Algebraic geo in Char $= p > 0$

Frobenius morphism: A comm ring: $F: A \to A$ $F(a) = a^p$

$F^{\#}: \mathrm{Spec}(A) \to \mathrm{Spec}(A)$ (Identity on points.)

If X is a scheme $/\mathbb{F}_p$, then $F: X \to X$ (id on space, p^{th}-power on functions).

Suppose $X \xrightarrow{\pi} S$,
$$\begin{tikzcd} X \arrow[r, "F_X"] \arrow[d] & X \arrow[d] \\ S \arrow[r, "F_S"] & X \end{tikzcd}$$
not in general cartesian.

$$\begin{tikzcd} X \arrow[r, "Fr_{X/S}"] \arrow[rr, bend left, "Fr_X"] & X \times_{F_S} X \arrow[r] \arrow[d] & X \arrow[d] \\ & S \arrow[r, "F_S"] & S \end{tikzcd}$$

e.g.: $S = \mathrm{Spec}(A)$, S

$X = \mathrm{Spec}(A[t_1, \dots, t_n])$.

$\mathrm{Spec}(A[t_1, \dots, t_n] \otimes_{Fr(A)} A)$ if $Fr: A \to A$ is an iso.

then $A[t_1, \dots, t_n] \otimes_{Fr(A)} A = A[t_1 \cdots t_n]$ (A-alg under p^{th} power)

$$A[t_1, \dots, t_n] \longleftarrow A[t_1 \dots t_n] \longleftarrow A[t_1 \dots t_n]$$

$$a^p t_i \longleftarrow\!\shortmid\, a t_i$$

$$t_i^p \longleftarrow\!\shortmid\, t_i$$

If X, S are reduce schemes, $X \times_{Fr\,S} S \cong X^{[p]} = \langle X, \mathcal{O}_X^p \otimes_{\mathcal{O}_S^p} \mathcal{O}_S \rangle$

Regular schemes and Cartier isomorphisms

$X \xrightarrow{\pi} S$ is quasiregular, if π is flat, finite type, and $\Omega^1(X/S)$ is flat over $\mathcal{O}_X$.

In this situation, we have $\Omega^k(X/S) = \Lambda^k(\Omega^1(X/S))$ de Rham differential.

$T(X/S) = (\Omega^1(X/S))^\vee$, we have lie derivative θ_ξ, interior derivative i_ξ

$$\theta_\xi = d i_\xi + i_\xi d.$$

Thm (Grothendieck).

If X is a smooth variety over $\mathbb{C}$, then in fact

i) If X is affine, $H^i_{sing}(X,\mathbb{C}) \simeq H^i_{dR}(X,\mathbb{C})$.

$$X \xrightarrow{Fr_{X/S}} X^{[p]}$$

Thm (Cartier): Define $\Omega^i_{[p]}(X/S) = Fr^*_{X^{[p]}}(\Omega^i(X/S))$ we have

$$\gamma: \bigoplus_{i\geq 0} \Omega^i_{[p]}(X/S) \longrightarrow \bigoplus_{i\geq 0} H^i((Fr_{X/S})_* \Omega^*(X/S))$$

at $i=0$, given by $F^*: \mathcal{O}_{X^{[p]}} \to \mathcal{O}_X$

at $i=1$, given by $1\otimes ds \longrightarrow s^{p-1} ds$.

Then γ extends to an "algebra homo" and if X/S is (quasi)-regular, γ is an isomorphism sheaf of algebra.

In particular, let $i=0$, this says $\mathcal{O}_{closed} \cong \mathcal{O}_{X^{[p]}}$ $\mathcal{O}_X$ is finite over $\mathcal{O}_{closed}$.

In the regular case, by inverting γ, we get a morphism

$$\begin{array}{c} \bigoplus H^i(Fr(\Omega^*(X/S))) \longrightarrow \bigoplus \Omega^1_{[p]}(X/S) \\ \uparrow \\ \bigoplus Fr_*(\Omega^*_{cl}(X/S)) \end{array}$$

$$C^i: \bigoplus Fr_*(\Omega^*_{cl}(X/S)) \longrightarrow \Omega^i_{[p]}(X/S)$$

Quantization "Fedesov Quantization in positive Char".

