Quiver varieties (à la Nakajima)

Let (I, E) be a finite graph.

Let H be the set of pairs of an edge with an orientation. Fix an orientation $\Omega \subset H$

$\Omega \cup \bar{\Omega} = H \quad \Omega \cap \bar{\Omega} = \emptyset$. (I, Ω) is a quiver.

Let $V = (V_k)_{k \in I}$ be a collection of f.d. v.s. over $\mathbb{C}$. $\dim V = (\dim V_k)_{k \in I} \in \mathbb{Z}_{\geq 0}^I$

define for two such collections V^1, V^2

$L(V^1, V^2) := \bigoplus_{k \in I} \operatorname{Hom}(V^1_k, V^2_k) \quad E(V^1, V^2) = \bigoplus_{h \in H} \operatorname{Hom}(V^1_{out(h)}, V^2_{in(h)})$

if $a \in L(V^1, V^1)$ $\quad tr(a) = \sum tr(a_k)$

for two collections V, W with $v = \dim V$ and $w = \dim W$ define

$$M(v,w) := \underset{\substack{\downarrow \\ B}}{E(V,V)} \oplus \underset{\substack{\downarrow \\ i}}{L(W,V)} \oplus \underset{\substack{\downarrow \\ j}}{L(V,W)}$$

for a collection of subspaces $S = (S_k)_{k \in I}$ of V_k and $B \in E(V,V)$ we say S is B-invariant if $B_h(S_{out(h)}) \subset S_{in(h)}$

for $B = (B_h) \in E(V^1, V^2)$ $\quad C = (C_h) \in E(V^2, V^3)$

define

$$CB := \Big(\sum_{in(h)=k} C_h B_{\bar{h}} \Big)_k \in L(V^1, V^3)$$

also if $a \in L(V^1, V^2)$, $b \in L(V^2, V^3)$

$B \in E(V^2, V^3)$ we can define

ba, Ba.

fix $\varepsilon : H \to \mathbb{C}^*$ s.t. $\varepsilon(h) + \varepsilon(\bar{h}) = 0$

for $B \in E(V^1, V^2)$ $\varepsilon B \in E(V^1, V^2)$

$$(\varepsilon B)_h = \varepsilon(h) B_h \qquad h \in H$$

<u>Symplectic form:</u>

$$\omega\big((B, i, j), (B', i', j')\big) := \mathrm{tr}(\varepsilon B B') + \mathrm{tr}(ij' - i'j)$$

$$G = G_V = \prod GL(V_k)$$

$G \curvearrowright M$

$$g \cdot (B, i, j) = (gBg^{-1}, gi, jg^{-1})$$

preserving the sym. form.

A moment map vanishing at the origin is given by

$$\mu : M \to L(V,V) \approx L(V,V)^*$$

$$\mu(B,i,j) = \varepsilon BB + ij$$

Define $\quad \mathcal{M}_0 = \mathcal{M}_0(V,W) := \mu^{-1}(0) /\!\!/ G$

$$= \operatorname{Spec} \mathbb{C}[\mu^{-1}(0)]^G$$

GIT $\Rightarrow$ this is an affine alg. var.

points $\Leftrightarrow$ closed G-orbits.

Define $\quad \chi : G \to \mathbb{C}^* \qquad \chi(g) = \prod \det g_k^{-1}$

Set $\quad \mathbb{C}[\mu^{-1}(0)]^{G,\chi^n} :=$

$$\{ f \in \mathbb{C}[\mu^{-1}(0)] \mid f(g(B,i,j)) = \chi(g)^n f(B,i,j) \}$$

$$\mathcal{M} = \mathcal{M}(V,W) := \operatorname{Proj} \bigoplus_{n \geq 0} \mathbb{C}[\mu^{-1}(0)]^{G,\chi^n}$$

Quiver varieties.

Def: $(B, i, j) \in \mu^{-1}(0)$ is stable if:

$S = (S_k) \subset V = (V_k)$ is B-inv and $S \subset \ker j \Rightarrow S = 0$.

$\mu^{-1}(0)^s$ is the set of stable points.

define a G action on $\mu^{-1}(0) \times \mathbb{C}$ by

$g(B, i, j, z) = (g \cdot (B, i, j), \chi^{-1}(g) z)$

Prop: ① (B, i, j) is stable iff the closure of $G \cdot (B, i, j, z)$ does not intersect with the zero section for $z \neq 0$.

② if (B, i, j) is stable then the diff $d\mu$ is surjective. so $\mu^{-1}(0)^s$ is non sing.

③ $\mu^{-1}(0)^s / G$ has a structure of non sing. quasi-proj. variety and $\mu^{-1}(0)^s$ is a principal G-bundle over it.

