can be replaced any reductive alg

can be replaced by any associative commutative algebra.

Lie algebra of matrices

Finite dimensional reps of $gl_r \otimes \mathbb{C}[x^1, \ , x^d]$

can see gl matrices of polynomials with usual commutator

$$\begin{pmatrix} P_{11} & P_{12} & \cdot \\ \cdot & & \\ \cdot & & \\ \cdot & & \end{pmatrix}$$

$\lambda \in \mathfrak{h}^+$

$\lambda : b \to \mathfrak{h}$

$gl_r \supset b \supset \mathfrak{h}$

upper triangular

diagonal matrices

Definition

v is a highest weight (λ, z) vector

λ weight as above

$z \in \mathbb{C}^d$

$$(g \otimes P) v = \lambda(g) P(z) v$$

$g \in b \quad P \in \mathbb{C}[x^1, \ , x^d]$

Definition

Weyl module $W_\lambda(z)$ is the maximal f.d representation gen a h.w (λ, z) - vector where by maximal we mean that any fin dim rep generated by such a vector is a quotient of $W_\lambda(z)$.

Proposition

(i) $W_\lambda(z)$ exists

(ii) $W_\lambda(z) \neq 0 \iff \lambda$ is dominant $\lambda(E_{ii}) = \xi_i \quad \xi_1 \geq \xi_2 \geq \geq \xi_t$ $\xi_i - \xi_j \in \mathbb{Z}$

(iii) shift automorphism of $\mathbb{C}[x^1, , x^d]$ produces an isomorphism

$$W_\lambda(z_1) \xrightarrow{\sim} W_\lambda(z_2)$$

(iv) Any finite dimensional module gen by a common eigenvector of $b \otimes \mathbb{C}[\cancel{x}^1 \cancel{x}^d]$ is a quotient of $W_{\lambda_1}(z_1) \otimes \quad \otimes W_{\lambda_m}(z_m)$

$\vdash$

$n=2 \quad \mathfrak{gl}_2 \quad e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad c = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$v \qquad b = \langle e, h, c \rangle \qquad b \otimes \mathbb{C}[x^1, \ , x^d]$

multiplication in universal enveloping alg

$W_\lambda(z)$ is spanned by $(f P_1 \cdot f P_2 \cdot \quad \cdot f P_k) v$

$\lambda(h) = m \in \mathbb{Z}_{\geq 0}$

$k \leq m \qquad 0 = (f \otimes 1)^{m+1} v$

$0 = (e P_1 \ \ e P_n)(f \otimes 1)^{m+1} v = c (f \otimes P_1 P_2 \ \ P_n) v$

$p_i(z) = 0$

p has a zero of mult m at z

$(f \otimes P) v = 0$

Proposition

$W_\lambda(\mathbb{Z})$ is generated by v with relations

$$(g\otimes P)v = \lambda(g)P(z)v$$

$$(f\otimes 1)^{m+1}v = 0$$

$\mathfrak{sl}_2 \otimes \mathbb{C}[x^1, \dots, x^d]$

d	$\dim W_{m\omega}$ (fundamental weight)	$\dim W_{m\omega}^k$	Conj. $\exists$ polynomials s.t. $P_k(m) / P_k(\dots)$
0	$m+1$	1	$P_k(m)=1$
1	2^m	$\begin{bmatrix} m \\ k \end{bmatrix}$	$P_k(m) = m(m-1)\dots(m-k+1)$ Chari, Pressley
2	$C_{m+1} = \frac{(2m+2)!}{(m+2)!(m+1)!}$		$P_k(m) = (m+1)m \dots (m-k+2)\cdot m(m-1)\dots(m-k+1)$ [Feigin, SL]
3	$D_{m+1} = 2\frac{(3m+3)!}{(2m+3)!(m+2)!}$		$P_k(m) = (m+k+1)(m+k)\dots(m-k+2)\cdot(2m-k+2)(2m-k+1)\dots(2m)$
	A 000139		

lacking integer roots

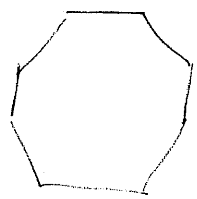
Conjecture

$d\geq 3$ polynomials become non-factorizable

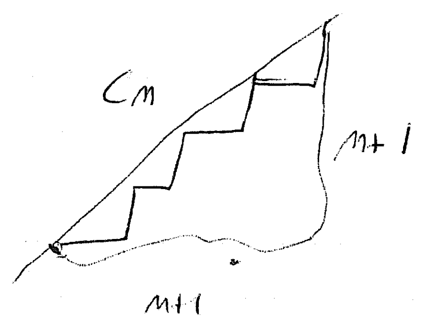
NOTATION $W^k_{m\omega}$ eigenspace of $h\otimes 1$ with eigenvalue $m-2k$

