$\mathfrak{g}$ – complex semisimple Lie algebra. G simply conn. complex Lie group.
$B \subset G$ Borel subgroup. $G/B = \mathcal{B}$ – flag variety.

Beilinson–Bernstein Corr. $\mathfrak{g}$-mod $\rightsquigarrow$ D-mod. on $\mathcal{B}$

① Differential Operators, quantization ② Hamiltonian reduction ③ Springer resolution.
④ BB-correspondence.

① X – smooth complex variety. $\mathcal{O}_X$ $\mathrm{Vect}_X = \mathrm{Der}_{\mathbb{C}}(\mathcal{O}_X) \subset \mathrm{End}_{\mathbb{C}}(\mathcal{O}_X)$.

D_X — generated by $\mathcal{O}_X$. Vect_X.

Ex. $X = \mathbb{A}_m$ $D_X = \mathbb{C}\langle x_1 \dots x_m, \partial_1 \dots \partial_m \rangle / \sim$ — m-th Weyl algebra.

Take the usual filtration on D_X (as: $F^0 = \mathbb{C}[x_1 \dots x_n] \subset F^1 = \mathbb{C}[x_1 \dots x_n] \oplus \bigoplus_i \mathbb{C}[x_1 \dots x_n]\partial_i \subset \dots$)

Define $\sigma(f \partial_{x_1}^{n_1} \cdots \partial_{x_m}^{n_m}) = f(x_1 \dots x_n)\, \xi_1^{n_1} \cdots \xi_m^{n_m} \in \mathbb{C}[\sim]$

Principal symbol.

$$\Rightarrow \mathrm{Gr}(D_{\mathbb{A}^n}) = \mathbb{C}[x_1 \dots x_m, \xi_1 \dots \xi_m]$$

Poisson-bracket: $g_1 \in D_{n_1}$, $g_2 \in D_{n_2}$ $g_1 g_2 - g_2 g_1 \in D_{n_1+n_2-1}$

$$g_1 g_2 - g_2 g_1 \mod D_{n_1+n_2-2} \in \mathbb{C}[x_1 \dots x_m, \xi_1 \dots \xi_m]_{n_1+n_2-1}$$

$$\{g_1, g_2\} = \sum_{i=1}^{m} \frac{\partial g_1}{\partial x_i}\frac{\partial g_2}{\partial \xi_i} - \frac{\partial g_1}{\partial \xi_i}\frac{\partial g_2}{\partial x_i}$$

$D_{\mathbb{A}^m}$ is a quant. of $\mathbb{A}^{2m}$.

Recall: Let (X, ω) be a symplectic variety, a formal quanti. of (X,ω) is a sheaf of ass. flat $\mathbb{C}[[h]]$-algebras $\mathcal{O}_h$ s.t. $\mathcal{O}_h / h\mathcal{O}_h \simeq \mathcal{O}_X$.

$\mathcal{O}_X$ has a poisson str. from $\mathcal{O}_h$: $\{\bar{f}, \bar{g}\} = \frac{1}{h}[f, g] \mod h$.

where $f, g \in \mathcal{O}_\hbar$ s.t. $\bar{f}\ \bar{g} \in \mathcal{O}_X$

This Poisson str. is the same from the symplectic str.

Example: X – smooth affine variety. Define $D_\hbar(X)(0)$ on T^*X to be the h-completion of the algebra generated by $\mathcal{O}_X$, Vect_X, h with rel:

$$\cancel{f_1 f_2 = f_2 f_1 \quad f\xi = f}$$

$$f_1 \circ f_2 = f_1 f_2 \qquad f \circ \xi = f\xi \qquad \xi \circ f - f \circ \xi = h\,\xi(f) \qquad \xi_1 \circ \xi_2 - \xi_2 \circ \xi_1 = h\underbrace{}_{[\xi_1, \xi_2]}$$

$$D_h(X) = D_h(X)(0)[h^{-1}]$$

On T^*X we have a $\mathbb{C}^*$-action. $D_\hbar(X)$ is equivariant w.r.t. that action.

