Definitions of dAHA and dDAHA.

Let $\mathfrak{h}$ be a finite-dim. real vector space with a positive defined symmetric bilinear form $(\cdot,\cdot)$.
Let $\{\varepsilon_i\}$ be an orthonormal basis for $\mathfrak{h}$. Let $R \subset \mathfrak{h}$ be an irreducible root system on $\mathfrak{h}$.
We use the following notations:

R – roots, R_+ – pos. roots. $\Pi = \{\alpha_i\}$ simple roots.

$\alpha^\vee$ – coroots defined by $\alpha^\vee = 2\alpha/(\alpha,\alpha)$; $Q = \bigoplus_{\alpha \in R} \mathbb{Z}\alpha$ root lattice $Q^\vee$ – coroot lattice

$P = \mathrm{Hom}_{\mathbb{Z}}(Q^\vee, \mathbb{Z})$ weight lattice. W – the Weyl group of R. Σ – set of reflections.

$\forall \alpha \in R$, S_α is the corresponding reflection. $\forall \alpha_i \in \Pi$, $S_i = S_{\alpha_i}$.

dAHA Let $K : \Sigma \to \mathbb{C}$ be a function which is invariant under the conjugate action of W.
The dAHA $\mathcal{H}(K)$ is the quotient of the free product $\mathbb{C}W * S\mathfrak{h}$ by the relations.

$$S_i y - y^{S_i} S_i = K(S_i)\,\alpha_i(y) \quad y \in \mathfrak{h}.$$

Remark: $\forall a \neq 0$ we have an isomorphism. $\mathcal{H}(aK) \simeq \mathcal{H}(K)$

dDAHA We define it by using the trigonometric Cherednik-Dunkl operators.

Def (Cherednik). $\forall \varepsilon \in \mathfrak{h}$, we define the differential operator.

$$D_\varepsilon^{t,k} = t\partial_\varepsilon - \sum_{\alpha \in R_+} \frac{k_\alpha\,\alpha(\varepsilon)}{1-e^{-\alpha}}(1-S_\alpha) + \rho(k)(\varepsilon)$$

where:
- k is a function on R defined by $\alpha \mapsto k_\alpha$ and invariant under the W-action. i.e. $k_{g(\alpha)} = k_\alpha$ $\forall g \in W$.
- $\rho(k) = \frac{1}{2}\sum_{\alpha \in R_+} k_\alpha \alpha$.
- ∂_ε is the differential along ε.
- $D_\varepsilon^{t,k}$ is an operator acts on the space of trigonometric polynomials on $\mathfrak{h}/Q^\vee$.

Prop: $D_\xi(t,k)$ are commutative to each other.

Def: (dDAHA). The dDAHA $\mathcal{H}(t,k)$ is generated by W, $D_\xi(t,k)$ and the elements e^λ. $\lambda \in P$.

Remark: ① The original def. for dDAHA is from the affine / extended affine root system. Here is called the differential repn of dDAHA. Cherednik shows that they are equivalent.

② $\forall a \neq 0$, we have $\mathcal{H}(at, ak) \simeq \mathcal{H}(t,k)$. So there're two cases.

$t=0$, the classical case. $t \neq 0$ the quantum case.

Prop: The subalgebra of $\mathcal{H}(t,k)$ generated by W and Dunkl-Cherednik operators is isomorphic to the dAHA $\mathcal{H}(K)$ with $K(S) = \sum_{S:\,S=S_\alpha} k_\alpha$.

Now let we see some examples.

Example 1: Type A_{n-1} dAHA and dDAHA.

Type A_{n-1}. Root system: $\tilde{\mathfrak{h}}$ — n-dimensional real vector space with $(\,,\,)$. and ε_i.

Let $\mathfrak{h} \subset \tilde{\mathfrak{h}}$ to be the subspace of $\tilde{\mathfrak{h}}$ consisting all the vectors orthogonal to $\varepsilon_1 + \cdots + \varepsilon_n$.

Define $R = \{\varepsilon_i - \varepsilon_j \mid i \neq j\}$ $R_+ = \{\varepsilon_i - \varepsilon_j \mid i<j\}$ $\Pi = \{\alpha_i = \varepsilon_i - \varepsilon_{i+1}\}$.

type A_{n-1} root system. There are only one parameter k.

So the dAHA is defined to be the associative algebra over $\mathbb{C}$ generated by

s_i $i=1,\dots,n-1$, y_i $i=1,\dots,n$ with relations.