$k = 0, \dots, m$

C_{n+2} = # triangulations of a polygon with $2n+2$ sides

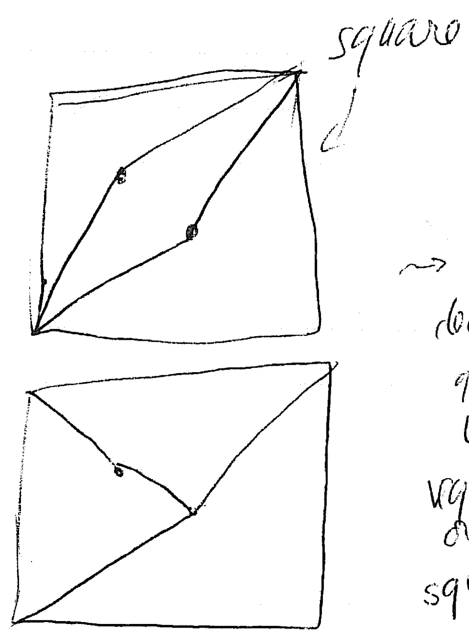


$2n+2=8 \quad n=3 \rightsquigarrow C_6$



→ number of paths that are below but can touch diagonal

Narayana numbers ⇝ paths with k peaks



⇝ count possible decomposition in quadrilaterals with vertices equal to those points or to vertices of the square

Generating function of $W_{n,d}$

$$W_d(X) = \sum X^n \dim W_{n,d}$$

$W_1(x) = \frac{1}{1-2X}$ $\left(W_2(x)\right)^2 =$

diagonal harmonics, diagonal coinvariance $\mapsto$

$DH_n = \mathbb{C}[x_1^1, \ldots, x_1^d, x_2^1, \ldots, x_2^d, \ldots, x_n^d] \Big/ I\left(\begin{array}{c}\Sigma_n\text{-invariants} \\ \text{with zero} \\ \text{at the origin}\end{array}\right)$

$\circlearrowleft$
Σ_n (interchanges groups of variables)
$\downarrow$
symmetric group

$d=1$ $DH_n = \mathbb{C}[\Sigma_n]$ Chevalley theorem

$d=2$ $DH_n \simeq \mathbb{C}PF_n \otimes \text{Sign}$ Mark Haiman

Definition

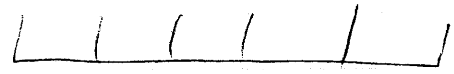
$f: \{1, \ldots, n\} \to \{1, \ldots, n\}$ is called parking if $|f^{-1}(\{1, \ldots, k\})| \geq k$

$PF_n \circlearrowleft \Sigma_n$

$\mathbb{C}PF_n$

n

car



n parking spot

f says for each car its favourite spot

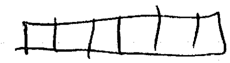
each car approaches lot and each goes to its favourite spot or to the nearest spot

cars can park successfully iff f is parking function

$$\mathrm{Rep}\,\Sigma_n \xrightarrow{\text{Schur Weyl dual}} \mathrm{Rep}\,\mathfrak{gl}_r \qquad V \text{ vector rep of } \mathfrak{gl}_r$$

$$\pi \to (\pi \otimes V^{\otimes n})^{S_n}$$

$$\Sigma_n \circlearrowright \ \circlearrowleft \mathfrak{gl}_r$$

$n\omega_1$ 

$\lambda(E_{ii}) = \lambda(E_{jj})$ for $i, j > 1$ 

Theorem (B Feigin, SU)

$W_{n\omega_1}(z)$ as a module over $\mathfrak{gl}_r \otimes 1$ is Schur Weyl dual to DH_n

consider $d = 1$

$$(\mathbb{C}[\Sigma_n] \otimes V^{\otimes n}) \simeq V^{\otimes n}$$

Theorem (conj by Chari Pressley
Thm Chari-Pressley
Fourier-Littelmann, simply-laced
Nakajima general case)

$$\boxed{W_\lambda(z) \simeq V^{\otimes \lambda_1} \otimes (\Lambda^2 V)^{\otimes \lambda_2} \otimes \cdots \otimes (\Lambda^{n-1} V)^{\otimes \lambda_{n-1}} \otimes (\Lambda^n V)^{\otimes \lambda(E_{nn})}}$$