$T_t : T^*X \to T^*X$ equi. means: $T_t^{-1}(D_\hbar(X)) \overset{\mathcal{F}_t}{\cong} D_\hbar(X)$.

$$\mathcal{F}_t(g) = g \qquad \mathcal{F}_t(\xi) = t\xi \qquad \mathcal{F}_t(h) = th.$$

② Hamiltonian reduction

G – connected algebraic group. $G \circlearrowright (X, \omega)$ Hamiltonian action.

momentum map $\mu : X \longrightarrow \mathfrak{g}^*$ G-equi. $\mu^{-1}(0)/G$.

Examples: $G \circlearrowright X$ then can lift this action to $G \circlearrowright T^*X$

(reason: $u \in \mathfrak{g}$, gives a vector fields u' on X

construct $\tilde{u}'$: $\tilde{u}'(\xi + f) = [u', \xi] + u'(f)$) and this is a Hamiltonian action.

Suppose $G/B = X$ Lie$(B) = \mathfrak{b}$

$$\mu : T^*(G/B) \longrightarrow \mathfrak{g}^* \qquad \mathfrak{b}^\perp \subseteq \mathfrak{g}^*.$$

We define $G \times_B \mathfrak{b}^\perp$ this is a quotient of $G \times \mathfrak{b}^\perp$ by the right action of B. $(g, \xi) \cdot b = (gb, ad^*_{b^{-1}} \xi)$ This is a bundle over G/B and $G \times_B \mathfrak{b}^\perp \cong T^*(G/B)$

$$\mu(g,\xi) = ad_g^*(\xi)$$

<u>Hamiltonian reduction.</u>

$$B \subset G. \quad \text{look at } T^*G \supset B$$

The moment map: $T^*G = G \times \mathfrak{g}^* \xrightarrow{\mu} \mathfrak{g}^* \xrightarrow{res} \mathfrak{b}^*$

$$(g,\xi) \longmapsto ad_g^*(\xi) \longmapsto ad_g^*(\xi)$$

$$\mu^{-1}(0): \quad res^{-1}(0) = \mathfrak{b}^\perp \quad \mu^{-1}(\mathfrak{b}^{\perp}) = \{(g, (ad_g^*)^{-1}\mathfrak{b}^\perp)\}$$

$$\mu^{-1}(0)/B = G \times_B \mathfrak{b}^\perp = T^*(G/B)$$

③. G - complex semisimple Lie group. B.

$$\mathcal{B} = G/B \qquad \tilde{\mathfrak{g}} = \{(x,\mathfrak{b}) \in \mathfrak{g} \times \mathcal{B} \mid x \in \mathfrak{b}\}.$$

$$\tilde{\mathfrak{g}} \xrightarrow{\pi} \mathcal{B} \qquad \tilde{\mathfrak{g}} \xrightarrow{\mu} \mathfrak{g}. \qquad \mathfrak{b} \in \mathcal{B} \quad \pi^{-1}(\mathfrak{b}) = \text{a copy of } \mathfrak{b}.$$

So $\tilde{\mathfrak{g}}$ is a $\mathfrak{b}$-bundle over $\mathcal{B}$.

Given the space $G \times \mathfrak{b} \supset B$ as before.

$$f: G \times \mathfrak{b} \longrightarrow \tilde{\mathfrak{g}}. \qquad f(g,k) = (ad_g k,\ g\cdot B/B)$$

$$\text{So } G \times_B \mathfrak{b} \xrightarrow{\sim} \tilde{\mathfrak{g}}$$

Let $N \subset \mathfrak{g}$, be the nilpotent cone, define $\tilde{N} = \mu^{-1}(N)$

$$\mu^{-1}(N) = \{(x,\mathfrak{b}) \in N \times \mathcal{B} \mid x \in \mathfrak{b}\} = G \times_B \mathfrak{n} \qquad \mathfrak{n} = [\mathfrak{b},\mathfrak{b}]$$