① $\sum_{i=1}^n y_i = 0$. ② $[y_i, y_j] = 0$ $i \neq j$ ③ $s_i y_i - y_{i+1} s_i = K_1$

④ $s_i y_j - y_j s_i = 0$ $\forall j \neq i, i+1$, ⑤ s_i satisfies the Coxeter relations in S_n.

the dDAHA is defined to be the associative algebra over $\mathbb{C}$ generated by x_i, y_i, s_i with relations. ① ~ ⑤ as dAHA.

⑥ $\prod x_i = 1$. ⑦ $s_i x_i = x_{i+1} s_i$ ⑧ $s_i x_j = x_j s_i$ $j \neq i, i+1$. ⑨ $[x_i, x_j] = 0$ $\forall i, j$

⑩ $[y_i, x_j] = k x_i s_{ij}$ $i<j$ $[y_i, x_j] = k_1 x_j s_{ij}$ ⑪ $[y_i, x_i] = t x_i - k x_i \sum s_{ik} - k \sum x_i s_{ik}$

We use another set of generators.

Define $\tilde{y}_i = y_i - k\sum_{j<i} S_{ij}$ Then use $\tilde{y}_i$, x_i, S_i, the dDAHA is defined by relations: $\prod X_i = 1$ $\sum\left(\tilde{y}_i + k\sum_{j<i} S_{ij}\right) = 0$ $S_{ij}\tilde{y}_i = \tilde{y}_j S_{ij}$

$[S_{ij}, X_\ell] = [S_{ij}, \tilde{y}_\ell] = 0$ $[\tilde{y}_i, \tilde{y}_j] = kS_{ij}(\tilde{y}_i - \tilde{y}_j)$

$[\tilde{y}_i, X_j] = (kS_{ij} - \frac{1}{n})X_j$

Type BC_n. dAHA and dDAHA

see paper.

Now let we recall the Arakawa-Suzuki construction for the type A_{n-1} dAHA.

Let we consider A_{n-1} dAHA $\mathcal{H}_n(\kappa)$.

Let $\mathfrak{sl}_N$ – complex Lie algebra. Then we know that we have a triangular decomposition

$$\mathfrak{sl}_N = \mathfrak{n}_+ \oplus \mathfrak{h}_N \oplus \mathfrak{n}_-$$

Let $\{e_\alpha \in (\mathfrak{sl}_n)_\alpha \mid \alpha \in R_N\}$ be the set of root vector with $(e_\alpha, e_{-\alpha}) = 1 \ \forall \alpha \in R^+$.

Let $\{\underset{h_i}{\underset{\parallel}{\alpha_i^\vee}} \in \mathfrak{h}_N\}$ be the set of coroot and $\{h^i\}$ be the dual basis of h_i s.t.

$$(h^i, h_j) = \delta_{ij}.$$

Now define $$\Omega = \sum_{i=1}^{n-1} h^i \otimes h_i + \sum_{\alpha \in R_n^+} (e_\alpha \otimes e_{-\alpha} + e_{-\alpha} \otimes e_\alpha)$$

Remark: Ω can be constructed in another way. Consider $\{b_i\}$ as an orthonormal basis for $\mathfrak{sl}_n$. Then $\Omega = \sum b_i \otimes b_i$

Category $\mathcal{O}$

Let $U(\mathfrak{sl}_N)$ be the universal enveloping algebra of $\mathfrak{sl}_N$. $\mathcal{O}(\mathfrak{sl}_N)$ is the category with obj. are $\mathfrak{sl}_N$-modules. M such that

① X is finitely generated over $U(\mathfrak{sl}_N)$. ② M is $\mathfrak{n}_+$-locally finite i.e. $U(\mathfrak{n}_+)v$ is f.d. $\forall v \in M$.

③ $M = \bigoplus_{\lambda \in \mathfrak{h}_N^*} M_\lambda$ where $M_\lambda = \{v \in M \mid hv = \langle \lambda, h \rangle v \ \forall h \in \mathfrak{h}_N\}$.

We have another category. Let we consider $(\mathfrak{g}\ \mathfrak{sl}_N \times \mathfrak{sl}_N, \mathfrak{sl}_N)$.

where $\mathfrak{sl}_N \subset \mathfrak{sl}_N \times \mathfrak{sl}_N$ as the diagonal image. Let $\mathfrak{g} = \mathfrak{sl}_N$.

Then we can define $HC(\mathfrak{g})$ – the category of Harish-Chandra bimodules

i.e. the category of $\mathfrak{g}$-bimodules which is finite under the adjoint action of $\mathfrak{g}$.