Chari Pressley conjecture

$$W_\lambda(z) \cong W_{\omega_1}^{\otimes \lambda_1} \otimes \cdots \otimes W_{\omega_n}^{\otimes \lambda_n} \qquad \lambda = \lambda_1\omega_1 + \cdots + \lambda_n \omega_n$$

as a rep of $g \otimes 1$

consider $W_\lambda(0) \simeq W_{\omega_1}^{\otimes \lambda_1} \otimes \cdots \otimes W_{\omega_n}^{\otimes \lambda_n}$ $\qquad g \otimes \mathbb{C}[t] \to$ graded by deg of t

graded

$$W = W_{\omega_1}(z_1) \otimes W_{\omega_1}(z_2) \qquad \simeq U(g \otimes \mathbb{C}[t])(v_1 \otimes v_2 \cdots \otimes v_N)$$

$$2 \times 1 \quad v_1 \otimes v_2 \qquad F^i(W) = U^{\leq i}(g \otimes \mathbb{C}[t])(v_1 \otimes \cdots \otimes v_N)$$

$\downarrow \quad \downarrow$

h.w vectors

$$\mathrm{gr}\, F^i(W) \cong W_\lambda(z)$$

study this and go back

$\mathfrak{gl}_r \otimes \mathbb{C}[x,y] \rightsquigarrow \mathfrak{gl}_r \otimes \mathbb{C}\langle x,y\rangle / (yx - xy = x)$ still a Lie algebra

relation

$x \rightsquigarrow t$

$y \rightsquigarrow t\frac{\partial}{\partial t}$

$\mathbb{C}[t] / t^N \mathbb{C}[t]$ rep of $\mathbb{C}\langle x,y\rangle$

stable under t $t\frac{\partial}{\partial t}$

vector space of dim r

$V \otimes \left(\mathbb{C}[t] / t^N \mathbb{C}[t]\right) \rightarrow$ vectors with entries in $\mathbb{C}[t] / t^N \mathbb{C}[t]$

ξ partition $\xi_1 \geq \xi_2 \geq \cdots \geq \xi_r$ $\xi_r \in \mathbb{Z}$

$|\xi| = \xi_1 + \cdots + \xi_r$

$\Lambda^{|\xi|}\left(V \otimes \left(\mathbb{C}[t] / t^N \mathbb{C}[t]\right)\right) \ni v_\xi = \bigwedge_{i=1}^{r} \bigwedge_{j=1}^{\xi_i} v_i \otimes t^{N-j}$

$N > \xi_1$

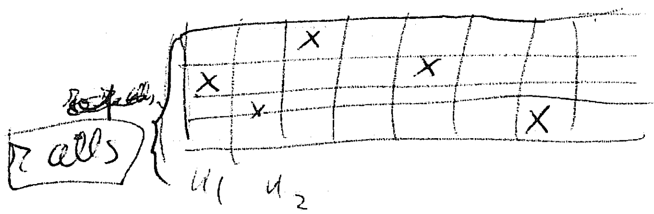
$V = \langle v_1, \ldots, v_r \rangle$ $v_i = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ \vdots \end{pmatrix} e_i$

Definition

W_ξ $\quad \mathfrak{gl}_r \otimes \mathbb{C}\langle x, y\rangle$-module gen by v_ξ

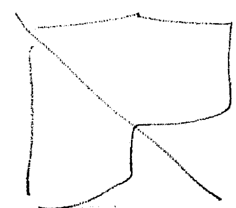
~~Proposition~~

$H \subset \{1, \dots, r\} \times \mathbb{N}$



$|H| = |\xi|$

$v_H = \bigwedge_{i \times j \in H} v_i \otimes t^{n-j}$



Proposition W_ξ is spanned by vectors

Vectors v_H for $|H| = |\xi|$ $\qquad |H \cap \{1, \dots, r\} \times \{1, \dots, n\}| \geq \xi^t_1 + \dots + \xi^t_n$

Corollary W_ξ is Schur Weyl dual to Sign $\otimes\, \mathbb{C}PF_n(\xi)$

$\mathbb{C}PF_n(\xi) = \{ f : \{1, \dots, |\xi|\} \to \mathbb{N} \mid |f^{-1}(\{1, \dots, n\})| \geq \xi^t_1 + \dots + \xi^t_n \}$