$\mathfrak{g} \simeq \mathfrak{g}^*$ since $\mathfrak{g}$ semisimple. $\mathfrak{b}^\perp \in \mathfrak{g}^*$ $\Rightarrow \mathfrak{b}^\perp = \mathfrak{n}$ $(\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n})$

$$\Rightarrow G \times_B \mathfrak{n} = G \times_B \mathfrak{b}^\perp = T^*(G/B)$$

$\mu: T^*(G/B) \longrightarrow N$ is the momentum map for $G \curvearrowright T^*(G/B)$

In fact, N has finitely many G-orbits and $\exists!$ open orbit consists the regular elements : $x \in \mathfrak{g}$ is regular if $\dim C_{\mathfrak{g}}(N) = \mathrm{rk}\,\mathfrak{g}$.

$$\underset{\substack{\uparrow \\ \text{reg. nilp}}}{\mu^{-1}(n)} = (n, \underset{\substack{\uparrow \\ \text{only}}}{b}) \quad \Rightarrow \mu \text{ is a resolution of singularities.}$$

N is a normal variety

④ B-B. thm.

$$Z(U(\mathfrak{g})) = \mathbb{C}[\mathfrak{h}]^W \quad \text{so} \quad \operatorname{maxspec}(Z(U(\mathfrak{g}))) = \mathfrak{h}/W$$

i.e. can para. by χ_λ, $\chi_\lambda = \chi_\mu$ iff $\exists\, w$ s.t. $w\lambda = \mu$.

If M_λ is a verma module with wt. λ, $z \in Z(U(\mathfrak{g}))$, $m \in M_\lambda$.

$$z \cdot m = \chi_{\lambda+\rho}(z) \cdot m$$

$$\mathbb{C}[N] = \mathbb{C}[\mathfrak{g}] \Big/ \underbrace{(\mathbb{C}[\mathfrak{g}]^G_+)}_{\text{space of } G\text{-inv. poly. without constant term}} \cdot \mathbb{C}[\mathfrak{g}]$$

$$Z(U(\mathfrak{g})) = U(\mathfrak{g})^G$$

Let $\mathrm{Ker}(\chi_\lambda)$ be any maximum ideal in $Z(U(\mathfrak{g}))$.

$$J_\lambda = U(\mathfrak{g}) \cdot \mathrm{Ker}(\chi_\lambda)$$

$$\mathrm{Gr}\left(U(\mathfrak{g}) / U(\mathfrak{g}) J_\lambda\right) = \mathbb{C}[\mathfrak{g}] / \mathbb{C}[\mathfrak{g}]^G_+ \mathbb{C}[\mathfrak{g}] = \mathbb{C}[N]$$

(Konstant : $U(\mathfrak{g})$ is a free module over $Z(U(\mathfrak{g}))$.)

$$\begin{tikzcd}
T^*(G/B) \arrow[r] \arrow[d] & D(G/B) \\
N \arrow[r] & U(\mathfrak{g})/U(\mathfrak{g}) J_{-\rho} \arrow[u, "\text{localization}"']
\end{tikzcd}$$

Recall: $D(G) = \mathcal{U}(\mathfrak{g}) \# \mathcal{O}(G)$.

[h-version: $D_h(G) = \mathcal{U}_h(\mathfrak{g}) \# \mathcal{O}(G)$.

$\mathcal{U}_h(\mathfrak{g})(0)$ — h-completion of $T(\mathfrak{g})[h] \big/ (xy - yx - \hbar[x,y])$.

$\mathcal{U}_h(\mathfrak{g}) := \mathcal{U}_h(\mathfrak{g})(0)[\hbar^{-1}]$.

Hamiltonian reduction on diff operators.

$$\mu' : \mathfrak{g} \longrightarrow D_h(G).$$

Step 1: $D_\hbar(G) \big/ D_\hbar(G)\cdot\mu'(\mathfrak{b})$ (restriction to $\mu^{-1}(0)$)

$$\left(D_\hbar(G) \big/ D_\hbar(G)\mu'(\mathfrak{b}) \right)^B \qquad (\mu^{-1}(0)/B).$$

This is $D_h(G/B)$.

We have a map $\varphi: U(\mathfrak{g}) \longrightarrow \Gamma(G/B, D_{G/B})$.

