① Symp. and Poisson mflds, moment maps

Let A be a comm. alg over a field k

Def: ① A is Poisson if it is equipped with a Lie bracket $\{,\}$ which is a derivation

$$\{a, bc\} = \{a,b\}c + b\{a,c\}.$$

② an ideal $I \subset A$ is Poisson if $\{A, I\} \subset I$. then A/I is Poisson.

③ a mfd M is Poisson if $C^\infty(M)$ is Poisson (alg., analytic)

④ a Poisson map is a regular map $M \xrightarrow{\pi} N$ s.t. $C^\infty(M) \xrightarrow{\pi^*} C^\infty(N)$ is Poisson.

the Poisson structure is defined by a bivector $\Pi \in \Gamma(M, \Lambda^2 TM)$ s.t.

$[\Pi, \Pi] = 0$ $\qquad \{f, g\} = (df \otimes dg)(\Pi)$

if M is symp. then $\Pi = \omega^{-1}$ is Poisson and nondeg Poisson is symp.

we have a map of Lie algebras

$$v : C^\infty(M) \to Vect_\pi(M)$$

Poisson preserving v.f.

$f \mapsto \{f, ?\}$ $\quad$ $v(f)$ is the Hamiltonian v.f.

Ex: ① $M = T^*X$ symp.

$$\omega = \sum dp_i \wedge dq_i$$

② $\mathfrak{g}$ f.d. Lie alg, $\pi : \mathfrak{g}^* \to \Lambda^2 \mathfrak{g}^*$ is Poisson on $\mathfrak{g}^*$

if $O \subset \mathfrak{g}^*$ is a coadjoint orbit π is nondeg. so O is symp.

Moment maps

Let G, Lie group, act on M by Poisson automorphisms. we have $\varphi : \mathfrak{g} \to Vect_\pi(G)$

Def: A G equivariant regular map $\mu : M \to \mathfrak{g}^*$ is a moment map for the action if $\mu^* : \mathfrak{g} \to C^\infty(M)$ satisfies

$$v(\mu^*(a)) = \varphi(a).$$

Ex: ~~Show that if~~ if M is s.c. and symp and G is compact it exists.

Ex: $M = T^*X \qquad G \circlearrowright X \qquad \mu : T^*X \to \mathfrak{g}^*$

$\mu(x,p)(a) = p(\psi(a)) \qquad \psi : \mathfrak{g} \to \mathrm{Vect}(X)$

μ is a moment map.

Hamiltonian Reduction

$G \circlearrowright M$ as before $\mu : M \to \mathfrak{g}^*$

$C^\infty(M)^G$ is a Poisson alg.

Let J be the ideal in $C^\infty(M)$ gen. by $\mu^*(a)$ $a \in \mathfrak{g}$

J is invariant under Poisson bracket with $C^\infty(M)^G$ hence Poisson ideal in $C^\infty(M)^G$

so $A := C^\infty(M)^G / J^G$ is Poisson.

Meaning: Assume G acts properly i.e. for any two compact sets K_1, K_2

$\{g \in G \mid gK_1 \cap K_2 \neq \emptyset\}$ is compact.

Assume the action is free then

M/G is a manifold and $C^\infty(M)^G = C^\infty(M/G)$

also μ is a submersion (Ex. need action to be loc. free stab is discrete) so $\mu^{-1}(0)$ is a smooth submfd.

and the ideal J^G corresponds to the submfd. $M/\!/G = \mu^{-1}(0)/G$

so $A = C^\infty(M/\!/G)$ and $M/\!/G$ is Poisson. (Check that G acts freely near $\mu^{-1}(0)$)

if M is symp. so is $M/\!/G$

in alg. case $M = \operatorname{spec}(B)$ and G affine alg. group.

suppose G reductive then $M/\!/G \overset{= \operatorname{spec}(A)}{}$ is affine Poisson scheme (possi. non red. and sing.).

Example: Let G act properly and freely on X and $M = T^*X$

then $T^*X/\!/G \cong T^*(X/G)$

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if the action is not free it can be complicated.

