Quantum Hamiltonian reduction

Recall: A_0 a Poisson mfd. $G \curvearrowright A$

moment map is $\varphi: \mathfrak{g} \to Der_\pi(A)$

$\mu: S\mathfrak{g} \to A$ s.t.

$\{\mu_0(a), b\} = \varphi(a) b$ $\quad b \in A$, $a \in \mathfrak{g}$

Def: Suppose A is an ass. algebra with a $\mathfrak{g}$ action $\varphi: \mathfrak{g} \to Der A$

1) A quantum moment map is a map $\mu: U(\mathfrak{g}) \to A$ of algebras s.t.

$[\mu(a), b] = \varphi(a) b$ $\quad a \in \mathfrak{g}$, $b \in A$

2) if A is filtered and grA is Poisson and $gr\,\mu = \mu_0$ we call μ a quantization

Ex:

X mfd. with an action of G

$A = D(X)$ diff op. on X. $\mu: \mathfrak{g} \to Vect\,X(\subset D(X))$

induces $\mu: U(\mathfrak{g}) \to D(X)$ is a moment map,

quantization of of action of G on T^*X.

Q Hamiltonian Reduction

$\mu : U(\mathfrak{g}) \to A$

$A^{\mathfrak{g}} = \{ b \in A \mid [\mu(a), b] = 0 \ \forall a \in \mathfrak{g} \} \subset A$

subalgebra. Let $J \subset A$ be the left ideal generated by $\mu(a)$ $a \in \mathfrak{g}$.

Claim: $J^{\mathfrak{g}} := J \cap A^{\mathfrak{g}}$ is a two-sided ideal in $A^{\mathfrak{g}}$

proof: $c \in A^{\mathfrak{g}}$ $b \in J^{\mathfrak{g}}$ $b = \sum b_i \mu(a_i)$

$$bc = \sum b_i \mu(a_i) c = \sum b_i c \mu(a_i) \in J^{\mathfrak{g}}$$

Def: $A /\!/ \mathfrak{g} := A^{\mathfrak{g}} / J^{\mathfrak{g}}$

Ex: X smooth affine alg. variety with a free action of a connected reductive alg. group G. $A = D(X)$

$$A /\!/ \mathfrak{g} = D(X/G)$$

Ex: $B \hookrightarrow G$ $D(G) /\!/ B = D(G/B)$

Let $I \subset U(\mathfrak{g})$ be a two sided ideal.

Let $J(I) = A\mu(I) \subset A$ then $J(I)^{\mathfrak{g}}$ is a 2-sided ideal in $A^{\mathfrak{g}}$.

set $R(A, \mathfrak{g}, I) := A^{\mathfrak{g}} / J(I)^{\mathfrak{g}}$

Quantization of Calogero-Moser space

$\mathfrak{g} = \mathfrak{gl}_n$ $A = D(\mathfrak{g})$ with adjoint action. Let W_k be the rep. of $\mathfrak{sl}_n$ on the space of functions of the form $(x_1 \cdots x_n)^k f(x_1, \ldots, x_n)$ where f is a Laurent polynomial of deg 0. regard W_k as a $\mathfrak{g}$ module by the map $\mathfrak{g} \to \mathfrak{sl}_n$ and let I_k be its annihilator in $U(\mathfrak{g})$

Let $B_k = R(A, \mathfrak{g}, I_k)$. B_k has a filtration induced from the Bernstein filt. on $D(\mathfrak{g})^{\mathfrak{g}}$. Let $HC_k : D(\mathfrak{g})^{\mathfrak{g}} \to B_k$ and $K(k) = \ker HC_k$.

Thm (Etingof, Ginzburg): (i) $K(0) = K$ $\quad B_0 = D(\mathfrak{h})^W$ $\quad HC_0 = HC$

(ii) $\operatorname{gr} K(k) = \ker \operatorname{gr} HC_k = K_0$ $\forall k \in \mathbb{C}$. Thus HC_k is a flat family of maps.

(ii) $\operatorname{gr} B_k$ is comm. and iso to
$\mathbb{C}[\mathfrak{h}\oplus\mathfrak{h}^*]^W$ as a Poisson alg.