φ is surjective $\quad \mathrm{Ker}(\varphi) = \mathcal{U}(\mathfrak{g}) J_{-\rho}$

$$\mathcal{U} \big/ \mathcal{U} J_{-\rho} \xrightarrow{\sim} \Gamma(G/B, D_{G/B}).$$

$\Gamma: D_{G/B}\text{-mod} \longrightarrow \mathcal{U}/\mathcal{U}\cdot J_{-\rho}\text{-mod}$ is an equivalence of categ.

$M \longrightarrow D_{G/B} \otimes_{\Gamma(D_{G/B})} M$ is the inverse functor.

(Replace $\mathbb{C}^*$-equiv. $D_h(G/B)$-mod.)

Topics: I. W-algebras — quantum Hamiltonian reduction.

II. adjoint quotients — transverse slices

III. resolution of singularities

IV. Localization theorem.

Let A be an ass. algebra $/k=\bar{k}$. and let M be a connected algebraic group $/k$.

$\mathrm{Lie}(M) := \mathfrak{m}$. We suppose that M acts on A algebraically (locall f.d. repn of M).

further: suppose given $\rho: \mathcal{U}(\mathfrak{m}) \longrightarrow A$ algebra morphism. satisfying the following

condition: $\mathrm{ad}(m)(a) = \rho(a)m - a\rho(m)$ agree with the diff of the M-action.

Take $I \subset \mathcal{U}(\mathfrak{m})$ Ad-stable two sided ideal.

Defn: The quantum Hamiltonian reduction of A w.r.t. I is $\left(A/A\cdot\rho(I)\right)^M$.

(This algebra $\simeq \mathrm{End}_A\left(A/A\cdot\rho(I)\right)^{op}_{?}$) (BGF)

E.g.: Let M acts on X (variety), then we have M acts on T^*X Hamiltonian.

Take A = regular function on T^*X.

$\rho(m)$ = Symbol (left inv. vector field ass. to m)

$A/A\cdot\rho(I) \longleftrightarrow$ functions on $\mu^{-1}(0)$ and $\left(A/A\rho(I)\right)^M \longleftrightarrow \mu^{-1}(0)/M$.

Let $\mathfrak{g}$ be a complex s.s. Lie algebra. $e \in \mathfrak{g}$ — a nilpotent element.

Then $\exists\, f, h$ s.t. $\{e, f, h\}$ s.t. sl_2. fix one triple.

Let sl_2 acts on $\mathfrak{g}$, $\mathfrak{g} = \bigoplus_i \mathfrak{g}(i)$ $\quad \mathfrak{g}(i) = \{x \in \mathfrak{g} \mid [h,x] = ix\}$

Define $S = e + \mathrm{Ker}(\mathrm{ad} f)$ affine subspace of $\mathfrak{g}$.

Let $\chi \in \mathfrak{g}^*$ be the functional ass. to e via Killing form: $\chi(x) = (e, x)$

We can define a skew-symm form on $\mathfrak{g}(-1)$ as follows:

$\langle x, y\rangle = \chi([x,y]) = (e, [x,y])$

This is nondegenerate : Let $0 \neq X \in \mathfrak{g}(-1)$.

① $[e,x] \neq 0$ $\quad [e,x] \in \mathfrak{g}(1)$

② Claim : $\exists\, y \in \mathfrak{g}(-1)$ s.t. $\quad ([e,x], y) \neq 0 = (e, [x,y])$

Choose $L \subset \mathfrak{g}(-1)$ a Lagrangian subspace. ($\dim L = \frac{1}{2}\dim \mathfrak{g}(-1)$ $\quad \langle\,,\rangle|_L = 0$)

Define $m_L = L \oplus \left(\bigoplus_{i\leq -2} \mathfrak{g}(i)\right)$ this is a nilpotent Lie algebra and

$\chi\ m_L$ is a character.

Let M_L be the subgroup of G s.t. $\mathrm{Lie}(M_L) = m_L$.