Ex: $M = T^* Mat_n(\mathbb{C})$ $G = PGL_n(\mathbb{C})$ $\mathfrak{g} = sl_n(\mathbb{C})$

using trace identify $\mathfrak{g}^*$ with $\mathfrak{g}$ and

M with $Mat_n(\mathbb{C}) \oplus Mat_n(\mathbb{C})$ then a

moment map is given by $\mu(X, Y) = [X, Y]$

$\mu^{-1}(0)$ is the comm. scheme $[X, Y] = 0$
(don't know if reduced, the variety is irred but very sing).

Thm (GG): $Comm(n)/G$ is reduced and iso.

to $\mathbb{C}^{2n}/S_n$. $\left(\mathrm{spec}\ \mathbb{C}[x_1 \ldots x_n, y_1 \ldots y_n]^{S_n}\right)$

~~where~~ with the usual Poisson structure
on $\mathbb{C}^{2n}$.

can do the same for any $\mathfrak{g}$ simple
Lie alg.

variety $\overline{Comm(\mathfrak{g})}$ is irred. (Richardson) and

$\overline{Comm(\mathfrak{g})}/G = \mathfrak{h} \oplus \mathfrak{h}/W$ (Joseph)

Hamiltonian reduction along an orbit

Let $\mathcal{O} \subset \mathfrak{g}^*$ be a closed coadjoint orbit

we can look also at $\mu^{-1}(\mathcal{O})/G$

If $I_{\mathcal{O}}$ is the ideal in $S\mathfrak{g}$ of $\mathcal{O}$

and J_G the ideal in $\mathbb{C}[M]$ gen by $\mu^*(I_\mathcal{O})$ then $\mathbb{C}[\mu^{-1}(\mathcal{O})/G] =$

$\mathbb{C}[M]^G / J_G^G$

$= \mu^{-1}(z)/G_z$ $\quad z \in \mathcal{O}$ $\quad G_z$ is the stab of z.

Notation $\mu^{-1}(\mathcal{O})/G = R(M, G, \mathcal{O})$

We can define reduction along any G-inv. Zariski closed subset.

Ex: Calogero-Moser space

Let $M = T^* Mat_n(\mathbb{C})$ $\quad G = PGL_n(\mathbb{C})$

and $\mathcal{O}$ the orbit of $\begin{pmatrix} -1 & & & \\ & -1 & & \\ & & \ddots & \\ & & & n-1 \end{pmatrix}$

traceless matrices T s.t. $T + I$ has rank 1.

Kazhdan-Kostant-Sternberg

Def: $C_n := R(M, G, \mathcal{O})$ is the Calogero-Moser space

$\{(X, Y) \mid XY - YX + I \text{ has rank } 1\} / G$

Thm: ~~the~~ G acts freely on $\mu^{-1}(0)$ and thus C_n is a smooth sym. variety (of dim $2n$).

proof: if X, y s.t. $[X, y] + I$ has rank 1 then (X, y) is irred. set of matrices.

then if B is s.t. $BX = XB$, $By = yB$, B is a scalar so the stab. is trivial.

if W is an inv subspace the eigenvalues of $[X, y]$ on W are a subcollection of $\{\underline{-1, -1, \ldots}, -1, n-1\}$ the sum is zero ($\mathrm{tr}[X, y]^{n-1} = 0$) so the collection is the entire collection $W = \mathbb{C}^n$.

Thm: C_n is connected.

GIT and the moment map

Let G be an alg group acting on a variety X. $\varphi: X \to M$ is G-equivariant

Def: (φ, M) is called a categorical quotient if it is universal

$$\begin{tikzcd} X \arrow[r] \arrow[dr] & M \arrow[d, dotted] \\ & M' \end{tikzcd}$$

a categorical quotient is an orbit space if the ~~fibers~~ fibers of φ are the G-orbits in X.

(φ, M) is a geometric quotient if φ is open, fibers are orbits and for any open $U \subset M$ $\varphi^{-1}(U) \to U$ induces an iso $R[\varphi^{-1}(U)]^G$ and $R[U]$.

Let G be reductive

Thm: Let $X = \operatorname{Spec}(R)$ be an affine variety with a G action. R^G is a f.g. k-alg. and $\varphi: X \to \operatorname{Spec}(R^G) = X /\!\!/ G$ is a categorical quotient. φ sends disjoint G-inv closed sets to disjoint closed sets in ~~Sp~~ $X /\!\!/ G$.