For any. $M \in HC(\mathfrak{g})$. we define

$$F_n(M) = (M \otimes (\mathbb{C}^N)^{\otimes n})^{\mathfrak{g}}.$$

where the $\mathfrak{g}$ action on M by adjoint action and action on $\mathbb{C}^N$ by vector repn.

Now. define S_n acts on $F_n(M)$ by permutation of the n-tensors.

Define. $\tilde{y}_i = \frac{N}{n}\Omega_{0i}$ where Ω_{0i} acts on $F_n(M)$ as follows.

$\Omega_{0i} = \sum (b_j)_0 \otimes (b_j)_i$ the 1-st factor acts on M. the second one acts on i-th $\mathbb{C}^N$ as the vector repn.

Thm (AS): The action of $\tilde{y}_i$ and S_n on $F_n(M) = (M \otimes (\mathbb{C}^N)^{\otimes n})^{\mathfrak{g}}$ defines a repn of $\tilde{\mathcal{H}}_n(1)$, the repn of dAHA (central extension) and $F_n: HC(\mathfrak{g}) \longrightarrow \tilde{\mathcal{H}}_n(1)$. is a functor.

Remark ① Here, the reason why the above defines a repn of $\tilde{\mathcal{H}}_n(1)$ not $\mathcal{H}_n(1)$ is. the operator $\sum(\tilde{y}_i + k\sum_{j<i} S_{ij})$ need not to be zero. In fact, in the AS paper it is zero by some translations. But here we use a more general statement from CEE.

② This is also not the ~~general~~ original statement of AS. In fact they construct a functor $\beta \in \mathfrak{h}_N^*$ $\tilde{F}_{n,\beta}: \mathcal{O}(\mathfrak{sl}_N) \longrightarrow \mathcal{H}_n(1)$. not for $HC(\mathfrak{g})$. But we can use Gelfand-Naimark construction, i.e. a functor from $\mathcal{O}(\mathfrak{sl}_N) \longrightarrow HC(\mathfrak{g})$.

For any $M \xmapsto{\Phi_\beta} \mathrm{Hom}_f(M_\beta, M)$ f- mean finite under adjoint action of $\mathfrak{g}$. Thus. The functor they constructed is the following.

$$\mathcal{O}(\mathfrak{sl}_N) \longrightarrow \mathcal{H}_n(1)\text{-mod.}$$

$$M \longmapsto \left(\mathrm{Hom}_{\mathfrak{g}}(M_\beta, M \otimes (\mathbb{C}^N)^{\otimes n})\right)^{\mathfrak{g}} = \left(\mathrm{Hom}_f(M_\beta, M) \otimes (\mathbb{C}^N)^{\otimes n}\right)^{\mathfrak{g}}$$

especially, when we take $M = M_\lambda$ to be a verma module

Then $\mathrm{Hom}_{\mathfrak{g}}(M_\beta, M_\lambda) := \mathcal{H}_{\beta,\lambda}$ the principal series of $SL(n,\mathbb{C})$.

and the functor will be

$$\mathcal{H}_{\beta,\lambda} \xmapsto{F_\beta} (\mathcal{H}_{\beta,\lambda} \otimes V^{\otimes n})^{\mathfrak{g}}.$$

<u>Images of this functor.</u> Let we first describ. one kinds of repn of dAHA.

standard modules and their simple quotients.

For $a, b \in \mathbb{C}$, we consider a pair $[a,b]$. Call $[a,b]$ is a segment if $b-a+1 \in \mathbb{Z}_{\geq 0}$

For a segment $[a,b]$ such that $b-a+1 = n$, there exists a unique one-dimensional repn

$\mathbb{C}_{[a,b]} = \mathbb{C}\mathbb{1}_{[a,b]}$ of $\mathcal{H}_n$ such that

$$w \cdot \mathbb{1}_{[a,b]} = \mathbb{1}_{[a,b]} \qquad y_i \mathbb{1}_{[a,b]} = (a+i-1)\mathbb{1}_{[a,b]} \quad i = 1, \dots, n.$$

Now let $\Delta := ([a_1, b_1], \dots, [a_k, b_k])$ be an ordered sequence of segments such that

$b_i - a_i + 1 = n_i$ and $n = \sum_{i=1}^{k} n_i$

Regard $H_{n_1} \otimes \cdots \otimes H_{n_k} \subset H_n$ as a subalgebra.

Define
$$M(\Delta) = H_n \otimes_{H_{n_1} \otimes \cdots \otimes H_{n_k}} (\mathbb{C}_{[a_1,b_1]} \otimes \cdots \otimes \mathbb{C}_{[a_k,b_k]}).$$
$$\simeq \mathbb{C}\left[W_n / W_{n_1} \times \cdots \times W_{n_k} \right]$$

as $\mathbb{C}[W_n]$-mod.