④ $\mathfrak{M} = \mu^{-1}(0)^s / G$

⑤ $\mu^{-1}(0)^s / G$ has a holo. symp. structure.

we have a proj. morphism $\pi : \mathfrak{M} \to \mathfrak{M}_0$

if $\pi [B, i, j] = [B^0, i^0, j^0]$ then $G \cdot (B^0, i^0, j^0)$ is the unique closed orbit contained in the closure of $G (B, i, j)$

Hyper Kähler structure

Def; Let X be a $2n$-dim mfd.

a Kähler structure on X is a Riem. metric g and an almost complex str. I s.t.:

(i) g is Hermitian for I $g(Iv, Iw) = g(v, w)$
(ii) I is integrable.
(iii) define $\omega(v, w) = g(Iv, w)$ then

$d\omega = 0$ (ω is the Kähler form).
holonomy of $\nabla \subset U(n)$

Def! Let X be a $4n$-dim. mfd. A

Hyper Kahler structure is given by g and I, J, K s.t:

(i) $g(Iv, Iw) = g(Jv, Jw) = g(Kv, Kw) = g(v, w)$
(ii) $I^2 = J^2 = K^2 = IJK = -1$
(iii) I, J, K are parallel w.r.t the Levi-Civita connection of g $\nabla I = \nabla J = \nabla K = 0$.

(holonomy of $\nabla \subset Sp(n)$)

define $\omega_{\mathbb{C}} = \omega_2 + \sqrt{-1}\,\omega_3$ is of type $(2,0)$

$d\omega_{\mathbb{C}} = 0$ meaning $\omega_{\mathbb{C}}$ is a holo. symp. form.

Hyper-Kähler reduction

Let (X, g, I, J, K) be a hK mfd, $\omega_1, \omega_2, \omega_3$ Kähler forms. Suppose a compact Lie group ~~G~~ K acts on X preserving g, I, J, K.

Def: A map $\mu := (\mu_1, \mu_2, \mu_3) : X \to \mathbb{R}^3 \otimes \mathfrak{g}^*$ is a hK moment map if

1. μ is G-equivariant.
2. ~~$\langle d\mu_i(v), \xi \rangle = \omega_i(\xi^*$~~ μ_i is a moment map for ω_i.

Take $\gamma = (\gamma_1, \gamma_2, \gamma_3) \in \mathbb{R}^3 \otimes \mathfrak{g}^*$ which is inv. for the coadj. action.

Thm: (Hitchin) Suppose the G-action on $\mu^{-1}(\gamma)$ is free then $\mu^{-1}(\gamma)/G$ is a smooth mfd. and has a hK structure.

if we fix hermitian inner products on $\vec{V}$ and W and an orientation $\mathcal{N}$ we get a Hermitian inner product and a quaternion structure on M.

put $K = \prod U(V_k)$

We have

$$\mu : M \to L(V,V)$$

$$\mu_{\mathbb{R}} : M \to \mathfrak{u}(V_k)$$

$$\mu_{\mathbb{R}}(B,i,j) = \frac{\sqrt{-1}}{2}\Big(\sum_{\substack{h\in H \\ k=\mathrm{in}(h)}} B_h B_h^* - B_{\bar h}^* B_{\bar h} + i_k i_k^* - j_k^* j_k\Big)_k$$

$$\mu_{\mathbb{R}}^{-1}(0) \cap \mu^{-1}(0)/K \cong \mu^{-1}(0)//G = \mathfrak{M}_0(v,w)$$

$$(\gamma)_k \in \sqrt{-1}\,\mathbb{R}_{>0}$$

$$\mu_{\mathbb{R}}^{-1}(-\gamma) \cap \mu^{-1}(0)/K \cong \mu^{-1}(0)^s/G = \mathfrak{M}(v,w)$$

Quantum Hamiltonian reduction

<u>Recall</u>; A_0 a Poisson mfd. $G \curvearrowright A$

moment map is $\varphi: \mathfrak{g} \to Der_\pi(A)$

$$\mu: S\mathfrak{g} \to A \quad \text{s.t.}$$

$$\{\mu_0(a), b\} = \varphi(a)b \qquad \begin{matrix} b \in A \\ a \in \mathfrak{g} \end{matrix}$$

<u>Def</u>: Suppose A is an ass. algebra with a $\mathfrak{g}$ action $\varphi: \mathfrak{g} \to Der A$

1) A quantum moment map is a map $\mu: U(\mathfrak{g}) \to A$ of algebras s.t.

$$[\mu(a), b] = \varphi(a)b \qquad \begin{matrix} a \in \mathfrak{g} \\ b \in A \end{matrix}$$

2) if A is filtered and grA is Poisson and $gr\mu = \mu_0$ we call μ a quantization

<u>Ex</u>:

X mfd. with an action of G

$A = D(X)$ diff op. on X. $\mu: \mathfrak{g} \to Vect X \subset D(X)$

induces $\mu: U(\mathfrak{g}) \to D(X)$ is a moment map, quantization of of action of G on T^*X.

Q Hamiltonian Reduction

$$\mu : U(\mathfrak{g}) \to A$$

$A^{\mathfrak{g}} = \{ b \in A \mid [\mu(a), b] = 0 \ \forall a \in \mathfrak{g} \} \subset A$ subalgebra. Let $J \subset A$ be the left ideal generated by $\mu(a)$ $a \in \mathfrak{g}$.

