Defn: A quantized algebra is an algebra A over k. also an ass. $k[[h]]$-alg with a $k[h]$-linear linear bracket $\{,\}$, which is a derivation in each argument

$\hbar\{a,b\} = ab-ba$

If $\hbar$ acts by 0 on A, A is commutative, $(A,\{,\})$ is Poisson.

If A is a flat $k[[h]]$-algebra, A is a quantization of $A/(\hbar)$.

$\exists$ an algebra $Q^\circ V$ which is the "free quantized algebra over V^* (V is a v.s. over k)

we have relations: $x, y \in V$ $\exists$ a Lie polynomial $L(x,y)$ s.t.

$$h^{p-1}L(x,y) = (x+y)^p - x^p - y^p \qquad (\mathrm{Char}\, k = p).$$

and $\exists$ polynomial $P(x,y)$, s.t. $h^{p-1}P(x,y) = (xy)^p - x^p y^p$

Defn: Restricted poisson algebra A/k is a poisson algebra over k. (has $\{,\}$), equipped with $x \longrightarrow x^{[p]}$ s.t. $\{,\}$ is a restricted Lie algebra with respect to $x \mapsto x^{[p]}$. and $(xy)^{[p]} = x^p y^{[p]} + x^{[p]}y^p + P(x,y)$ $\forall x, y \in A$.

Defn: Let A be a ~~restricted~~ quantilized alg over k. A restricted str. on A is an operation.

$X \longrightarrow X^{[p]}$ s.t. (i) $(A,\{,\})$ is a restricted Lie alg

(ii) $h^{[p]} = h$

(iii) $(xy)^{[p]} = x^p y^{[p]} + x^{[p]}y^p - h^{p-1}x^{[p]}y^{[p]} + P(x,y)$.

Suppose h acts 0 and $\{,\}=0$, then $x \mapsto x^{[p]}$ is a "Frob. derivation" of A

<u>Recall</u> If A is flat over $k[[\hbar]]$, then "A is Frob. constant quantization of $A/(h)$" is the same as A is restricted quantized algebra

Thm: Let X/S a symplectic mfd, $\Omega \in \Lambda^2(\Omega^1(X/S))$ Then TFAE:

① $C^2(\Omega)=0$ ② the subalgebra of Hamiltonian v.f. is closed under $\xi \longmapsto \xi^{[p]}$

Further if X admits a central quant. then (1) and (2) holds.

Central quant. A alg. $A_\hbar$ a quant. $Z(A_\hbar) \xrightarrow{\text{mod } \hbar}$ possion center of A.

If this is surjective, it is called central. quant.

Suppose X/S symplectic and condition of thm 1 are satisfied.

So $C^2(\Omega)=0 \Rightarrow \exists\ \alpha \in \Omega^1(X/S)$ s.t. $d\alpha = F_*\Omega$

In this case, the set of such α form a principal homogeneous space over

$F_*(\Omega^1_{cl}(X/S))$ M_Ω denote this set.

We have $C^1 : F_*(\Omega^1_{cl}(X/S)) \longrightarrow \Omega^1_{[p]}(X/S)$

we can push-forward $C^1_* M_\Omega$ over $\Omega^1_{[p]}(X/S)$

Suppose that we have $K : \mathcal{O}_S \to \mathcal{O}_S$ a Frob. derivation. A restricted structure on $\mathcal{O}_X$ is compactible with K, if $\pi^{-1}\mathcal{O}_S \to \mathcal{O}_X$ is a restricted poisson alg map.

Notation. $\exists\ \theta_K \in H^1(X, \mathcal{T}(X/S))$ ass. to K. Let $\{U_i\}$ an oper cover of X.

the set $\{d : \mathcal{O}(U) \to \mathcal{O}(U) \mid d \text{ compactible with } K\}$

If d, d' are in this set, $d-d' \in \mathcal{T}(X/S)(U)$. Pick d_i a derivation on U_i compactible with K. Then $d_i - d_j|_{U_i \cap U_j} \in \mathcal{T}(X/S)(U_i \cap U_j)$

Then you get an element of $\check{H}^1(\{U_i\}, \mathcal{T}(X/S)) \simeq H^1(X, \mathcal{T}(X/S))$ for affine cover.