M_L acts on $\mathcal{U}(\mathfrak{g})$ by adjoint action. Let $I \subseteq \mathcal{U}(m_L)$ be the ideal gen. by $\langle x - \chi(x) \mid x \in m_L\rangle$.

$\mathcal{U}(\mathfrak{g}, e) = \left(\mathcal{U}(\mathfrak{g}) / \mathcal{U}(\mathfrak{g})\cdot I\right)^{M_L}$ $\quad$ the finite W-alg of $e \in \mathfrak{g}$

How $\mathcal{U}(\mathfrak{g}, e)$ is a quantization of $\mathbb{C}[S]$.

Kazhdan filtration : $\exists$ a $\mathbb{C}^*$-action on $\mathfrak{g}$ ~~vit~~ via.

$$\mathfrak{sl}_2(\mathbb{C}) \hookrightarrow \mathfrak{g} \Rightarrow SL_2(\mathbb{C}) \hookrightarrow G.$$

$\gamma(t) = \tilde{\gamma}\begin{pmatrix} t & \\ & t^{-1}\end{pmatrix}$ $\qquad \mathrm{Ad}\,\gamma(t)\, e = t^2 e$

We will look at $\quad \rho(t) = t^{-2}\mathrm{Ad}(\gamma(t))$ This action stabilizes S and fixes e.

(It not keep the $[\,,]$). Use this action to define a grading on $\mathbb{C}[S]$.

Describ explicitly on $\mathbb{C}[\mathfrak{g}^*] = S\mathfrak{g}$:

Let $S(\mathfrak{g}) = \bigoplus_n S^n\mathfrak{g}$ usual grading. define $S^n(\mathfrak{g})(i) = \{x \in S^n\mathfrak{g} \mid [h,x] = ix\}$

Define $S(\mathfrak{g})[n]$ = span of $S^j[i]$ st. $i + 2j = n$.

$\mathbb{C}[S]$ is a graded algebra by the inherites grading. it has only ≥ 0 degrees.

We define a filtration on $\mathcal{U}(\mathfrak{g})$. Let $\mathcal{U}_n(\mathfrak{g})$ be the usual filtration. Define $F_n(\mathcal{U}(\mathfrak{g}))$ = span of $\mathcal{U}_j(\mathfrak{g})(i)$ s.t. $i+2j \leq n$. So ~~$\mathcal{U}(\mathfrak{g})$~~ $\mathcal{U}(\mathfrak{g},e)$ inherits this filtration.

Thm: $Gr(\mathcal{U}(\mathfrak{g},e)) \cong \mathbb{C}[S]$

$\exists$ a poisson bracket on $\mathbb{C}[\mathfrak{g}^*]$: $F_1, F_2 \in \mathbb{C}[\mathfrak{g}^*]$, $\{F_1, F_2\}(\xi) = \xi(dF_1(\xi), dF_2(\xi)]$ $\xi \in \mathfrak{g}^*$

$\mathbb{C}[\xi]$ inherits this bracket and it agree with the one from $Gr(\mathcal{U}(\mathfrak{g},e))$

If e is regular nilpotent, $\mathcal{U}(\mathfrak{g},e) \cong Z(\mathcal{U}(\mathfrak{g}))$

II. transverse slices.

X – smooth alg. var. over $\mathbb{C}$. $Y \subseteq X$ a locally closed subvar. of X. $y \in Y$.

$S \subset X$ a locally closed subset is transverse slice to $y \in Y$ if $\exists\ U \in X$ open in analytic topo. and an iso: $f: (Y \cap U) \times S \xrightarrow{\sim} U$

s.t. $f|_{\{y\} \times S} : S \to S$ is id $\quad f|_{Y \cap U \times \{y\}} : Y \cap U \xrightarrow{id} Y \cap U$

If $G \curvearrowright X$, and $\mathbb{O} \subseteq X$ is an orbit, $e \in \mathbb{O}$.

example: Let $e \in \mathfrak{g}$ nilpotent. $\mathbb{O}_e$ be the orbit of e under Ad. action then S is transverse slice to $\mathbb{O}_e$ at e.