Thm (AS):
$$F_\lambda(M_\mu) \simeq \begin{cases} M(\lambda,\mu) & \lambda-\mu \in P(V_N^{\otimes n}) \\ 0 & \text{otherwise} \end{cases} \qquad V_N = \mathbb{C}^N$$

where $P(V_N^{\otimes n})$ is the weight space as $\mathfrak{sl}_n$-module.

Type A_{n-1} dDAHA.

In this case, we need to define two kinds of operators x_i, y_i.

Let $G = SL_N(\mathbb{C})$. $\forall\, b \in \mathfrak{g} = \mathrm{Lie}(G)$. Let L_b be the right invariant vector field on G.

Now consider the category of D-mod on G. Let $M \in D(G)$. (it is on $\frac{G\times G}{G}$)

define $F_n(M) = (M \otimes (\mathbb{C}^N)^{\otimes n})^{\mathfrak{g}}$ where $\mathfrak{g}$ acts on itself by conjugation.

Define the following operators: $X_i = \sum_{j,l} A_{jl} \otimes (E_{lj})_i \quad \tilde{Y}_i = \frac{N}{n} \sum_p L_{b_p} \otimes (b_p)_i$

where A_{jl} is a function on G. as take the jl-th element of the matrix.

~~Re~~

Thm: (CEE) We have an exact functor $F_n : D(G) \longrightarrow H_n(1)$-module.

~~Now let we consider the type BC_n dAHA and dDAHA.~~

~~Notations We will use the following notations:~~

~~$p, q \in \mathbb{N}$. $N = p+q$. E_{ij} — $N\times N$ matrix $J = \begin{pmatrix} I_p & \\ & -I_q \end{pmatrix}$~~

~~Consider the group $U(p,q)$. Then $\mathfrak{g} = \mathrm{Lie}(U(p,q))_{\mathbb{C}} = \mathfrak{gl}_N(\mathbb{C})$.~~

~~Let $K \subset U(p,q)$ $k_{\mathbb{C}} := k = \mathfrak{gl}_p(\mathbb{C}) \times \mathfrak{gl}_q(\mathbb{C})$.~~

~~Let k_0 be the subalgebra of k with trace 0 matrix.~~

~~We define a character of k χ_μ $\forall \mu \in \mathbb{C}$ by follows:~~

~~$\chi_\mu\left(\begin{pmatrix} X_1 & \\ & X_2 \end{pmatrix}\right) = \mu(q\,\mathrm{tr} X_1 - p\,\mathrm{tr} X_2)$.~~

Now let we consider the type BC_n dAHA and dDAHA.

dAHA: we use the following notations: $p, q \in \mathbb{N}$ $N = p+q$.

$\mathfrak{g} = \mathfrak{gl}_N(\mathbb{C})$ be the complexification of the Lie alg Lie group $U(p,q)$.

$\mathfrak{k} = \mathfrak{gl}_p(\mathbb{C}) \times \mathfrak{gl}_q(\mathbb{C})$ be the complexification of $K = U(p) \times U(q)$

$\mathfrak{k}_0$ = subalgebra of trace zero matrices.

For any $\mu \in \mathbb{C}$, we define a character χ_μ of $\mathfrak{k}$ by $\chi_\mu\begin{pmatrix} X_1 & \\ & X_2 \end{pmatrix} = \mu(q\,\mathrm{tr}X_1 - p\,\mathrm{tr}X_2)$

For any $\mathfrak{k}_0$-module Y, the μ-invariants is defined by

$$Y^{\mathfrak{k}_0,\mu} = \{ v \in Y \mid k \cdot v = \chi_\mu(k) v \quad \cancel{\forall v \in Y} \quad k \in \mathfrak{k}_0 \}$$

Let $V = \mathbb{C}^N$ be the vector repn of $\mathfrak{g}$. For any $\mathfrak{g}$-module M, we define

$$F_{n,p,\mu}(M) = (M \otimes V^{\otimes n})^{\mathfrak{k}_0,\mu}.$$

The Weyl group action is defined as follows.

We have generators S_i $i = 1, \ldots, n-1$ and γ_n.

action of S_i on $F_{n,p,\mu}$ as before. γ_n acts as multiply J to the n-th factor.

Check this defined a W_{BC_n}-module structure on $F_{n,p,\mu}(M)$.

Now define $\tilde{y}_k = -\sum_{i \mid j} E_{ij} \otimes (E_{ji})_k$ $\quad$ $i|j$ means $i \le p$ $j > p$ or $i > p$ $j \le p$.

and action of $\tilde{y}_k$ on $F_{n,p,\mu}(M)$ as before.