Now given $\alpha \in H^0(X, \Omega^k(X/S))$, you can consider $\mathcal{T}(X/S) \longrightarrow \Omega^{k-1}(X/S)$

$\xi \longmapsto \alpha \lrcorner \xi$ Then $[\theta_K]$ induces a homogeneous space over

$\Omega^{k-1}(X/S)$, $\alpha \lrcorner [\theta_K]$

(For θ_K, we have a $\mathcal{T}(X/S)$-homogeneous space $[\theta_K] = \{d : \mathcal{O}_X \to \mathcal{O}_X \mid d \text{ compact with } K\}$

Theorem 2. Let X/S satisfied the conditions in Thm 1. Then we have a bijection

$$\{\text{restricted structures on } X/S\} \longleftrightarrow \{\text{isomorphisms over } \Omega^1_{[p]}(X/S)$$

$$\cancel{C^*M_\Omega} \quad C^!_* M_\Omega \xrightarrow{\sim} Fr^*_{X[p]}\Omega \lrcorner [\Theta_K]\}$$

<u>Cor</u>: X/S symplectic, $K: \mathcal{O}_S \to \mathcal{O}_S$ a Frob. derivation. $(\mathcal{O}_\hbar, S)$ is a Frob. constant quantization, compactible with K.

$f \in \Gamma(U, \mathcal{O}_\hbar)$ s.t. $\bar{f} \in \Gamma(U, \mathcal{O}_X)$ is a pullback of $b \in H^0(S, \mathcal{O}_S)$

We demand $\frac{1}{\hbar^{p-1}}(f^p - s(f)) = \pi^*(K(b))$

Then $C^2(\Omega) = 0 \quad C^1(\Omega) = (Fr^*_{X[p]}[\Omega]) \lrcorner \Theta_K$

If K is trivial, $C^1(\Omega) = 0$.

<u>Thm 1 Sketch of the proof</u>

(1) $\Leftrightarrow$ (2) is local. Affine case. A/B are algebras.

define $\iota^{[p]}_\xi(\alpha) = \xi^{[p]} \lrcorner \alpha - ad^{p-1}_\xi(\xi \lrcorner \alpha)$

$C^i(\iota^{[p]}_\xi(\alpha)) = C^{i+1}(\alpha) \lrcorner \xi$

Put $\alpha = \Omega$, $\xi = H_f$ for $f \in A$.

$C^1(H^{[p]}_f \lrcorner \Omega - ad^{p-1}_{H_f}(H_f \lrcorner \Omega)) = C^2(\Omega) \lrcorner H_f$

$C^1(H_f \lrcorner \Omega) = C^1(df) = 0$

So $C^1(H^{[p]}_f \lrcorner \Omega) = C^2(\Omega) \lrcorner H_f$

ξ is hamiltonian $\Leftrightarrow \xi \lrcorner \Omega = df \Leftrightarrow C^1(\xi \lrcorner \Omega) = 0$

So we only need to show RHS vanishes iff $C^2(\Omega) = 0$

H_f generates the tangent space at each point.

Suppose X admits a central quantization. given $f \in A$, $f^p \in A^p$ = Poisson center of A.

So $\exists\ \hat{f}^p$ lifting f^p. Can write $\hat{f}^p = f^p + f_1 h + f_2 h^2 + \dots + f_{p-2} h^{p-2} + g h^{p-1}$

$f_i \in$ Poisson center of A. $\Rightarrow H_f^{[p]} = H_g$ (mod h^p)

$\square$

Motivation

Deformation of ass. algebras.

R is an alg over K. Let M be an R-R bimodule. A square-zero extension E of R by M is: E is an alg, $0 \to M \to E \underbrace{\to}_{\text{map of algs}} R \to 0$ and $M^2 = 0$

Thm: {equivalent classes of such extensions} $\longleftrightarrow H^2(R, M)$

In quantization theory: $A = A_0[[h]]$. $H^2(A_0, K)$ classifieds alg str. on $A_0[h]/h^2$ the obstruction to lifting further live in $H^3(A_0, K)$

Def: A quantization base B is a commutative noetherian complete local $K[[h]]$-algebra where $m_B \subseteq B$ maximal ideal, $h \in m_B$ $K \simeq B/m_B$. ~~Thus~~ it has a Frob. derivation $K: B \to B$ $K(b) = b^{[p]}$

If $X|_K$ is symplectic, a B-quantization of X is $\mathcal{O}^B$ a sheaf of restricted quantized flat B-algebras s.t. $\mathcal{O}^B/m_B \simeq \mathcal{O}_X$

If $B = K$, restricted Poisson str. on $\mathcal{O}_X$.