<u>Def:</u> B_k is the spherical rational Cherednik alg of type A_1

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<u>HC</u>

Let $\mathfrak{g}$ be a reductive Lie alg
$A = D(\mathfrak{g})$. want to describe $D(\mathfrak{g})/\!/\mathfrak{g}$

we have a map $\overset{\text{restriction map}}{}$ $\mathbb{C}[\mathfrak{g}]^{\mathfrak{g}} \to \mathbb{C}[\mathfrak{h}]^W$

this induces an ~~map~~ action of $D(\mathfrak{g})^{\mathfrak{g}}$ on $\mathbb{C}[\mathfrak{h}]^W$ and gives a map

radial part $= HC' : D(\mathfrak{g})^{\mathfrak{g}} \to D(\mathfrak{h}_{reg})^W$ since the diff. op will have poles.

Let $\delta(x) = \prod_{\alpha>0} (\alpha, x)$ $x \in \mathfrak{h}$ α pos. root

$HC(D) := \delta \circ HC'(D) \circ \delta^{-1} \in D(\mathfrak{h}_{reg})^W$

Thm: ① Im $HC \subset D(\mathfrak{h})^W \subset D(\mathfrak{h}_{reg})^W$

② (Levasseur, Stafford) HC defines an isomorphism $D(\mathfrak{g})/\!\!/\mathfrak{g} = D(\mathfrak{h})^W$

in $\mathfrak{gl}_n$ case This is a quantization of Gan-Ginzburg and can be proved using GG.

Let U be a Hopf alg. ~~[illegible]~~

Suppose A is a module algebra

$A \otimes A \to A$ is a U module map

$u(ab) = u_1(a)\, u_2(b)$

a moment map for the action is an algebra map $\mu : U \to A$ s.t.

$X \cdot a = \mu(X_1)\, a\, \mu(SX_2)$

Example: adjoint action $U \curvearrowright U$

μ is equivariant w.r.t the adj. action on U and action on A.

Define: $A^U = \{ a \in A \mid xa = \varepsilon(x) a \quad \forall x \in U \}$

Let $I \subset U$ be a two-sided ideal

Let $J(I)$ be the left ideal in A

gen. by $\varphi(I)$ $\quad J(I) = A\,\varphi(I)$

claim: $J(I)^U \subset A^U$ is a two-sided ideal

$J(I)^U = J(I) \cap A^U$

Def: $R(A, I, \varphi) = A^U / J(I)^U$

In case $I = \ker \varepsilon$ the aug. ideal

we write $A/\!/U$.

Quantum ham. reduction and localization

$G \circlearrowright X \qquad T^*X \to D_X$

$$T^*X /\!\!/ G = T^*(X/G) \to D_{X/\!\!/G} = D_{X/G}$$

$$G \circlearrowright T^*X \qquad T^*X /\!\!/ G \to \;?$$

W-algebras

Let M be a complex mfd. $\mathcal{O}_M$ - sheaf of holo. functions. D_M - sheaf of diff. op.

On $T^*\mathbb{C}^n$ we can construct a sheaves over

$$W_{T^*\mathbb{C}^n}(m) = \{ \text{formal series } \sum_{k \geq -m} h^k a_k \quad a_k \in \mathcal{O}_{T^*\mathbb{C}^n} \}$$

$$W_{T^*\mathbb{C}^n} = \bigcup_m W_{T^*\mathbb{C}^n}$$

The sheaf $W_{T^*\mathbb{C}^n}$ has a structure of a $\mathbb{C}((h))$ algebra

$$a \circ b = \sum_{\alpha \in \mathbb{Z}^n_{\geq 0}} h^{|\alpha|} \frac{1}{\alpha!} \partial_p^\alpha a \, \partial_x^\alpha b$$

over $\mathbb{C}((h))$

in other words it is gen by x_i and p_i with relations $[p_i, x_j] = h\delta_{ij}$

we have a ring homo. $D_{\mathbb{C}^n}(\mathbb{C}^n) \to W_{T^*\mathbb{C}^n}(T^*\mathbb{C}^n)$

$x_i \to x_i \quad \frac{\partial}{\partial x_i} \to h^{-1} p_i$

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Let X be a complex symp. mfd with form ω_X.

a W-alg. is a $\mathbb{C}((h))$-alg. W on X

s.t. for any point $x \in X$ $\exists\, U \subset X$ $x \in U$

a symp. map $f : U \to T^*\mathbb{C}^n$ and a $\mathbb{C}((h))$-alg iso $g : W|_U \xrightarrow{\sim} f^{-1} W_{T^*\mathbb{C}^n}$

a W-alg W satisfies:

1. it is coherent and noetherian
2. W contains a canonical subalg. $W(0)$ loc. iso. to $W_{T^*\mathbb{C}^n}(0)$. set

$W(n) = h^{-n} W(0)$.