$\exists$ open $U \subset X /\!\!/ G$ (maybe empty) s.t. $\varphi^{-1}(U)$ consists of orbits of maximal dimension which are closed. $\varphi|_{\varphi^{-1}(U)}$ is a geometric quotient.

if G not reductive need not be affine

G/B

If X not affine (G reductive) and action is free a quotient need not exist.

If $X = \cup U_i$ open affine G-inv

we can glue local quotients but may get non-separated.

$\mathbb{C} \times \mathbb{C} \setminus \{(0,0)\} / \mathbb{C}^*$ $\quad \lambda(a,b) = (\lambda a, \lambda^{-1} b)$

we get line with origin doubled.

(*) $\varphi(x) = \varphi(y) \iff \overline{Gx} \cap \overline{Gy} \neq \emptyset$

the points of $V/\!/G$ is the set of closed G orbits modulo (*)

Let V be a $\mathbb{C}$ v.s. with hermitian metric G conn. compact Lie sub group $G^{\mathbb{C}}$ complexification

define $\mu : V \to \mathfrak{g}^*$

$$\mu(x)(a) = \frac{1}{2}(\sqrt{-1}\, a x, x) \qquad x \in V \quad a \in \mathfrak{g}$$

Thm: (Kempf Ness) $\exists$ a bijection between $G^{\mathbb{C}}$-closed orbits in V and

$\mu^{-1}(0)/G$

Example: $V = \mathrm{End}(\mathbb{C}^n)$ $U(n) \curvearrowright V$

$$B \mapsto g^{-1} B g$$

$$\mu(B) = \frac{-\sqrt{-1}}{2} [B, B^*]$$

$$\{\text{closed } GL(n,\mathbb{C}) \text{ orbits}\} \cong \{B \mid [B, B^*] = 0\} / U(n)$$

$\updownarrow$ $\updownarrow$

diag. Normal.

Example: V, W hermitian v.s. $\dim V = n$, $\dim W = 1$

$$M = \mathrm{End}(V) \oplus \mathrm{End}(V) \oplus \mathrm{Hom}(W,V) \oplus \mathrm{Hom}(V,W)$$

$U(V)$ acts on M

$$(B_1, B_2, i, j) \mapsto (g^{-1}B_1 g, g^{-1}B_2 g, g^{-1} i, j g)$$

the moment map is

moment map $\mu_{\mathbb{R}}(B_1, B_2, i, j) = \frac{\sqrt{-1}}{2}\left([B_1, B_1^*] + [B_2, B_2^*] + i i^* - j^* j\right)$

$(\mathbb{C}^2)^n / S_n$

$$\mu_{\mathbb{C}}(B_1, B_2, i, j) = [B_1, B_2] + ij$$

$\|$

$S^n(\mathbb{C}^2) = \{\text{closed } GL(V) \text{ orbits in } \mu_{\mathbb{C}}^{-1}(0)\} \cong$

$$\mu_{\mathbb{R}}^{-1}(0) \cap \mu_{\mathbb{C}}^{-1}(0) / U(V)$$

(hyper Kahler reduction ... later)

Can we get the Hilbert scheme $(\mathbb{C}^2)^{[n]}$?

$$(\mathbb{C}^2)^{[n]} = \{ \underset{\text{ideal}}{I \subset \mathbb{C}[x,y]} \mid \dim \mathbb{C}[x,y]/I = n \}$$

<u>Thm:</u> $\cancel{\#}$ $$(\mathbb{C}^2)^{[n]} = \Big\{ (B_1, B_2, i) \;\Big|\; B_1, B_2 \in \mathrm{End}(\mathbb{C}^n) \quad i \in \mathrm{Hom}(\mathbb{C}, \mathbb{C}^n)$$

① $[B_1, B_2] = 0$

② $\nexists$ subspace ~~in~~ $S \subsetneq \mathbb{C}^n$ s.t. $B_1 S \subset S$ $B_2 S \subset S$ and $\mathrm{im}\, i \subset S \Big\} \Big/ GL_n(\mathbb{C})$

$$g(B_1, B_2, i) = (gB_1g^{-1}, gB_2g^{-1}, gi).$$

Suppose we have I then $V = \mathbb{C}[x,y]/I$

x_1 x_2 give B_1 B_2 and 1 give i.