A "small extension" of B_0 (quant. base) is a quant. base B with restricted ideal I. ($I^{[p]} \subseteq I$) with $B/I \simeq B_0$, $m_B \cdot I = 0$

Main ~~exple~~ example: $B/m_B^{n+1} \longrightarrow B/m_B^n$

Consider the sheaf $Fr_* \Omega^1_{cl} \otimes_K I$. Let $H\langle I\rangle = \{\alpha \in Fr_*(\Omega^1_{cl}) \otimes_K I \mid C'(\alpha) = K_I(\alpha)\}$ where $K_I : I \to I$, $K_I(m) = m^{[p]}$

Examples : Suppose $I = K$. $K_I(a) = a^p \lambda$ for some fixed λ.

If $\lambda = 0$, $H\langle 0\rangle$ = sheaf of closed 1-forms with $C'(\alpha) = 0$.

So we have $0 \to \mathcal{O}_X^{[p]} \to \mathcal{O}_X \xrightarrow{d} H\langle 0\rangle \to 0$

If $\lambda = 1$. $H\langle 1\rangle$ = sheaf of closed 1-forms with $C'(\alpha) = \alpha$

these are of type $\frac{df}{f}$.

we have : $0 \to (\mathcal{O}_X^p)^* \to \mathcal{O}_X^* \xrightarrow{d\log} H\langle 1\rangle \to 0$

Thm3. Let $\langle B, I\rangle$ a small extension of B_0. $\mathcal{O}_h$ a regular B_0-quantization of X/K. Let $Q(B; \mathcal{O}_h) = \{$ iso. classes of B-quant. $\mathcal{O}_h'$ s.t.

$$\mathcal{O}_h'/I = \mathcal{O}_h.\}$$

① $\exists$ a "canonical obstruction class" $c(\mathcal{O}_h) \in H^2_{et}(X, H\langle I\rangle)$

s.t. $Q(B, \mathcal{O}_h) \neq \phi$ iff $c(\mathcal{O}_h) = 0$.

② If $c(\mathcal{O}_h) = 0$, $Q(B, \mathcal{O}_h) \longleftrightarrow H^1_{et}(X, H\langle I\rangle)$.

Thm4. Take same assumptions plus $B_0 = K$, $I = m_B$ $m_B^2 = 0$

Suppose X/K has a restricted Poisson str. then $c(\mathcal{O}_X) = 0$ and $Q(B, \mathcal{O}_X) \simeq H^1_{et}(X, H\langle m_B\rangle)$ tautologically.

Defn. If $f \in \Gamma(U, \mathcal{O}^B)$ s.t. $\bar{f} \in \Gamma(U, \mathcal{O}^B/m_B)$ has $\bar{f}^p = 0$ then $f^p = h^{p-1} f^{[p]}$

Then its called regular B-quantization.

Cor: Let X/K a symplectic mfd s.t. $Fr : H^i(X^{[p]}, \mathcal{O}_X^{[p]}) \to H^i(X, \mathcal{O}_X)$ is bije. for $i = 1, 2, 3$. equipped with a restricted Poisson str. B a quant. Base

Let $Q(B,X) = \{$ isomorphic ~~ext~~ classes of regular B-quantization $\mathcal{O}^B$ with $\mathcal{O}^B/_{m_B} \cong \mathcal{O}_X \}$

then $Q(B,X) \longleftrightarrow H^1_{et}(X, H\langle m_B/m_B^2\rangle)$.

pf: Set $B_n = B/\cancel{B}m_B^{n+1}$ then $B_{n+1} \twoheadrightarrow B_n$ are small extensions.

Now if $b \in m_B^{n+1}$, $b^{[p]} \in m_B^{n+2}$.

So $K: I \to I$ is trivial for $n \geq 1$.

Look $H^i_{et}(X, H\langle 0\rangle)$. this vanishes by admissibility ($i=1,2$) There are no obstructions for B_1 to B_2, B_2 to B_3, ..., to B.

Look at $B_1 = B/m_B^2$ these are classified by $H^1_{et}(X, H\langle m_B/m_B^2\rangle)$. $\square$