3. we have a canonical iso. $W(0)/W(1) \xrightarrow{\sim} \mathcal{O}_X$.

call the mor. $\sigma_n : W(n) \to h^{-n}\mathcal{O}_X$ the symbol map.

4. $\sigma_0(h^{-1}[a,b]) = \{\sigma_0(a), \sigma_0(b)\}$ (Poisson)

5. Given a $\mathbb{C}((h))$-auto of W φ loc. $\exists$ an invertible section a of $W(0)$ s.t. $\varphi = Ad(a)$ - unique upto scaler so we have

$$W(0)^*/\mathbb{C}[[h]]^* \xrightarrow[Ad]{\sim} \mathrm{Aut}(W(0))$$
$$\wr\downarrow \qquad\qquad \downarrow\wr$$
$$W^*/\mathbb{C}((h))^* \xrightarrow[Ad]{\sim} \mathrm{Aut}(W)$$

(5) Let r be a $\mathbb{C}((h))$-linear filtration preserving derivation of W. $\exists$ be a section a of $W(1)$ s.t. $r = ad(a)$. unique up to scalar

$$W(1)/h^{-1}\mathbb{C}[[h]] \xrightarrow[ad]{\sim} \mathrm{Der}_{filt}(W).$$

Assume $\exists\ a_i, b_i \in W(0)$ s.t. $[a_i, a_j] = [b_i, b_j] = 0$

$[b_i, a_j] = h\,\delta_{ij}$, They induce a map

$$f = (\sigma_0(a_1) \cdots, \sigma_0(a_n); \sigma_0(b_1) \cdots \sigma_0(b_n)) ; X \to T^*\mathbb{C}^n$$

$$W \xrightarrow{\sim} f^{-1} W_{T^*\mathbb{C}^n} \qquad a_i \mapsto x_i \quad b_i \mapsto p_i$$

Let M be a complex mfd. $\pi_M : T^*M \to M$

We have a canonical W-alg W_{T^*M} with a morphism $\pi^{-1}D_M \to W_{T^*M}$

Suppose we have an action of $\mathbb{C}^* = \mathbb{G}_m$ on X $\quad \mathbb{C}^* \ni t \to T_t \in Aut(X)$.

Assume $T_t^* w_X = t^m w_X \quad \mathbb{Z} \ni m > 0$

Def: An F-action with exp. m on W is an action of $\mathbb{C}^*$ on the $\mathbb{C}$-alg W

$F_t : T_t^{-1} W \xrightarrow{\sim} W$ s.t. $F_t(h) = t^m h$ and $F_t(a)$ depends holo. on t for any $a \in W$.

We can define F-equivariant modules.

Ex: Let M be a mfld $X = T^* M$

$$Mod_F^{good}(W) \xrightarrow{\sim} Mod_{good}(\mathcal{D}_M)$$

$$M \mapsto M^{F_m}$$

Equivariant W-modules

Let G be a complex Lie group acting on a symp. manifold X, $g \mapsto T_g$ the symp. auto. $\mu_X : X \to \mathfrak{g}^*$ a moment map.

A W-alg. with G-action is: we have $\mathbb{C}((h))$ alg. iso $\rho_g : W \xrightarrow{\sim} T_g^{-1} W$, $\rho_g(a)$ depends holo. on $g \in G$.

We assume we have a quantized moment map: $\mu_W : \mathfrak{g} \to W(1)$ s.t.

$$[\mu_W(A), a] = \frac{d}{dt} \rho_{\exp(tA)}(a)\Big|_{t=0}$$

$$\sigma_0(h\mu_W(A)) = A \circ \mu_X$$

$$\mu_W(Ad(g)A) = \rho_g(\mu_W(A)).$$

μ_W is a Lie alg. homo.

