trigonometric degeneration. But you only recover the "basic" $D(G)$-module of functions on G. A technique to wed these two theorems will be the focus of this talk.

Definition 5 *Let (H, Δ, ϵ, S) be a Hopf algebra. A normal left 2-cocycle on H (a.k.a a twist) is an invertible element $c \in H \otimes H$ such that*

$$(\epsilon \otimes id)(c) = (id \otimes \epsilon)(c) = 1 \otimes 1, \text{ and } (\Delta \otimes id)(c)(c \otimes 1) = (id \otimes \Delta)(c)(1 \otimes c).$$

Definition 6 *The twisted Hopf algebra, H_c, is the algebra H*

- *with twisted comultiplication $\Delta_c(h) = c^{-1}\Delta(h)c$*
- *and antipode $S_c(h) = QS(h)Q^{-1}$, where $Q = \mu \circ (id \otimes S)(c)$*

Remark 2 *H-mod $\cong$ H_c-mod, as tensor categories.*

Definition 7 *An algebra F is H-equivariant if*

- *F is an H-module*
- *$\mu : F \otimes F \to F$ is a map of H-modules*

Definition 8 *Let F be H-equivariant, and let c be a 2-cocycle for H. Then F_c is H_c-equivariant, where*

$$\mu_c(f_1 \otimes f_2) := \mu(c(f_1 \otimes f_2))$$

. F_c is called the H_c-equivariant algebra associated *to F.*

Let U denote a quasi-triangular Hopf algebra with universal R-matrix $\mathcal{R} = \sum_k r_k^+ \otimes r_k^-$.

$$\begin{tikzcd}
& U \otimes U \arrow[dl, dash, "\mathcal{R}_{13}"'] \arrow[dr, dash, "\mathcal{R}_{23}"] & \\
U^o \otimes U \arrow[rr, dash, "\mathcal{R}_{13}\mathcal{R}_{23}"'] & & U \tilde{\otimes} U
\end{tikzcd}$$

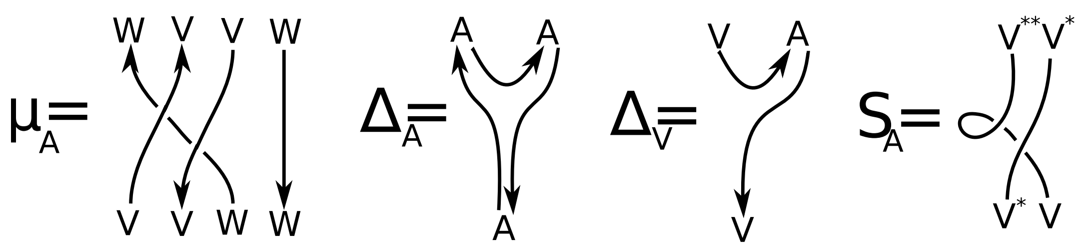
- U^o: algebra U, with opposite co-multiplication.
- $U \otimes U$: tensor product Hopf algebra.
- $U^o \otimes U$: tensor product Hopf algebra.
- $U \tilde{\otimes} U := (U \otimes U)_{\mathcal{R}_{23}} = (U^o \otimes U)_{\mathcal{R}_{13}\mathcal{R}_{23}}$.

Remark 3 $\tilde{\Delta} \circ \Delta(x) = \mathcal{R}_{23}^{-1}(x_1 \otimes x_3 \otimes x_2 \otimes x_4)\mathcal{R}_{23} = x_1 \otimes x_2 \otimes x_3 \otimes x_4$

- U: a ribbon Hopf algebra (e.g. $U = U_q(\mathfrak{g})$),
- F: dual Hopf algebra relative to the category of finite dimensional U-modules:

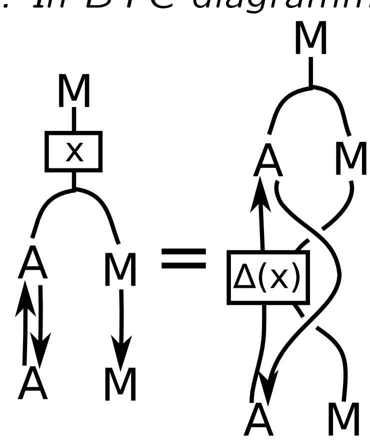
$$F = \{c_{f,v}(u) := f(uv), | f \in V^*, v \in V, V \text{ f.d. } U\text{-mod}\}.$$

- F is $U^o \otimes U$-equivariant action by $(x \otimes y)c_{f,v} = c_{xf,yv}$
- $A := F_c$, the $U\tilde{\otimes}U$-module associated to F via $c = \mathcal{R}_{13}\mathcal{R}_{23}$.
- Why all the fuss? A is $U\tilde{\otimes}U$ equivariant, and thus U-equivariant for the adjoint action! A is generated by $\{V^* \otimes V\}$, which are U-submodules. In BTC diagrammatical language,



- Let $\partial_\triangleleft : U = U \otimes 1 \hookrightarrow U\tilde{\otimes}U$ denote the inclusion into the right tensor factor.

