2/5/08.

- Alg gps (hyperalgebras)
- Finite dim'l repns of Affine kac-Moody Alg (loop alg)

Fix a simple Lie algebra $\mathfrak{g}$ finite dim over $\mathbb{C}$ and a Chevalley basis $x_\alpha^\pm$, h_i $\alpha \in R^+$ (pos.) $i \in I$

$\mathfrak{h} = \bigoplus_{i \in I} \mathbb{C} h_i$ $\quad \mathfrak{n}^\pm = \bigoplus_{\alpha \in R^+} \mathbb{C} x_\alpha^\pm$ $\quad \mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$.

Let $\mathfrak{g}_\mathbb{Z}$ be the $\mathbb{Z}$-span of this basis.

Given a field $\mathbb{k}$. let $\mathfrak{g}_\mathbb{k} = \mathfrak{g}_\mathbb{Z} \otimes_\mathbb{Z} \mathbb{k}$. $\mathbb{k}$-split form of $\mathfrak{g}$ / chevalley algs. ($\mathfrak{g}_\mathbb{C} = \mathfrak{g}$)

let $U(\mathfrak{g})$ be the UEA of $\mathfrak{g}$ and $U(\mathfrak{g})_\mathbb{Z}$ be the $\mathbb{Z}$-subalgebra generated by $(x_\alpha^\pm)^{(k)} = \frac{(x_\alpha^\pm)^k}{k!}$

Thm (Kostant) $U(\mathfrak{g})_\mathbb{Z}$ is an integral form of $U(\mathfrak{g})$

Hence we can construct $U(\mathfrak{g})_\mathbb{k} = U(\mathfrak{g})_\mathbb{Z} \otimes_\mathbb{Z} \mathbb{k}$

$U(\mathfrak{g})_\mathbb{k}$ is called the hyperalgebra of $\mathfrak{g}$ over $\mathbb{k}$.

In general, $\exists$ an alg map: $U(\mathfrak{g}_\mathbb{k}) \to U(\mathfrak{g})_\mathbb{k}$.

If $\mathrm{char}(\mathbb{k}) = 0$ this map is an iso.

When $\mathbb{k} = \bar{\mathbb{k}}$ (alg. closed) $U(\mathfrak{g})_\mathbb{k}$ is isomorphic to the so-called algebra of distributions (at e) of the corresponding Chevalley gp of simply connected universal type. ($\mathfrak{g} = \mathfrak{sl}_n \rightsquigarrow SL_n(\mathbb{k})$)

Basis of $U(\mathfrak{g})_{\mathbb{k}}$ (and of $U(\mathfrak{g})_{\mathbb{k}}$)

$U(\mathfrak{n}^{\pm})_{\mathbb{Z}} \rightsquigarrow (x^{\pm}_{\alpha_1})^{(k_1)} \cdots (x^{\pm}_{\alpha_n})^{(k_n)} \quad k_i \in \mathbb{Z}_+$

$U(\mathfrak{h})_{\mathbb{Z}} = U(\mathfrak{g})_{\mathbb{Z}} \cap U(\mathfrak{h}) \rightsquigarrow \prod_{i \in I} \binom{h_i}{k_i} \quad k_i \in \mathbb{Z}_+.$

$$\binom{h_i}{k_i} = \frac{h_i(h_i-1)\cdots(h_i-(k_i-1))}{k!}$$

loop algebras. $\tilde{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}]$ with bracket

$$[x \otimes f(t), y \otimes g(t)] = [x, y] \otimes (f(t) g(t))$$

Analogue of Chevalley basis.

$x^{\pm}_{\alpha, r} = x^{\pm}_{\alpha} \otimes t^r \qquad h_{i,r} = h_i \otimes t^r.$

$\tilde{\mathfrak{g}}_{\mathbb{Z}} \rightsquigarrow$ the $\mathbb{Z}$-span $\rightsquigarrow \tilde{\mathfrak{g}}_{\mathbb{k}} = \tilde{\mathfrak{g}}_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{k} \cong \mathfrak{g}_{\mathbb{k}} \otimes_{\mathbb{k}} \mathbb{k}[t, t^{-1}]$

let $U(\tilde{\mathfrak{g}})_{\mathbb{Z}}$ the $\mathbb{Z}$-subalg of $U(\tilde{\mathfrak{g}})$ generated by

$(x^{\pm}_{\alpha, r})^{(k)} \quad \alpha \in R^+, r \in \mathbb{Z}_+ \quad k \in \mathbb{Z}_+$

Thm (Garland) $U(\tilde{\mathfrak{g}})_{\mathbb{Z}}$ is a $\mathbb{Z}$-form of $U(\tilde{\mathfrak{g}})$

$\rightsquigarrow U(\tilde{\mathfrak{g}})_{\mathbb{k}} \rightsquigarrow$ hyper loop alg of $\mathfrak{g}$ over $\mathbb{k}$.