Thm (EFM): The above action of $\tilde{y}_k$ and W_{BC_n} defines a repn of $\mathcal{H}(\kappa_1, \kappa_2)$ with parameters. $\kappa_1 = 1$ $\kappa_2 = p - q - \mu N$.

Then we have a functor from the category of $\mathfrak{g}$-modules to the category of $\mathcal{H}(\kappa_1, \kappa_2)$-modules with above parameters.

Now let we consider the dDAHA case. We use the following setting.

Consider $G = GL_N(\mathbb{C})$ $K = GL_p \times GL_q$. Consider the homogeneous space G/K.

Let L_x $x \in \mathfrak{g}$ be the vector field on G generated by the left action of x

i.e. $L_x f(g) = \frac{d}{dt}\Big|_{t=0} f(e^{tx}A)$

Let $D^{\chi_\lambda}(G/K)$ be the sheaf of differential operators on G/K twisted by the character χ_λ (local sections of $D^{\chi_\lambda}(G/K)$ act naturally on χ_λ-twisted functions on G/K.

i.e. $R_z f = \chi_\lambda(z) f$ $z \in \mathfrak{k}$. where R_z is the left inv. vector field corresponding to the right Translation.)

L_x can be regard as the global sections of $D^{\chi_\lambda}(G/K)$

Now let M be a $D^{\chi_\lambda}(G/K)$-module. Then via $x \longmapsto L_x$, we can give M a $\mathfrak{g}$-module structure. Define $F^{\lambda}_{n,p,\mu}(M) = (M \otimes V^{\otimes n})^{k_0, \mu}$.

We define the W-action on $F^{\lambda}_{n,p,\mu}(M)$ as before.

For $k = 1, \dots, n$, we define.

$$X_k = \sum_{i,j} (AJA^{-1}J)_{ij} \otimes (E_{ij})_k \qquad \tilde{y}_k = \sum_{i,j} L_{ij} \otimes (E_{ji})_k.$$

Here. $(AJA^{-1}J)_{ij}$ is a function on G/K. $L_{ij} = L_{E_{ij}}$.

Thm (EFM). X_k, $\tilde{y}_k$, W defines a $\mathcal{H}(t, k_1, k_2, k_3)$-module structure on $F^{\lambda}_{n,p,\mu}(M)$ with para:

$$t = \frac{2n}{N} + (\lambda+\mu)(q-p) \quad k_1 = 1 \quad k_2 = p - q - \lambda N \quad k_3 = (\lambda - \mu) N$$

Then we have a functor from the category of $D^{\chi_\lambda}(G/K)$-modules. to the category of repn of type BC_n dDAHA with such para.

Images of the functor $F_{n,p,\mu}$. on Principal Series modules.

Let us recall the definition of Principal Series modules for reductive groups.

Let G be a reductive gp $K \subset G$ be maximal compact subgroup.

Let $\mathfrak{g}_0$, $\mathfrak{k}_0$ be the real Lie algebra of G and K.

For $\mathfrak{g}_0$ we have ~~an~~ Cartan decomposition $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}$.

Let $\mathfrak{a} \subset \mathfrak{p}$ be the maximal abelian algebra. in $\mathfrak{p}$. A be the corresponding Lie group.

M be the centralizer of A in K.

Then we have Iwasawa decomposition $G = K \times N \times A$.

For $\nu \in \mathfrak{a}_{\mathbb{C}}^*$, we can define a character of A. by $\nu(\exp X) = \exp \nu(X)$. $X \in \mathfrak{a}$.

Let Δ = nonzero roots of $\mathfrak{a}$ in $\mathfrak{g}$ — restricted roots. and let $\rho = \frac{1}{2} \sum_{\alpha \in \Delta^+} \alpha \in \mathfrak{a}^*$.

Let W be an irreducible repn of M.

Consider the space. $H_{\pi\otimes\nu} = \{ f : G \to W \mid f$ is measurable, $f|_K$ is square integrable

$\forall\, m \in M,\ a \in A\ n \in N\ g \in G$

$f(gman) = a^{-(\nu+\rho)}\, \pi(m^{-1}) f(g) \}$

Define a repn of G, $\pi_{\pi\otimes\nu}$ on $H_{\pi\otimes\nu}$ by

$$\pi_{\pi\otimes\nu}(g) f(x) = f(g^{-1}(x)).$$

This repn. is called the principal series module of G with parameter π and ν.