$i(1) = 1$.

Let $\chi : G \to U(1)$ be a character
$\chi : G^{\mathbb{C}} \to \mathbb{C}^*$.

consider $V \times \mathbb{C}$ the trivial line bundle

with action $g(x, z) = (gx, \chi(g)^{-1} z)$

Let $\mathcal{O}(V)^{G^{\mathbb{C}}, \chi} = \{ f \in \mathcal{O}(V) \mid f(gv) = \chi(g) f(v) \}$

$G^{\mathbb{C}}$ inv. sections of the line bundle.

<u>Define</u>: $V /\!\!/_{\chi} G^{\mathbb{C}} = \mathrm{Proj} \left(\bigoplus_{n \geq 0} \mathcal{O}(V)^{G^{\mathbb{C}}, \chi^n} \right)$

$\mathcal{O}(V)^{G^{\mathbb{C}}} \subset \bigoplus \mathcal{O}(V)^{G^{\mathbb{C}}, \chi^n}$

so we get a proj. morphism

$V /\!\!/_{\chi} G^{\mathbb{C}} \to V /\!\!/ G^{\mathbb{C}}$

$x \in V$ is χ-semistable if $\exists f \in \mathcal{O}(V)^{G^{\mathbb{C}}, \chi^n}$
$n \geq 1$ s.t. $f(x) \neq 0$. Let $V^{ss}(\chi)$ be
the set of semistable points.

Say $x \sim y$ iff $\overline{G \cdot x}$ and $\overline{G y}$ intersect
in $V^{ss}(\chi)$. Then $V /\!\!/_{\chi} G^{\mathbb{C}} = V^{ss}(\chi) / \sim$

$\nabla \times \mathbb{C}$ - zero section

closed orbits in

Claim: $\mu^{-1}(\sqrt{-1}\, d\chi)/G = \{ \cancel{x} \ G^{\mathbb{C}}(x,z)$ is closed for $z \neq 0 \}$

$$(\mathbb{C}^2)^{[n]} = \mu_{\mathbb{C}}^{-1}(0) /\!\!/_{\chi} GL_n(\mathbb{C})$$

$$= \mu_1^{-1}(\sqrt{-1}\, d\chi) \cap \mu_{\mathbb{C}}^{-1}(0) / U(n)$$

$\chi(g) = (\det g)^{l} \qquad l \geq 0$, $l \in \mathbb{Z}$

Lemma: (B_1, B_2, i, j) satisfy the stability condition iff $G^{\mathbb{C}}(x, z)$ is closed for $z \neq 0$.

Hilbert - Chow morphism:

$$(\mathbb{C}^2)^{[n]} = \mu_{\mathbb{C}}^{-1}(0) /\!\!/_{\chi} GL_n(\mathbb{C}) \longrightarrow \mu_{\mathbb{C}}^{-1}(0) /\!\!/ GL_n(\mathbb{C})$$

$$\mu_{\mathbb{C}}^{-1}(0) /\!\!/ GL_n(\mathbb{C}) = S^n \mathbb{C}^2.$$

Quivers: directed graph Q

fixed dimension vector v : Vertices $\to \mathbb{N}$

$$Rep_Q^v := \prod_{\substack{\alpha \\ \text{edge}}} Hom_k\left(k^{v(begin(\alpha))}, k^{v(end(\alpha))}\right)$$

$$\parallel$$

$$Gl_v := \prod_{i \text{ vertex}} Gl_{v(i)} \qquad \prod Hom_k\left(k^{v(s\alpha)}, k^{v(t\alpha)}\right)$$

$$g \cdot (f_\alpha) = \left(g_{t\alpha} f_\alpha g_{s\alpha}^{-1}\right)$$

if Q has no oriented cycles then there is exactly one closed orbit - the unique semisimple rep. of dim v.

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framed moduli - an additional dimension vector w

$$Rep_Q^{v,w} := Rep_Q^v \times \prod_{i \in Q_0} Hom_k\left(k^{w(i)}, k^{v(i)}\right)$$

$$g \cdot \left((f_\alpha)_{\alpha \in Q_1}, (h_i)_{i \in Q_0}\right) = \left(g \cdot (f_\alpha), (g_i h_i)\right)$$

Double of Quiver.