Remark: The hopf alg str. on $U(\mathfrak{g})$ or $U(\tilde{\mathfrak{g}})$ induces Hopf alg str. on the hyperalg.

Basis of $U(\tilde{\mathfrak{g}})_{\mathbb{Z}} \rightsquigarrow U(\tilde{\mathfrak{n}}^{\pm})$ obvious $(x^{\pm}_{\alpha, r})$

$U(\tilde{\mathfrak{h}})_{\mathbb{Z}}$ consider elts $\Lambda_{i,r} \in U(\tilde{\mathfrak{h}})$ defined by equality of formal power series in u

$$\Lambda^{\pm}(u) = \sum_{r\geq 0} \Lambda_{i,\pm r}\, u^r = \exp\left(-\sum_{s>0} \frac{h_{i,s}}{s} u^s\right)$$

Let $\Lambda^{\pm} = \{\Lambda_{i,\pm r},\ i\in I\ r\in\mathbb{N}\}$ and $\mathbb{Z}[\Lambda^{\pm}]$ the $\mathbb{Z}$-sub alg of $U(\hat{h})$ generated by $\Lambda^{\pm}$.

Prop: $\mathbb{Z}[\Lambda^{\pm}]$ is the polynomial alg in the set $\Lambda^{\pm}$ and

$$U(\hat{h})_{\mathbb{Z}} \underset{\text{as alg.}}{\cong} \mathbb{Z}[\Lambda^-] \otimes_{\mathbb{Z}} \underset{\binom{h_i}{k_i}}{\underbrace{U(h)_{\mathbb{Z}}}} \otimes_{\mathbb{Z}} \mathbb{Z}[\Lambda^+]$$

$$\rightsquigarrow U(\hat{h})_{\mathbb{K}} \cong \mathbb{K}[\Lambda^-] \otimes_{\mathbb{K}} U(h) \otimes_{\mathbb{K}} \mathbb{K}[\Lambda^+].$$

Finite diml reps of $U(\mathfrak{g})_{\mathbb{K}}$.

$$V = \bigoplus_{\substack{\mu\in P}} V_\mu \qquad V_\mu = \left\{v\in V:\ \binom{h_i}{k} v = \binom{\mu(h_i)}{k} v \quad \forall i\in I\ \forall k\in\mathbb{N}\right\}$$

P ← wt lattice of $\mathfrak{g}$.

$\langle$ char $= p$ $\binom{h}{p}$ not gen'd by h $\rangle$

Relation:

$$\binom{h_i}{k} (x^{\pm}_{\alpha,r})^{\ell} = (x^{\pm}_{\alpha,r})^{(\ell)} \binom{h_i \pm \ell\alpha(h_i)}{k}$$

we can talk about highest wt modules.

- Every irr. f.d. $U(\mathfrak{g})_K$-mod is highest wt ~~of~~ with h.w lying in P^+.
- For each $\lambda\in P^+$, $\exists$ a universal finite diml highest wt module of h.w λ $\rightsquigarrow W_{\mathbb{K}}(\lambda)$ (Weyl modules)

$\rightsquigarrow W_{\mathbb{K}}(\lambda)$ has unique irr. quotient $\rightsquigarrow V_{\mathbb{K}}(\lambda)$

- If $\mathbb{P}$ is the prime field of K, ~~and~~ then $W_{\mathbb{P}}(\lambda)\otimes_{\mathbb{P}}\mathbb{K} \cong W_{\mathbb{K}}(\lambda)$

$$(V \longmapsto V\otimes_{\mathbb{P}}\mathbb{K})$$

f.d $U(\mathfrak{g})_{\mathbb{P}}$ f.d $U(\mathfrak{g})_{\mathbb{K}}$.

Finite-dim Repn of $U(\tilde{\mathfrak{g}})_{\mathbb{k}}$.

One easily sees that if V is an irr. f.d. $U(\tilde{\mathfrak{g}})_{\mathbb{k}}$-mod, then $\exists\ \lambda \in P^+$

s.t. (i) $V = \bigoplus_{\mu \leq \lambda} V_\mu$

$h_\alpha = [x_\alpha^+, x_\alpha^-]$

(ii) $(x_{\alpha,s}^-)^{(l)} V_\lambda = 0$ if $l > \lambda(h_\alpha)$

(iii) V_λ is an irr. $U(\tilde{h})_{\mathbb{k}}$-module.

(iv) $V = U(\tilde{n}^-)_{\mathbb{k}} V_\lambda$.

Reps satisfying (i) (iii) (iv) are called highest quasi-l-wt modules.

Thm1. Let V be a highest qlw $U(\tilde{\mathfrak{g}})_{\mathbb{k}}$-module satisfying (ii). $\leadsto \lambda \in P^+$

s.t. $\dim(V_\lambda) < \infty$.