Definition 9 (Varagnolo, Vasserot) *A quantum D_U module is a module over $A \rtimes \partial_\triangleleft(U)$. In BTC diagrammatical language,*



Remark 4 *Varagnolo, Vasserot showed that this agrees with usual definition in terms of Heisenberg double, and explained how to construct adjoint action of vector fields as a quantum moment map.*

Results

Theorem 10 *Let M be a D_U-module, and let V be a U-module. Consider the space*

$$W = (\underbrace{V \otimes \cdots \otimes V}_{n} \otimes M)^{inv},$$

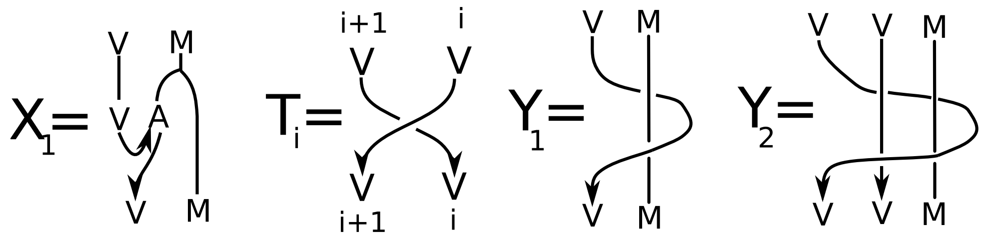
where we take invariants with respect to the adjoint action on M. W carries an action of the elliptic braid group on n strands, which gives rise to a functor:

$$F_{n,V} : D_U\text{-mod} \rightarrow B_n^{Ell}\text{-mod}.$$

Corollary 11 *If V is Hecke, the above descends to a representation of $\mathcal{H}\!\mathcal{H}_n(q,t)$ with corresponding parameters.*

Proposition 12 *If M is the "basic" D_U-module A, we recover the construction of Lyubashenko and Majid. If $U = U_t(sl_N)$, $V = \mathbb{C}^N$ and M is arbitrary, we recover the construction of Calaque, Enriquez, and Etingof as a quasi-classical limit when $t \rightarrow 1$.*

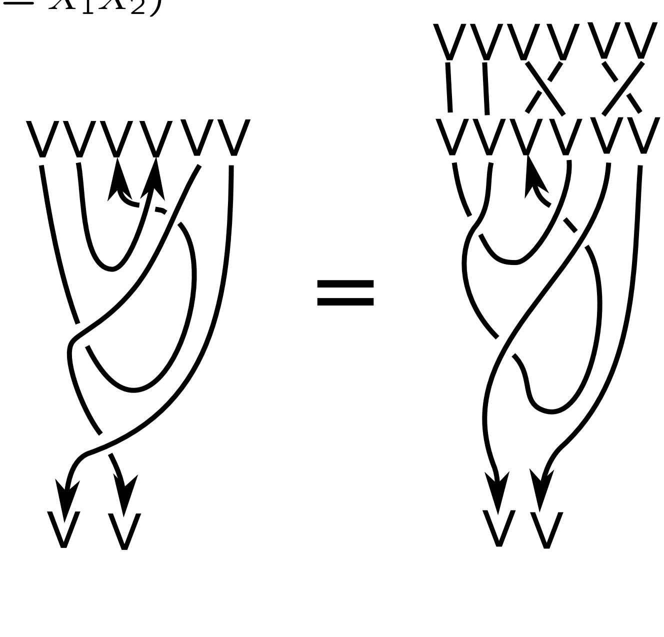
Proof of Theorem 8. We define operators as follows:



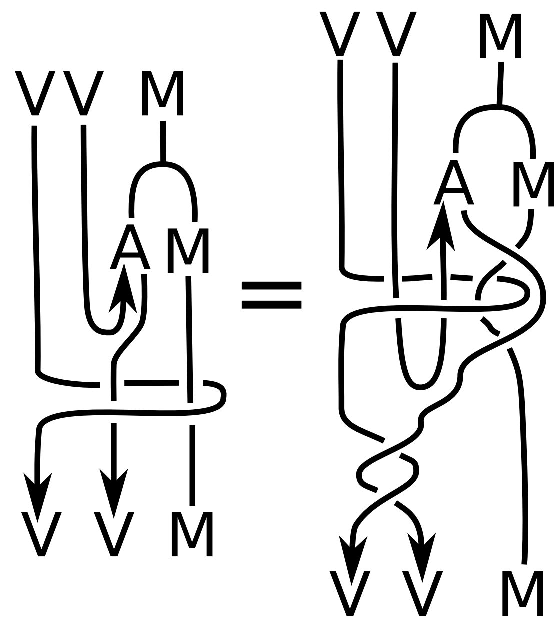
- The T_i act in the obvious way on the copy of $V^{\otimes n}$.
- X_1 first comultiplies V as an A-comodule, then absorbs the new A-factor into M.
- Y_1 braids V around the back of M.
- Remaining X_i and Y_i determined by relations $X_{i+1} = T_i X_i T_i$, $Y_{i+1} = T_i Y_i T_i$.

That these operators preserve W is clear from functorial description. Now we have to check all the relations.

- X's commute: This is the well-known reflection equation for A: (e.g. $X_2 X_1 = X_1 X_2$)



- Y's commute: This is the well-known QYBE equation for R-matrix.
- $X_1 Y_2 = Y_2 X_1 T_1^2$.



- Express $\tilde{Y}$ via the ribbon element, find that it multiplies by a constant, so it's trivially central.

Forward directions

- Eliminate Hopf algebra U, work in BTC
- Thus extend to Deligne's $U_t(sl_z)$, where z is an arbitrary complex number, and build DAHA representations for any parameters.
- Higher genus generalizations in braided and trigonometric case, modular group.
- Geometric/operadic interpretation of Elliptic braid relations? Categorical interpretation of Heisenberg double analogous to Drinfeld double?
- Quantum quiver varieties?