Def: A quasi-G-equi. W-module is a W-module M with an action of G:

$$\rho_g : M \xrightarrow{\sim} T_g^{-1} M \quad \text{(holo.)}$$

$$\rho_g(au) = \rho_g(a)\rho_g(u)$$

This gives a Lie alg homomorphism $\alpha : \mathfrak{g} \to End_{\mathbb{C}((h))}(M)$

$$\alpha(A)(u) = \frac{d}{dt} \rho_{\exp(tA)} u \Big|_{t=0} \quad A \in \mathfrak{g} \quad u \in M.$$

It satisfies

$$\alpha(A)(au) = [\mu_W(A), a]u + a \cdot \alpha(A)(u)$$

so we have

$$\gamma_M : \mathfrak{g} \to \mathrm{End}_W(M) \quad A \mapsto \alpha(A) - \mu_W(A)$$

Let $\lambda \in (\mathfrak{g}^*)^G$ if

$$\gamma_M = \lambda : \mathfrak{g} \xrightarrow{\lambda} \mathbb{C} \to \mathrm{End}_W(M)$$

we say that M is a twisted G-equi. W-module with twist λ. If M is coherent we will have $\mathrm{supp}(M) \subset \mu_X^{-1}(0)$.

Denote by $\mathrm{Mod}(W, G)$ the cat. of quasi-G-equi. and by $\mathrm{Mod}^G_\lambda(W)$ the full sub cat. of twisted ~~G~~ G-equiv with twist λ.

The embedding $\mathrm{Mod}^G_\lambda \to \mathrm{Mod}(W, G)$ has a left adjoint

$$\phi_\lambda : \mathrm{Mod}(W, G) \to \mathrm{Mod}^G_\lambda(W)$$

$$\phi_\lambda(M) = M \Big/ \sum_{A \in \mathfrak{g}} (\gamma_M(A) - \lambda(A))M$$

Symplectic reduction

~~Def~~ Assume G acts prop and freely
$G\times X \to X\times X$ $\quad (g,x)\mapsto (gx,x)$ is a closed embedding.

Let $Z = \mu_X^{-1}(0)/G$ $\quad p : \mu_X^{-1}(0) \to Z$

Let W be a W-alg on X with a G-action.

Let $\lambda \in (\mathfrak{g}^*)^G$. set

$$\mathcal{L}_\lambda := \Phi_\lambda(W) = W\Big/\Big(\sum_{A\in\mathfrak{g}} (\mu_W(A)+\lambda(A))W\Big)$$

the support of $\mathcal{L}_\lambda$ is $\mu_X^{-1}(0)$.

Let $\quad W_Z := \big((p_* \mathcal{E}nd_W(\mathcal{L}_\lambda))^G\big)^{op}$

a sheaf of $\mathbb{C}((h))$-algebras on Z.

Prop: ① W_Z is a W-alg on Z

② we have equiv

$$\mathrm{Mod}^{(good)}(W_Z) \leftrightarrows \mathrm{Mod}^{G\text{-good}}_\lambda(W)$$

$$N \mapsto \mathcal{L}_\lambda \otimes_{p^{-1}W_Z} p^{-1}N$$

$$(p_* \mathcal{H}om_W(\mathcal{L}_\lambda, M))^G \leftarrow M$$

Localization

X symp.

Let $X \to S$ be a proj morphism.

L rel. ample line bundle on X. Let W be a W-alg on X.

<u>Thm</u> (BB): for $n \gg 0$ let $\mathcal{L}_n(0)$ be a loc. free $W(0)$-mod of rank 1 such that $\mathcal{L}_n(0)/h\,\mathcal{L}_n(0) = L^{\otimes(-n)}$. Set

$$\mathcal{L}_n = W \otimes_{W(0)} \mathcal{L}_n(0).$$

<u>Suppose</u>: (*) for $n \gg 0$ $\exists$ v.s. V_n and a split epimorphism

$$\mathcal{L}_n \otimes V_n \twoheadrightarrow W \quad (W \text{ is a direct summand of fin many copies of } \mathcal{L}_n).$$

(**) for $n \gg 0$ $\exists$ v.s. V_n and a epi

$$W \otimes V_n \twoheadrightarrow \mathcal{L}_n$$

(*) $\Rightarrow$ $R^i\Gamma_X(M) = 0$ $i \neq 0$ M W-good

(**) $\Rightarrow$ every W-mod M is gen. by its global sections (loc. on S).

Kashiwara – Rouquier

$$\begin{tikzcd} \mathrm{Hilb}_n \arrow[r, "\text{quantize}"] \arrow[d] & W\text{-alg} \arrow[d] \\ (\mathbb{C}^2)^n / S_n \arrow[r, "\text{quantize}"] & \text{spherical rational cherednik} \end{tikzcd}$$

$$\begin{tikzcd} T^*G/B \arrow[r] \arrow[d] & D_{G/B} \arrow[d] \\ N \arrow[r] & U_0 \end{tikzcd}$$