$\deg y = 1$

$\deg X = 0$

$F^i \mathbb{C}\langle x, y\rangle = \langle x^k y^l \mid l \le i \rangle$

$\mathrm{gr}\, F^i \mathbb{C}\langle x, y\rangle \simeq \mathbb{C}[x, y]$

$F^i (\mathfrak{gl}_r \otimes \mathbb{C}\langle x, y\rangle) \simeq \mathfrak{gl}_r \otimes F^i \mathbb{C}\langle x, y\rangle$

$F^i W_\xi \cong U^{\le i}(\mathfrak{gl}_r \otimes \mathbb{C}\langle x, y\rangle)\, v_\xi$

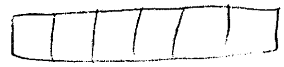
<u>Prop</u> $\mathrm{gr}\, W_\xi$ highest weight module over $\mathfrak{gl}_r \otimes \mathbb{C}[x, y]$

satisfies Weyl relation up to elements of smaller degree in filtration

check $(E_{ij} \otimes x^k y^l)$ $i \ge j$

$$\begin{cases} 0 & i > j \quad \text{any } k \\ 0 & i = j \quad k > 0 \\ \text{proportional to } v_\xi & i = j \quad k = 0 \quad l > 0 \\ \xi_i\, v_\xi & i = j \quad k = l = 0 \end{cases}$$

Conjecture $\mathrm{gr}\, W_\xi \simeq W_\xi$ known for $d = 1$

$\xi = (n)$ 

$\mathfrak{gl}_2 \otimes \mathbb{C}[x,y]$

$$W_\xi = \bigoplus_{\substack{0 \leq k_1, k_2 \leq n \\ k_1 + k_2 = n}} W_\xi^{k_1 k_2} \qquad E_{ii} W_\xi^{k_1 \overset{i}{\downarrow}{k_i} k_2} = k_i$$

<u>Prop</u>

$$\dim W_\xi^{k_1 k_2} = \frac{\begin{bmatrix} n+1 \\ k_1 \end{bmatrix} \begin{bmatrix} n+1 \\ k_2 \end{bmatrix}}{n+1}$$

$$\dim W_\xi = \frac{\begin{bmatrix} 2(n+1) \\ n \end{bmatrix}}{n+1}$$

<u>Lemma</u>

$f : (\{1, \ldots, n\} \longrightarrow \{1, \ldots, n+1\})$

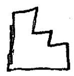
$\sigma = \begin{pmatrix} \text{cyclic} \\ \text{permutation} \end{pmatrix} = \begin{pmatrix} 1 & & \cdots & n+1 \\ 2 & 3 & \cdots & 1 \end{pmatrix}$

$$\exists!\ m \in \{0, \ldots, n\} \mid |(\sigma^m \circ f)^{-1}(\{1, \ldots, k\})| \geq k$$

$k = 1, \ldots, n$

$\rightsquigarrow$ do computation for arbitrary function and divide by #permutations.

$$c_{n+1} = c_0 c_n + c_1 c_{n-1} + \quad + c_n c_0$$

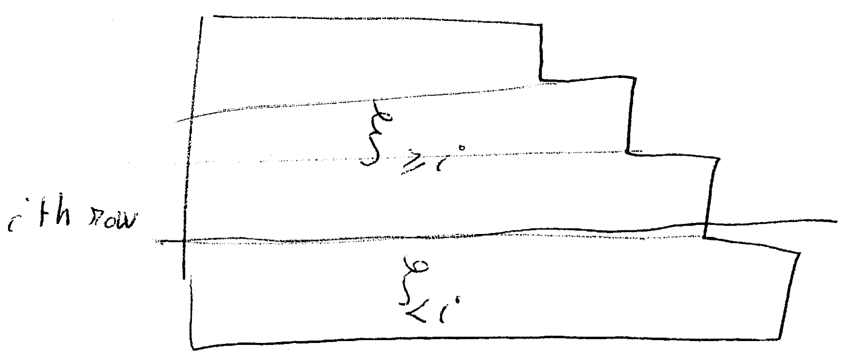
c  $= \dim W$ 

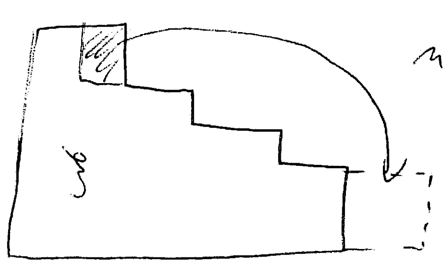
also a quadratic relation

Proposition

$$c_\xi = \sum_{i=1}^{\xi_1} c_{\xi_{\geq i}} \cdot c_{\xi_{< i}}$$

$\rightarrow$ this relation also relates characters.





move last cell here