Let A be a commutative algebra over k. A quantization $A_\hbar$ is an ass. flat, $k[[\hbar]]$-alg with a given iso: $A_\hbar/(\hbar) \xrightarrow{\sim} A$. complete w.r.t. $\hbar$-adic topo.

For a scheme $(X, \mathcal{O}_X)$, a quantization is $(X, \mathcal{O}_\hbar)$

Flat: as $k[[\hbar]]$-modules, $A_\hbar \cong A[[\hbar]]$. (agree with $A_\hbar/(\hbar) \xrightarrow{\sim} A$).
not an isomorphism of rings.

$a, b \in A \hookrightarrow A[[\hbar]]$.

$a * b = ab + \varphi_1(a,b)\hbar + \cdots$

given $a, b \in A$ $\quad a*b - b*a \in h \cdot A_\hbar$. define $\{a,b\} = \frac{1}{\hbar}(a*b - b*a) \bmod (\hbar)$

Then $\{,\}$ is a Poisson bracket on A.

Convention: start with $(A, \{,\})$ and ask that $A_\hbar$ covers $\{,\}$.

Recall: if X is (mfd, scheme) smooth, a symplectic form is a nondeg closed 2-form, Ω.

In this case, $f, g \in \mathcal{O}_X$, $\quad \{f,g\} = \Omega^{-1}(df, dg)$

Def. A quantization $\mathcal{O}_\hbar$ is called central if the natural map $Z_\hbar = \mathrm{Cent}(\mathcal{O}_\hbar)$

$Z_h \longrightarrow$ Poisson center of $\mathcal{O}_X$ is surjective.

If X is (smooth) regular and symplectic, Poisson center of $\mathcal{O}_X = \mathcal{O}_{\mathcal{O}^p} = \mathcal{O}_X[p]$.

Def: A Frobenius-Constant quantization of A consists of: a quant. $A_\hbar$ with

$S: A_\hbar \to A_\hbar$ which is p-semilinear, multiplicative, central ($\mathrm{im}(S) \subseteq$ center)

has $S(h) = 0$ and $S(a) \equiv a^p$ ~~and~~ $\bmod (h^{p-1})$ $\quad (S(\lambda a) = \lambda^p S(a))$

So in particular, $S: A \to A_\hbar$ and the quantization is called regular if in fact S factors through A^p

$$\begin{tikzcd} A \arrow[rr, "S"] \arrow[dr] & & A_\hbar \\ & A^p \arrow[ur] & \end{tikzcd}$$

So if A is reduced alg, we can always have this.

In the regular case,

$$\begin{tikzcd} A^p \arrow[r, "s"] \arrow[dr, bend right] & A_\hbar \arrow[d] \\ & A \end{tikzcd}$$

- Suppose $X \longrightarrow \mathrm{Spec}(k)$, X smooth, $\mathcal{O}_\hbar$ is Frob. const. $\Leftrightarrow$ ① it is central
  ② $Z_\hbar \simeq \mathcal{O}_X^p[[\hbar]]$ locally

pf: If $\mathcal{O}_\hbar$ is Frob. const.
$$\begin{tikzcd} \mathcal{O}^p \arrow[r] \arrow[dr] & \mathcal{O}_\hbar \arrow[d] \\ & \mathcal{O} \end{tikzcd}$$
and $\mathcal{O}^p \hookrightarrow \mathcal{O}$ is the poisson center.

and then $Z_\hbar$ is a comm alg over $\mathcal{O}^p$, flat, complete, $Z_\hbar|_\hbar \simeq \mathcal{O}_X^p$ $\square$.

Examples: X is a smooth variety, then T^*X is a symplectic.

Consider $\mathcal{D}_\hbar(X)$: h-completion of the alg generated by $\mathcal{O}_X$, $\mathcal{T}_X$ with relations.

