Rational dunkl operators and quantum integrable system. • math.QA/0606233.

Let W be a finite Coxeter group and $\mathfrak{h}$ be its reflection representation. Suppose $\dim \mathfrak{h} = r$. · Raphaël Rouquier.

Let $\mathcal{S}$ = set of reflections in W. Rep. of RCA.

For $s \in \mathcal{S}$, a reflection, then there is a reflection hyperplane of s.

Suppose this hyperplane is defined by $\alpha_s = 0$ where $\alpha_s \in \mathfrak{h}^*$ be a linear function and such α_s is unique upto scaling.

Let $\alpha_s^\vee$ be a (-1)-eigenvector in $\mathfrak{h}$ of s, and we normalize it by asking

$$\langle \alpha_s^\vee, \alpha_s \rangle = \alpha_s(\alpha_s^\vee) = 2.$$

Consider a parameter set $\{c_s\}_{s\in\mathcal{S}}$ where $c_s \in \mathbb{C}$ and suppose we have

c_s is conj invariant, i.e. $c_{gsg^{-1}} = c_s$.

Define the Dunkl operator D_a to be an operator on $\mathbb{C}(\mathfrak{h})$, the rational function on $\mathfrak{h}$ by

$$D_a = D_a(c) = \partial_a - \sum_{s\in\mathcal{S}} \frac{c_s\,\alpha_s(a)}{\alpha_s}(1-s) \qquad a \in \mathfrak{h}.$$

Then $D_a \in \mathbb{C}W \ltimes D(\mathfrak{h}_{reg})$ $\qquad \mathfrak{h}_{reg} = \mathfrak{h} - \{\text{union of reflection hyperplanes}\}$

Examples: Let we consider $W = \mathbb{Z}_2$, $\mathfrak{h} = \mathbb{C}$.

$$D = \partial_x - \frac{c}{x}(1-s).$$

Propositions of Dunkl operators:

1) $[D_a, x] = (a,x) - \sum_{a\in S} c_s\,\alpha_s(a)\,x(\alpha_s^\vee)\,s. \qquad \forall x \in \mathfrak{h}^*.$

② $\forall g \in W$, we have $g D_a g^{-1} = D_{ga}$.

Pf: All by direct computations. The key point is.

① $[\partial_a, x] = (a, x)$ and $(1-s)(x) = x - s(x) = (x, \alpha_s^\vee)\alpha_s$.

② c_s is conj. invariant. and $gSg = S$.

Theorem (Dunkl). The Dunkl operators are commutative to each other.

i.e. $\forall a, b \in \mathfrak{h}$, $[D_a, D_b] = 0$.

Pf: We will show $\forall f \in \mathbb{C}[\mathfrak{h}]$, $[D_a, D_b] \cdot f = 0$.

$\forall x \in \mathfrak{h}^*$, we have

$$[[D_a, D_b], x] = [[D_a, x], D_b] - [[D_b, x], D_a]$$

Notice that $[[D_a, x], D_b] = [-\sum_{s \in S} c_s \alpha_s(a) x(\alpha_s^\vee) s, D_b] = -\sum_s c_s \alpha_s(a) x(\alpha_s^\vee) \alpha_s(b) \cdot s D_{\alpha_s^\vee}$

by the symm of a and b, we have

$$[[D_a, D_b], x] = 0. \quad \square$$

Applications in Quantum Integrable System.

Olshanetsky - Perelomov operator.

$$L = \Delta_{\mathfrak{h}} - \sum_{s \in S} \frac{c_s(c_s+1)(\alpha_s, \alpha_s)}{\alpha_s^2}$$

Let $\mathfrak{h}$ be a irreducible rep and $(\,,\,)$ on $\mathfrak{h}$ be an invariant inner product.

Restriction of diff operators.

For any element $B \in (\mathbb{C}W \ltimes D(\mathfrak{h}_{reg}))^W$, it defines a linear operator on $\mathbb{C}(\mathfrak{h})$

For $\mathbb{C}(\mathfrak{h})$, we have a subspace $\mathbb{C}(\mathfrak{h})^W$. Then we can restrict B to $\mathbb{C}(\mathfrak{h})^W$

define $m(B)$ to be such restriction.