(a) V is f.d.

(b) $\Lambda_{i,\pm r} V_\lambda = 0 \quad \forall r > \lambda(h_i)$

(c) $\Lambda_{i,\pm\lambda(h_i)}$ act bijectively on V_λ.

(d) For every $i \in I$ and $r = 0, 1, \cdots, \lambda(h_i)$ $\exists\ f_{i,r} \in \mathbb{k}[u_1, \cdots u_{\lambda(h_i)}]$

s.t. $\Lambda_{i,-r} v = f_{i,r}(\Lambda_{i,1}, \cdots \Lambda_{i,\lambda(h_i)}) v \quad \forall v \in V_\lambda$.

Remark. $f_{i,r}$ depend only on $\mathfrak{g}$ and the isom. class of V_λ as a $\mathbb{k}[\Lambda^+]$-module.

Def: Let $\mathfrak{m}$ be a maximal ideal of $\mathbb{k}[\Lambda^+]$ containing $\Lambda_{i,r}$ for sufficiently large r and say $\lambda \in P^+$ is s.t. $\Lambda_{i,\lambda(h_i)}$ is not in $\mathfrak{m}$ and $\Lambda_{i,r} \in \mathfrak{m}$ if $r > \lambda(h_i)$

Define $W_{\mathbb{k}}(\mathfrak{m})$ to be the $U(\tilde{\mathfrak{g}})_{\mathbb{k}}$-module given by the quotient of $U(\tilde{\mathfrak{g}})_{\mathbb{k}}$ by the left ideal generated by $(x_{\alpha,r}^{+})^{(k)}$ $\forall \alpha \in R^+$ $r \in \mathbb{Z}, k \in \mathbb{N}$

$(x_{\alpha,0}^{-})^{(\ell)}$ $\ell > \lambda(h_\alpha)$ $\binom{h_i}{k} - \binom{\lambda(h_i)}{k}$ $\forall i \in I$ $\forall k \in \mathbb{N}$ $\mathfrak{m}$.

$\Lambda_{i,-r} - f_{i,r}(\Lambda_{i,1}, \cdots \Lambda_{i,\lambda(h_i)})$ where $f_{i,r}$ are the poly of the thm

if $0 \le r < \lambda(h_i)$ $f_{i,r} = 0$ otherwise.

Cor. $W_{\mathbb{k}}(\mathfrak{m})$ is the universal ℓ-h.w. module having $\mathbb{k}[\Lambda^+]/\mathfrak{m}$ as highest wt space.

– $W_{\mathbb{k}}(\mathfrak{m})$ has a unique irr. quotient $\rightsquigarrow V_{\mathbb{k}}(\mathfrak{m})$

Let $\mathbb{F}$ be an alg closure of $\mathbb{k}$. X a set $(\longrightarrow X = \Lambda^+)$

$\mathbb{F}_k[X]$ the poly. alg.

$\mathbb{F}$ alg. closed. $\Rightarrow$ irr. repn of $\mathbb{F}[X]$ are of the form

$V = \mathbb{F}v$ with $x\,v = \omega_x v$ for some $\omega_x \in \mathbb{F}$.

$\omega : \mathbb{F}[X] \to \mathbb{F}$ (unitary $\mathbb{F}$-alg homo) $\omega(x) = \omega_x$.

Given a field $\mathbb{L} \rightsquigarrow M_{\mathbb{L}} = \{\omega : \mathbb{L}[X] \to \mathbb{L}$ unitary $\mathbb{L}$-alg hom$\}$

Given $\omega \in M_{\mathbb{F}}$, let $\mathcal{F}(\omega)$ be the corresp. 1-dimℓ $\mathbb{F}[X]$-module.

and let $\mathcal{K}(\omega) = \mathbb{k}[X]v$ and $\deg(\omega) = \dim(\mathcal{K}(\omega)) = [\mathbb{k}(\omega):\mathbb{k}]$

where $\mathbb{k}(\omega) = \mathbb{k}(\omega_x : x \in X)$

Say $\omega \underset{\mathbb{k}}{=} \omega'$ if $\exists\, g \in \mathrm{Aut}(\mathbb{F}/\mathbb{k})$ s.t. $\omega_x = g(\omega_x)$ $\forall x \in X$.

$[\omega] \rightsquigarrow$ the conj. class of ω

$\rightsquigarrow M_{\mathbb{F},\mathbb{k}}$ all conj. classes

Thm. (a) $K(\omega)$ is irr. $\forall \omega \in M_{\mathbb{F}}$. as $\mathbb{k}[X]$-module.

Moreover, every finite-dim irr. $k[X]$-mod is of this form.

$X^F = X \otimes_k \mathbb{F}$.