Prop: ① $\pi_{\pi\otimes\nu} = \mathrm{Ind}_{MAN}^{G} (\pi\otimes\nu\otimes 1)$

② $\mathrm{Res}_K^G H_{\pi\otimes\nu} = \mathrm{Ind}_M^K W$.

Principal Series modules for $U(p,q)$.

Now let $G = U(p,q)$ $\quad p, q \in \mathbb{N}$. $\quad N = p+q$ $\quad$ assume $p \leq q$.

$K = U(p) \times U(q)$.

$$\mathfrak{a} \cong \mathbb{R}^p = \left\{ \left(\begin{array}{c|c|c} 0 & \begin{matrix} a_1 & & \\ & \ddots & \\ & & a_p \end{matrix} & \\ \hline \begin{matrix} a_1 & & \\ & \ddots & \\ & & a_p \end{matrix} & 0 & 0_{q-p} \\ \hline 0 & 0 & 0 \end{array} \right) \;\middle|\; a_i \in \mathbb{R} \right\}.$$

A be the Lie group corr. to $\mathfrak{a}$.

M as the centralizer of A in K is.

$$M = \left\{ \begin{pmatrix} g_1 & & & & & & \\ & \ddots & & & & & \\ & & g_p & & & & \\ & & & g_1 & & & \\ & & & & \ddots & & \\ & & & & & g_p & \\ & & & & & & S \end{pmatrix} \;\middle|\; g_i \in U(1),\ S \in U(q-p) \right\} = \Delta(\underbrace{U(1) \times \cdots \times U(1)}_{p}) \times U(q-p)$$

$$\Delta(U(1)\times\cdots\times U(1)) \subset \overbrace{U(1) \times \cdots \times U(1)}^{2p}$$

The restricted roots are
$$\left.\begin{array}{ll} \pm a_i \pm a_j,\ \pm a_i,\ \pm 2a_i & (p \neq q) \\ \pm a_i \pm a_j,\ \pm 2a_i & (p = q) \end{array}\right\} BC_n \text{ type.}$$

Now suppose we have an irreducible M-module W. Then we have a Principal serie module. $H_{\pi \otimes \nu}$ for $\nu \in \mathfrak{a}_{\mathbb{C}}^*$.

Thus. since it is admissble, we can take $(H_{\pi\otimes\nu})_{HC}$ as the Harish-Chandra module of $H_{\pi\otimes\nu}$ and then. it is a $(\mathfrak{g}, K)$-module.

applying the functor. we get
$$\left((H_{\pi\otimes\nu})_{HC} \otimes V^{\otimes n} \right)^{k_0, \mu} = \left(H_{\pi\otimes\nu} \otimes V^{\otimes n} \right)^{k_0, \mu}.$$

Now we need describ the above space. First notice that

(k_0, μ)-invariants is equivalent to consider the k_0 invariants of $H_{\pi\otimes\nu} \otimes V^{\otimes n} \otimes \mathbb{1}^{k_0}_{(-\mu)}$ here $\mathbb{1}^{k_0}_{(-\mu)}$ is the 1-dim module of k_0 defined by the character χ_μ.

Secondly, since $k = k_0 \oplus \mathfrak{u}(1)$ and the $\mathfrak{u}(1)$ action is described by the following

$g \in \mathfrak{u}(1)$. $\quad g \cdot (H_{\pi\otimes v} \otimes V^{\otimes n} \otimes \mathbb{1}^{k_0}_{(-\mu)})$.

$$= g \cdot (\mathrm{Ind}_M^K W \otimes V^{\otimes n} \otimes \mathbb{1}^{k_0}_{(-\mu)})$$

Suppose. $\quad W = V(n_1) \otimes \cdots \otimes V(n_p) \otimes V(r) \quad r = (r_1 \cdots r_{q-p})$.

So $\quad g \cdot \mathrm{Ind}_M^K W = \begin{pmatrix} g & & \\ & \ddots & \\ & & g \end{pmatrix} \mathrm{Ind}_M^K W = (\sum n_i + \sum r_i) g \cdot \mathrm{Ind}_M^K W$.

$$g \cdot V^{\otimes n} = n g \cdot V^{\otimes n}.$$

$$g \cdot \mathbb{1}^{k_0}_{(-\mu)} = 0.$$

Thus. $\quad H_{\pi\otimes v} \otimes V^{\otimes n} \otimes \mathbb{1}^{k_0}_{(-\mu)} \otimes \mathbb{1}^{\mathfrak{u}(1)}_{(-\sum n_i - \sum r_i - n)}$. $\quad$ Let $\alpha = -\sum n_i - \sum r_i - n$

has a trivial g action.