Then $m(B)$ is a differential operator. and we defined a map.

$$m : (\mathbb{C}W \ltimes D(h_{reg}))^W \longrightarrow D(h_{reg})^W.$$

Propsion (Heckman).

Let $\{y_1, \dots, y_r\}$ be an orthonormal basis of h. then we have.

$$m(\sum D_{y_i}^2) = \Delta_h - \sum_{s \in S} \frac{c_s(\alpha_s, \alpha_s)}{\alpha_s} \partial_{\alpha_s^\vee} := \bar{L}.$$

Pf: At first, let we extend the definition of m from.

$$(\mathbb{C}W \ltimes D(h_{reg}))^W \text{ to } \mathbb{C}W \ltimes D(h_{reg}).$$

$D_y^2 \in \mathbb{C}W \ltimes D(h_{reg})$. define $m(D_y^2)$ as an operator acts on $\mathbb{C}(h)^W$

(notice that the image not in $\mathbb{C}(h)^W$ but $\mathbb{C}(h)$.)

Then we have $m(D_y^2) = m(D_y \partial_y)$

by direct computation, we have $m(D_y^2) = \partial_y^2 - \sum_{s \in S} \frac{c_s \alpha_s(y)^2}{\alpha_s} \partial_{\alpha_s^\vee}$

So we get

$$m(\sum_{i=1}^r D_{y_i}^2) = \sum \partial_{y_i}^2 - \sum_{s \in S} c_s \frac{\sum_{i=1}^r \alpha_s(y_i)^2}{\alpha_s} \partial_{\alpha_s^\vee}. = \bar{L}. \quad \square$$

Cor: $\bar{L}_i = m(P_i(D_{y_1} .. D_{y_r}))$ form a quantum integrable system.

and $\bar{L}_1 = \bar{L}$.

Where $P_1, \dots, P_r$ are homogeneous generators of $(Sh)^W$.

(Chevalley's theorem : $(Sh)^W$ is free).

$P_1 = \sum x_i^2$

Pf: Trivial from the comm properties of D_{y_i}.
and the fact that if $b_1\ b_i$ inv. then $m(b_1)m(b_2) = m(b_1b_2)$. □

Now, let we consider element $\delta_c(x) = \prod_{s\in S} \alpha_s(x)^{c_s}$.

Then consider the family $\delta_c^{-1}\circ \bar{L}_i\circ \delta_c \quad i=1,\dots,r$.

define $L_i = \delta_c^{-1}\circ \bar{L}_i\circ \delta_c$

Then as a corollary we have.

L_i defines ~~an~~ quantum integrable system and

L_1 is the Olshanetsky - Perelomov operator.

Remark: ① When W is the Weyl group, the Dunkl operator can be extended to trignometric case, and elliptic case.

Then we have the trignometric Calogero-Moser System with Hamiltonian.

$$L_{trig} = \Delta_h - \sum_{s\in S} \frac{c_s(c_s+1)(\alpha_s,\alpha_s)}{\sin^2\alpha_s}$$

and elliptic Calogero-Moser System with Hamiltonian.

$$L_{ell} = \Delta_h - \sum_{s\in S} c_s(c_s+1)(\alpha_s,\alpha_s)\,\wp(\alpha_s,\tau).$$

② In the above case, if we let $W = S_n$

then
$$L_1 = \Delta_h - \sum_{i\neq j} \frac{k(k+1)}{(x_i-x_j)^2}$$

which is the quantum Calogero - Moser Hamiltonian.

Rational Cherednik algebra.

Definition: the rational Cherednik algebra $H_c = H_c(W, \mathfrak{h})$ associated to $(W, \mathfrak{h})$ is the algebra generated by (in $A = \mathrm{Rees}(\mathbb{C}W \ltimes \mathcal{D}(\mathfrak{h}_{reg}))$)

$x \in \mathfrak{h}^*$ $\quad g \in W \quad D_a(c, \hbar) = \hbar D_a(c/\hbar)$ renormalized Dunkl operator

If $\hbar = t \in \mathbb{C}$ then $H_{t,c}$ is the specialization of H_c at $\hbar = t$.