$(f_1, f_2 \in \mathcal{O}_X)$, $f_1 \cdot f_2 = f_1 f_2$

$(f \in \mathcal{O}_X, \xi \in \mathcal{T}_X)$, $f \cdot \xi = f\xi$. $\quad \xi f - f\xi = \hbar \xi(f)$

$$\xi_1 \xi_2 - \xi_2 \xi_1 = \hbar [\xi_1, \xi_2].$$

Claim: $D_\hbar(X)$ is Frob-constant

$$\mathcal{O}(T^*X)^p \longrightarrow D_\hbar(X) \qquad f^p \in \mathcal{O}_X^p \longrightarrow f^p \in D_\hbar(X)$$

$$\xi^p \in \mathcal{T}_X \longmapsto \xi^p - \hbar^{p-1}\xi^{[p]} \qquad \xi^{[p]} \sim p^{th}\text{-power of a vector field.}$$

Under lying algebraic lemma: (Char $=p$)

A - ass. alg, suppose that all p^{th} comm vanish. $A_{(2)} = [A, A]$, $A_{(2)}^p = 0$.

(ex: $A_n/\hbar^{p+1}$). then the function $x \longmapsto x^p$ is a mult. additive, central map.

Defn: A quantized alg $A|_K$ is an associative ~~K-alg~~ complete $K[[h]]$-alg with $K[h]$-linear lie bracket $\{,\}$, which is a derivation in each argument and such that $h\{a,b\} = ab - ba$.

One extreme h acts as 0: Poisson algebra.

Another extreme: A is flat over $K[[\hbar]]$, Then A is a quantization of $A/(\hbar)$.

let V be a f.d. vector space over K. Consider $T^\bullet V$. can consider

$$F^\bullet_{PBW}(T^\bullet V). \qquad Q^\bullet(V) = \mathrm{Rees}(F, T^\bullet V) = \bigoplus_{i=0} F_i(T^\bullet V)$$

h acts by $F_i(T^\bullet V) \hookrightarrow F_{i+1}(T^\bullet V)$.

$Q^\circ(V)$ — the free quantized algebra over V. Consider $Q^\circ(V)/h^{p-1}$

e.g. $(x+y)^p - x^p - y^p = h^{p-1} L(x,y)$ $\quad$ $L(x,y)$ — Lie polynomial in x, y.

$(xy)^p - x^p y^p = h^{p-1} P(x,y)$ $\quad$ $P(x,y)$ — polynomial

Defn: A restricted Lie alg over K. (char $K = p$) is a Lie alg with $X \to X^{[p]}$ s.t.

$(ax)^{[p]} = a^p x^{[p]}$ $\quad$ $[x^{[p]}, y] = \mathrm{ad}_{x^p}(y)$ $\quad$ $(x+y)^{[p]} = x^{[p]} + y^{[p]} + L(X,Y)$

e.g. If B is an ass. alg over K, $(B, [,])$ is a restricted Lie alg $X \to X^p$

Defn: A restricted Poisson alg (char $K = p$) is a Poisson alg equipped with $X \to X^{[p]}$ s.t. $(xy)^{[p]} = x^p y^{[p]} + x^{[p]} y^p + P(X,Y)$ s.t. $\{,\}$ is a restricted Lie algebra under $X \to X^{[p]}$.

Defn: Let A be a quantized alg. A restricted str. on A/K is an operation $X \to X^{[p]}$ s.t.
① $(A, \{,\})$ is a restricted Lie alg
② $h^{[p]} = h$
③ $(xy)^{[p]} = x^p y^{[p]} + x^{[p]} y^p - h^{p-1} x^{[p]} y^{[p]} + P(X,Y)$

If $A_\hbar$ is a Frob-const. quantization of A, with splitting $S: A_\hbar \to A_\hbar$
$X^{[p]} = \frac{1}{h^{p-1}}(X^p - S(X))$ gives $A_\hbar$ a restricted str.

Conversely, if $A_\hbar$ is flat over $K[[\hbar]]$, a restricted quantized alg, then
$S(X) = X^p - h^{p-1} X^{[p]}$ is a splitting

Main result:

Let X/S be a symplectic mfd. Then TFAE.

1) The space of Hamiltonian v.f. is closed under $\xi \longmapsto \xi^{[p]}$.

$H_f = \{f, \cdot\}$ is a derivation.

2) $C^2(\Omega) = 0$

Pf. X admits a central quantization, these are satisfied.