and. $\quad \left(H_{\pi\otimes v} \otimes V^{\otimes n} \otimes \mathbb{1}^{k_0}_{-\mu} \otimes \mathbb{1}^{\mathfrak{u}(1)}_{(-\sum n_i - \sum r_i - n)} \right)^{k} = (H_{\pi\otimes v} \otimes V^{\otimes n})^{k_0, \mu}$

$$\| $$

$$(H_{\pi\otimes v} \otimes V^{\otimes n} \otimes \mathbb{1}^{k_0}_{-\mu} \otimes \mathbb{1}^{\mathfrak{u}(1)}_{\alpha})^{K}.$$

Now we use the following identity.

$$(H_{\pi\otimes v} \otimes V^{\otimes n} \otimes \mathcal{I})^K = (\mathrm{Res}^G_K H_{\pi\otimes v} \otimes (\mathrm{Res}^G_K V)^{\otimes n} \otimes \mathrm{Res}^G_K \mathcal{I})^K$$

$$= (\mathrm{Ind}_M^K W \otimes V^{\otimes n} \otimes \mathcal{I})^K$$

$$= (W \otimes V^{\otimes n} \otimes \mathcal{I})^M.$$

Now the question is how to use this to find the $\mathcal{H}$-structure.

Some preknowelge of Repn. theory.

① Repn of unitary groups. For $U(n)$. the irr. repn is classified by dominant weights i.e. a sequence of integers $(m_1, \ldots, m_n)$ $\quad m_1 \geq m_2 \geq \cdots \geq m_n$. (highest wt / lowest wt)

For $r = (r_1, \ldots, r_n)$ we use $V(r)$ to denote such irr. repn. of $U(n)$

So for M, the irreducible repn will be.

$$\bigotimes_{i=1}^{p} V(\alpha_i+\beta_i)\otimes V(\xi)$$ where $V(\alpha_i+\beta_i)$ is a 1-dim rep of $\begin{pmatrix} g & \\ & g\end{pmatrix}$ with (α_i,β_i)-weight. It is only depends on $\alpha_i+\beta_i$. $\xi=(\xi_1,\dots,\xi_{q-p})$

$\mathbb{C}^N$ as a $U(N)$-module has weight $V=V(1,0,\dots,0) := V_N$ and we have the following lemma:

$$V(\xi)\otimes V_{q-p} = V(\xi_1+1,\dots,\xi_{q-p})+\dots+V(\xi_1,\dots,\xi_{q-p}+1).$$

The non dominent item will be deleted.

More generally we have $V(\xi)\otimes V_{q-p}^{\otimes n} = \sum_{\sum n_i=n} k_{n_1\dots n_{q-p}} V(\xi_1+n_1,\dots,\xi_d+n_d)$.

Now back to our case.

Suppose $W=V(n_1)\otimes\cdots\otimes V(n_p)\otimes V(\xi)$ as an irreducible repn of M.

Then. $W\otimes V^{\otimes n}\otimes \mathcal{I}$.

$$\sum \bigotimes_i V(n_i+\alpha_i+\beta_i)\otimes V(\xi_1+m_1,\dots,\xi_{q-p}+m_{q-p})\otimes \mathcal{I}\cdot k$$

$$=\sum \bigotimes_i V(n_i+\alpha_i+\beta_i-\mu(q-p)+\frac{\alpha}{N})\otimes V(\xi_1+m_1+\frac{\alpha}{N}+\mu p,\dots,\xi_{q-p}+m_{q-p}+\mu p+\frac{\alpha}{N})$$

Then it is K-inv. iff it is trivial module.

$$\begin{cases} n_i+\alpha_i+\beta_i=\mu(q-p)-\frac{\alpha}{N} & i=1,\dots,p \\ \xi_j+m_j+\frac{\alpha}{N}+\mu p=0 & j=1,\dots,q-p\end{cases}$$ all integers.

also we have $\sum n_i+\sum\xi_j = -\sum(\alpha_i+\beta_i)-\cancel{\sum\xi_j}\sum m_j-\alpha$.

$$=-n+\sum n_i+\sum\xi_j+n$$

$$=0.$$

Thus the only condition is need $\alpha=-\sum n_i-\sum\xi_i-n$ is an integer.

Now let we consider $\alpha=0$ i.e. $\sum n_i+\sum\xi_i=-n$ case

Conclusion is the following:

Suppose $W = V(n_1)\otimes\cdots\otimes V(n_p)\otimes V(\xi)$ is an irreducible repn of W and $\sum n_i + \sum \xi_i = -n$. Then.