(b) $K(\omega) \cong K(\omega') \Leftrightarrow \omega \equiv \omega'$

(c) If $\omega \in M_{\mathbb{F}}$ s.t $\deg(\omega) < \infty$. $K(\omega)^{\mathbb{F}} \underset{\mathbb{F}[X]}{\cong} \bigoplus_{\omega \in [\varpi]} V_\omega$ with V_ω an indecomp

self-extension of $\mathcal{F}(\omega)$ of length $\deg(\varpi)/|[\varpi]|$

Define $\mathcal{P}^+_{\mathbb{F}} = \{\omega \in M_{\mathbb{F}}\ \omega_x \neq 0$ for finitely many $x \in X\}$

In the case $X = \Lambda^+$, $\mathcal{P}^+_{\mathbb{F}}$ can be identified with an I-tuple of polynomials

$\omega \mapsto (\omega_i(u))$ $\quad \omega_i(u) : 1 + \sum_{r \in \mathbb{N}} \underbrace{\omega(\Lambda_i, r)}_{\omega_{i,r}} u^r$

Drinfeld polyomomials.

$\mathcal{P}^+_{\mathbb{F}}$ is a multiplicative abelian monoid.

Let $\mathcal{P}_{\mathbb{F}}$ be the corresponding abelian group. $\rightarrow$ I-tuples of rat. fns.

$\mathcal{P}_{\mathbb{F}} \subseteq M_{\mathbb{F}} \rightsquigarrow \frac{p(u)}{q(u)} \quad q(u) = \prod_{j=1}^{m} (1 - a_j u)$

$\frac{1}{q(u)} = \prod_{j=1}^{m} \big(\underbrace{\sum_{k \geq 0} (a_j u)^k}_{\text{g. p.s.}} \big) = 1 + \omega_{i,r} u + \cdots$

$\rightsquigarrow \omega \in M_{\mathbb{F}}$.

$\rightsquigarrow \mathcal{P}^+_{\mathbb{F},\mathbb{k}}$, $\mathcal{P}_{\mathbb{F},\mathbb{k}}$ the sets of corresponding conj. classes.

$W_{\mathbb{k}}(\pi) \leftrightarrow W_{\mathbb{k}}(\omega)$ with $\omega \in \mathcal{P}^+_{\mathbb{F}}$. (iso class depend on $[\omega]$)

$$\begin{array}{c} W_{\mathbb{k}}(\omega) \\ \downarrow \\ V_{\mathbb{k}}(\omega) \end{array}$$

Let V be a hl weight module for $U(\hat{\mathfrak{g}})_{\mathbb{F}}$ and v be a h.l.w vector of h.l.w. $\omega \in \mathcal{P}_{\mathbb{F}}^+$

Set $V^{\mathbb{F}_{\mathbb{K}}} := U(\hat{\mathfrak{g}})_{\mathbb{K}} v$ (ample form)

Thm 2:

(a) $\deg(\omega) \cdot \dim_{\mathbb{F}}(V) \leq \dim_{\mathbb{K}}(V^{\mathbb{F}_{\mathbb{K}}}) < \infty$.

(b) If $V = V_{\mathbb{F}}(\omega)$ then $=$ and $V^{\mathbb{F}_{\mathbb{K}}} \cong V_{\mathbb{K}}(\omega)$

(c) If $V = W_{\mathbb{F}}(\omega)$ $=$ and $V^{\mathbb{F}_{\mathbb{K}}} \cong W_{\mathbb{K}}(\omega)$

Warning: $W_{\mathbb{K}}(\omega)$ and $W_{\mathbb{F}}(\omega)$ may have different lengths.

$V_{\mathbb{K}}(\omega)$ and $V_{\mathbb{F}}(\omega)$ may have different length when regarded as $\mathbb{K}[\Lambda^+]$ and $\mathbb{F}[\Lambda^+]$-modules respectively.

Reduction modulo p.

$U(\mathfrak{g})_{\mathbb{F}}$ case: ($\operatorname{char} \mathbb{F} = p$) If $V = V_{\mathbb{C}}(\lambda)$ is an irr. $\mathfrak{g}$-module ($U(\mathfrak{g})_{\mathbb{C}}$-module)

let v be a h.w.v. and $L = U(\mathfrak{g})_{\mathbb{Z}} v$.

Thm: (a) L is a $\mathbb{Z}$-form of V.

(b) $L \otimes_{\mathbb{Z}} \mathbb{F} \cong W_{\mathbb{F}}(\lambda) \rightsquigarrow$ kempf vanishing thm.

Conj: If you reduce modulo p $W_{\mathbb{C}}(\omega)$ ($\omega \in \mathcal{P}_{\mathbb{C}}^+$) you get a Weyl module for $U(\hat{\mathfrak{g}})_{\mathbb{F}}$.