__claim:__ $J^{\mathfrak{g}} := J \cap A^{\mathfrak{g}}$ is a two-sided ideal in $A^{\mathfrak{g}}$

__proof:__ $c \in A^{\mathfrak{g}}$ $b \in J^{\mathfrak{g}}$ $b = \sum b_i \mu(a_i)$

$$bc = \sum b_i \mu(a_i) c = \sum b_i c \mu(a_i) \in J^{\mathfrak{g}}$$

__Def:__ $A /\!/ \mathfrak{g} := A^{\mathfrak{g}} / J^{\mathfrak{g}}$

__Ex:__ X smooth affine alg. variety with a free action of a connected reductive alg. group G. $A = D(X)$

$$A /\!/ \mathfrak{g} = D(X/G)$$

__Ex:__ $B \curvearrowright G$ $\quad D(G) /\!/ B = D(G/B)$

Let $I \subset U(\mathfrak{g})$ be a two sided ideal.
Let $J(I) = A\mu(I) \subset A$ then $J(I)^{\mathfrak{g}}$ is a 2-sided ideal in $A^{\mathfrak{g}}$.
set $R(A, \mathfrak{g}, I) := A^{\mathfrak{g}} / J(I)^{\mathfrak{g}}$

Quantization of Calogero-Moser space

$\mathfrak{g} = \mathfrak{gl}_n$ $A = D(\mathfrak{g})$ with adjoint action. Let W_k be the rep. of $\mathfrak{sl}_n$ on the space of functions of the form $(x_1 \cdots x_n)^k f(x_1, \ldots x_n)$ where f is a Laurent polynomial of deg 0. regard W_k as a $\mathfrak{g}$ module by the map $\mathfrak{g} \to \mathfrak{sl}_n$ and let I_k be its annihilator in $U(\mathfrak{g})$
Let $B_k = R(A, \mathfrak{g}, I_k)$. B_k has a filtration induced from the Bernstein filt. on $D(\mathfrak{g})^{\mathfrak{g}}$, Let $HC_k : D(\mathfrak{g})^{\mathfrak{g}} \to B_k$ and $K(k) = \ker HC_k$.

Thm (Etingof, Ginzburg) : (i) $K(0) = K$ $B_0 = D(\mathfrak{h})^W$
$HC_0 = HC$
(ii) $\operatorname{gr} K(k) = \ker \operatorname{gr} HC_k = K_0$ $\forall k \in \mathbb{C}$ Thus HC_k is a flat family of maps.

(iii) $\operatorname{gr} B_k$ is comm. and iso to
$\mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}^*]^W$ as a Poisson alg.

<u>Def:</u> B_k is the spherical rational Cherednik alg of type A_1

<u>HC</u>

Let $\mathfrak{g}$ be a reductive Lie alg
$A = D(\mathfrak{g})$. want to describe $D(\mathfrak{g})/\!/\mathfrak{g}$

we have a ~~map~~ restriction map $\mathbb{C}[\mathfrak{g}]^{\mathfrak{g}} \to \mathbb{C}[\mathfrak{h}]^W$
this induces an ~~map~~ action of $D(\mathfrak{g})^{\mathfrak{g}}$
on $\mathbb{C}[\mathfrak{h}]^W$ and gives a map
radial part $= HC' : D(\mathfrak{g})^{\mathfrak{g}} \to D(\mathfrak{h}_{reg})^W$ since the
diff. op will have poles.

Let $\delta(x) = \prod_{\alpha > 0} (\alpha, x)$ $x \in \mathfrak{h}$ α pos. root
$HC(D) := \delta \circ HC'(D) \circ \delta^{-1} \in D(\mathfrak{h}_{reg})^W$

Thm: ① I_h $HC \subset D(\mathfrak{h})^W \subset D(\mathfrak{h}_{reg})^W$
② (Levasseur, Stafford) HC defines an isomorphism $D(\mathfrak{g})/\!/_{\mathfrak{g}} = D(\mathfrak{h})^W$

in $\mathfrak{gl}_n$ case This is a quantization of Gan-Ginzburg and can be proved using GG.

Let U be a Hopf alg. ~~~~
Suppose A is a module algebra
$A \otimes A \to A$ is a U module map
$u(ab) = u_1(a)\, u_2(b)$
a moment map for the action is an algebra map $\mu : U \to A$ s.t.

$$X \cdot a = \mu(X_1)\, a\, \mu(S X_2)$$

Example: adjoint action $U \circlearrowright U$

μ is equivariant w.r.t. the adj. action on U and action on A.

Define: $A^U = \{ a \in A \mid xa = \varepsilon(x) a \quad \forall x \in U \}$

Let $I \subset U$ be a two-sided ideal

Let $J(I)$ be the left ideal in A

gen. by $\varphi(I)$ $\quad J(I) = A\,\varphi(I)$

claim: $J(I)^U \subset A^U$ is a two-sided ideal

$$\underset{\displaystyle J(I) \cap A^U}{\overset{\shortparallel}{J(I)^U}}$$

Def: $R(A, I, \varphi) = A^U / J(I)^U$

in case $I = \ker \varepsilon$ the aug. ideal

we write $A /\!\!/ U$.