Proporsition: H_c is the quotient of the algebra $\mathbb{C}W \ltimes T(\mathfrak{h} \oplus \mathfrak{h}^*)[\hbar]$ (where T means the tensor algebra) by the ideal generated by the relations: $[x, x'] = 0$. $[y, y'] = 0$

$$[y, x] = \hbar(y, x) - \sum_{s \in S} c_s (y, \alpha_s)(x, \alpha_s^\vee) s.$$

Pf: Let we denote the quotient alg to be H_c'.

Define $\phi : H_c' \to H_c$ by

$$\phi(x) = x \quad \phi(y) = D_y(c, \hbar) \quad \phi(g) = g$$

we extend ϕ to a surjective homomorphism $\phi : H_c' \to H_c$.

We want to show ϕ is injective.

Let $\{y_i\}$ be a basis for $\mathfrak{h}$. x_i be the dual basis of $\mathfrak{h}^*$.

Then it is clear that H_c' is spanned by the elements

$$g \prod_{i=1}^{r} y_i^{m_i} \prod_{i=1}^{r} x_i^{n_i}$$

It's image under ϕ is $\quad g \prod_{i=1}^{r} D_{y_i}(c, \hbar)^{m_i} \prod_{i=1}^{r} x_i^{n_i}$.

Since their symbols in $\mathbb{C}W \ltimes \mathbb{C}[\mathfrak{h}^* \times \mathfrak{h}_{reg}][\hbar]$ are linearly independent, we done. $\square$

From now on, we will use the second def. of Cherednik alg. i.e. by generators and relations

Thus, we have

Dunkl embedding: $\Theta_c : H_c \longrightarrow \mathrm{Rees}(\mathbb{C}W \ltimes D(\mathfrak{h}_{reg}))$

PBW theory for Rational Cherednik algebra.

Filtration: $x \in \mathfrak{h}^*$ $y \in \mathfrak{h}$ $\deg = 1$. $g \in W$ $\deg = 0$

Then we have a homomorphism $\xi : \mathbb{C}W \ltimes \mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}^*][\hbar] \longrightarrow gr(H_c)$.

PBW theorem: ξ is an isomorphism.

Pf: it is equivalent to see $g \prod y_i^{m_i} \prod x_i^{n_i}$ is a basis for H_c which is already proved.

This basis are called PBW basis for H_c.

Specialization of $H_{t,c}$.

~~Let $t \in \mathbb{C}$. complex number. we can define $H_{t,c}$ to be the alg H_c but replace all $\hbar$ by t.~~

Now we replace $\hbar$ by a ~~for~~ complex number t.

Then we get an algebra $H_{t,c}$, call the specialization of H_c at t.

It is easy to see that $\forall \lambda \in \mathbb{C}^*$ $H_{\lambda t, \lambda c} = H_{t,c}$.

Thus, we can reduce to the following cases

$H_{0,c}$, $H_{1,c}$.

The Dunkl operator embedding gives the following two embeddings.

$$\Theta_{1,c} : H_{1,c} \longrightarrow \mathbb{C}W \ltimes D(\mathfrak{h}_{reg}) \quad x \to x \quad g \to g \quad y \to D_y.$$

$$\Theta_{0,c} : H_{0,c} \longrightarrow \mathbb{C}W \ltimes \mathbb{C}[\mathfrak{h}^* \times \mathfrak{h}_{reg}] \quad x \to x \quad g \to g \quad y \to D_y^\circ(c)$$

where $D_y^\circ(c)$ is called the classical dunkl operators.

i.e. look. $D_a(c, \hbar) = \hbar D_a(c/\hbar)$ as in the Rees alg.

$$A = \mathrm{Rees}\,(\mathbb{C}W \ltimes D(\mathfrak{h}_{reg}))$$

Then $D_a^\circ(c) \in A_0 := A/\hbar A = \mathbb{C}W \ltimes \mathcal{O}(T^*\mathfrak{h}_{reg})$.

Here notice that D_y maps polynomials to polynomials.

So $\Theta_{1,c}$ defines a repn of $H_{1,c}$ on $\mathbb{C}[\mathfrak{h}]$

This is called the polynomial repn of $H_{1,c}$. we will meet it later.

Spherical subalgebra.

Now, let we consider an element $e \in \mathbb{C}W$. $e = \frac{1}{|W|} \sum_{g \in W} g$.

Then $e^2 = e$.