Recall $\tilde{\mathfrak{g}} = \{(x, b) \in \mathfrak{g} \times \mathcal{B} \mid x \in b.\} \simeq G \times_B b$ where B choosing borel.

$\exists$ a map $\tilde{\mathfrak{g}} \to \mathfrak{h} = b/[b,b]$

$(x, b) \longmapsto \bar{x} \in b/[b,b]$

$\mathfrak{h} \to \mathfrak{h}/W \quad \left(\mathbb{C}[\mathfrak{h}/W] = \mathbb{C}[\mathfrak{h}]^W\right).$

$$\begin{tikzcd} & \tilde{\mathfrak{g}} \arrow[dl] \arrow[dr] & \\ \mathfrak{g} \arrow[dr, dashed, "\rho"] & & \mathfrak{h} \arrow[dl] \\ & \mathfrak{h}/W & \end{tikzcd}$$

To obtain ρ, $\mathbb{C}[\mathfrak{h}]^W \simeq \mathbb{C}[\mathfrak{g}]^G \hookrightarrow \mathbb{C}[\mathfrak{g}]$

$\rho^{-1}(0) = \mathcal{N}$ — the nilpotent cone. $\left(I(\mathcal{N}) = \mathbb{C}[\mathfrak{g}]^G_+ \right)$.

In fact, every fiber of ρ satisfies a) it is normal b) union of f.m. G-orbits.

((b) $\Rightarrow$ c) has a dense open subset of regular element.)

Fact. $\rho|_S$ also has these properties (S being a transverse slice).

$(\rho|_S)^{-1}(0) = S \cap \mathcal{N}$ has these properties.

III IV. $\mu : \tilde{\mathfrak{g}} \to \mathfrak{g}$. $\mu^{-1}(\mathcal{N}) \longrightarrow \mathcal{N}$ is a resolution of singularities.

$$\mu^{-1}(\mathcal{N}) = \tilde{\mathcal{N}} \simeq T^*(G/B)$$

$\widetilde{S\cap\mathcal{N}} = \mu^{-1}(S\cap\mathcal{N})$ — resolution of sing. of $S\cap\mathcal{N}$.

We have $M_L \circlearrowright T^*(G/B)$ ~~$S\cap M$~~

$\widetilde{S\cap\mathcal{N}}$: Hamil. reduction of $T^*(G/B)$ w.r.t. M w. chara. χ.

We have $\mathcal{D}_\hbar(G/B)$ on $T^*(G/B)$, we can take the Hamil. reduction of this sheaf to $\widetilde{S\cap\mathcal{N}}$

$$\left(\mathcal{D}_\hbar(G/B) \big/ \mathcal{D}_\hbar(m - \chi(m)) \right)^{M'} \quad \text{call this } W_\hbar(\widetilde{S\cap\mathcal{N}})$$

Thm: We have an equivalence of categories

① Put a $\mathbb{C}^*$-action on $\widetilde{S\cap\mathcal{N}} = \{(x,b) \mid x \in S\cap\mathcal{N},\ x \in b\}$

$t\cdot(x,b) = (\rho(t)\cdot x, \rho(t)\cdot b)$ $\mu : \widetilde{S\cap\mathcal{N}} \longrightarrow S\cap\mathcal{N}$ is a $\mathbb{C}^*$-map.

② $\mathbb{C}^*$-equivariant good $W_\hbar(\widetilde{S\cap\mathcal{N}})$-mod $\xrightarrow{(\Gamma)^{\mathbb{C}^*}}$ f.g. $U(\mathfrak{g}, e)$-mod with central character $\cancel{\chi_e}$ ψ_{-e}

This is an equivalence of categories.

Thm (Skryabin)

C = f.g. $\mathcal{U}(\mathfrak{g})$-mod, on which $(m - \chi(m))_{m \in \mathfrak{m}_l}$ act ~~freely~~ locally nilpotently

$W_h : C \longrightarrow$ f.g. $\mathcal{U}(\mathfrak{g}, e)$-mod

$W_h(V) = \{ v \in V \mid m \cdot v = \chi(m) v \}$ is an equi. of categories.