① $\left(H_{\pi\otimes v}\otimes(\mathbb{C}^N)^{\otimes n}\right)^{k_0,\mu} \neq \phi$ iff

(i) all $n_i - \mu(q-p)$ are non-positive integers.

(ii) $\xi_\mu = (\xi_1+\mu p, \dots, \xi_{q-p}+\mu p)$ is a dominant weight for $U(q-p)$. with $\xi_1+\mu p \le 0$.

② If above condition satisfied, we have.

$$\dim F_{n,p,\mu}(H_{\pi\otimes v}) = n! \prod_{i=1}^{p} 2^{|n_i-\mu(q-p)|} \Big/ \prod_{i=1}^{p} |n_i - \mu(q-p)|! \ \prod h_k(\xi^\mu)$$

where ξ^μ is the hook length of the partition $(-\xi_{q-p}-\mu p, \dots, -(\xi_1+\mu p))$.

The next question is what is this module? How W and y_k acts on it. we still only consider $\alpha=0$ case.

look at. $(H_{\pi\otimes v}\otimes V^{\otimes n}\otimes I)^K = (W\otimes V^{\otimes n}\otimes I)^M$.

$$f\otimes v \longmapsto f|_K(e)\otimes v.$$

Thus. we can describ y_i acts on $f|_K(e)\otimes v$.

Step 1: Consider y_i's first tensor factor, decompost it into Iwasawa decomposition $\mathfrak{g} = k_0 \oplus \mathfrak{a} \oplus \mathfrak{n}$.

So the action of $\mathfrak{n}$ will be zero. $\mathfrak{a}$ will given by v.

only left with k_0 move it to ~~seeto~~ second factor by X_μ-invariant.

Step 2: Construct a common ~~weight eigh~~ eigenvector for y_k $k=1,\dots,n$.

This can be done. by using the Jucys-Murphy elements in the Specht module of S_n

Step 3. Use these eigenvectors construct a module $\mathfrak{P}$ and induces it to an $\mathcal{H}$-module $\tilde{\mathfrak{P}}$ prove that it is exactly. $F_{n,p,\mu}(H_{\pi\otimes v})$.

Thm: Let P be a vector space. Span $\{\omega_1,\dots,\omega_d\}$. $d=\dim S^{\xi^{\mu}}$.

and ω_i s.t. ① $y_k w_i = \lambda_{k,i} w_i$.

② w_i is $S_{n_1}\otimes\cdots\otimes S_{n_p}$ - invariant

③ $\mathbb{Z}_2^{n_3}$ acts by (-1) on w_i

~~Let P=~~ ④ S_{n_3} acts as the Specht-module of S_{n_3}.

So it defines a module of $\Gamma\ltimes\mathbb{C}[y_1,\dots y_n]$, $\mathfrak{P}$

where $\Gamma\subset S_n\ltimes(\mathbb{Z}_2)^n$

$\Gamma=(S_{n_1}\times\cdots\times S_{n_p}\times S_{n_3})\ltimes(1\times\cdots\times 1\times(\mathbb{Z}_2)^{n_3})$.

and. $F_{n,p,\mu}(H_{\pi\otimes v})\simeq \mathrm{Ind}_{\Gamma\ltimes\mathbb{C}[y_1\cdots y_n]}\mathfrak{P}$

A method to check this answer. — Infinitesimal characters.

We still consider $\alpha=0$. suppose $n=1$, rank 1 case.

Let we consider the Casimir elements in $\mathfrak{gl}_N(\mathbb{C})$.

$C_k=\sum E_{i_1i_2}\cdots E_{i_ki_1}$.

Lemma 1: $y_1^2=-\frac{1}{3}C_3+\frac{1}{2}\left(\frac{p+q}{3}+\mu(q-p)\right)C_2+\frac{(p+q)^2}{12}-\frac{1}{3}$

$+\frac{1}{4}(p-q)^2\mu^2-\frac{1}{6}pq(p-q)(p+q)\mu(1-\mu^2)$. $\quad C_i=C_i\otimes 1$.

Lemma 2 (Vogan): The infinitesimal character of $H_{\pi\otimes v}$ is (λ,v)

where $\lambda=\lambda_0+\rho(m)$ $\forall z\in Z(\mathfrak{g})$ ~~and~~. z acts on $H_{\pi\otimes v}$ by

$(\lambda,v)\,\xi(z)$ where ξ is the Harish-Chandra map for G

Then as operators on $F_{n,p,\mu}$, we have two way to compute y_1^2.

① From the eigenvector we find.

② From the infinitesimal character.

Two results are match!