Definition: ~~$e \in \mathbb{C}W$~~ The spherical subalgebra of H_c is the subalgebra.

$B_c = eH_ce$. The spherical subalgebra of $H_{t,c}$ is the subalg

$$B_{t,c} = eH_{t,c}e = B_c/(\hbar = t).$$

From the Dunkl embedding, we know.

$$e\Theta_{1,c}e : B_{t,c} \to (D(h_{reg})^W) \quad t \neq 0$$

and $e\Theta_{0,c}e : B_{0,c} \to \mathbb{C}[h^* \oplus h_{reg}]^W \quad t = 0$

$$\left(e(\mathbb{C}W \ltimes D(h_{reg}))e = D(h_{reg})^W \right).$$

So we know
- $B_{0,c}$ is commutative and does not have zero divisor.
- B_c is an algebraic deformation of $B_{0,c}$.

Now let $M_c = \mathrm{Specm}\, B_{0,c}$ it is an irreducible affine algebraic variety.

From the deformation, we can see that ~~B~~ M_c has a poisson str.

which is the restriction of the Poisson structure of $\mathbb{C}[h^* \times h_{reg}]^W$

to its subalg $B_{0,c}$.

M_c is called the Calogero-Moser space of W, h.

When $W = S_n$, $h = \mathbb{C}^n$, M_c is isomorphic as a Poisson variety to the Kazhdan-Kostant-Sternberg CM space.

Complex reflection group.

Let V be a complex vector space with dimension ℓ.

$GL(V)$ be the linear transformations on V. Let we define complex reflection.

Defn: A complex reflection (or pseudo-reflection) is a nontrivial element s in $GL(V)$ which acts trivially on a hyperplane. The hyperplane is called the reflecting hyperplane of s.

A complex reflection group is a finite subgroup of $GL(V)$ generated by complex reflections. we will denoted by W.

Properties of complex reflection groups.

- for each complex reflection, s, it's eigenvalues are 1 except 1 eigenvalue λ, which is a root of unity. i.e. $\lambda^{m_s}=1$ for $m_s>1$.
- Let H be the reflecting hyperplane of a complex reflection s. then G_H is a cyclic group of order m_s.
- m_s only depends on the conjugate class of s.

  In fact, we denote $\mathcal{H}$ = set of ~~by~~ reflecting hyperplanes.

  and $\mathcal{O}_{\mathcal{H}}$ be the set of orbits of $\mathcal{H}$ under the action of G.

  We have the following correspondence:

  m_s only depends on the orbit of S_H.

- Let α_s be the eigenvector of s in $\mathfrak{h}^*$, ~~then~~ with nontrivial eigenvector.

α_s is a linear function on $\mathfrak{h}$ and $H=\{h\in\mathfrak{h} \mid \alpha_s(h)=0\}$

This α_s is unique upto a scaling.

Choose $\alpha_s^\vee$ as eigenvector in $\mathfrak{h}$ s.t. $(\alpha_s, \alpha_s^\vee) = 2$.

Let S = set of complex reflections in W.

$\{c_s\}$ be a set of parameters which is invariant under conjugation.

Definition. (Dunkl operators for complex reflection groups)

For any $y \in \mathfrak{h}$,
$$D_y(c) = \partial_y - \sum_{s \in S} \frac{2c_s}{1-\lambda_s} \frac{(\alpha_s, y)}{\alpha_s}(1-s)$$

$$\left(= \partial_y - \sum_{s \in S} \frac{c_s(\alpha_s, y)}{\alpha_s}(1-s) \right)$$

where λ_s is the nontrivial eigenvalue of s.

Definition: The rational Cherednik alg $H_{1,c} = H_{1,c}(\mathfrak{h}^\vee, W)$ attached to (V, W, c) is a quotient of the algebra $\mathbb{C}W \ltimes T(\mathfrak{h} \oplus \mathfrak{h}^*)$ (T means tensor alg). by the ideal generated by.

$$[x, x'] = 0 \quad [y, y'] = 0 \quad [y, x] = (y, x) - \sum_{s \in S} c_s (y, \alpha_s)(x, \alpha_s^\vee) s.$$

The above definitions are generalization of rational dunkl operators and rational Cherednik algebra for the complex reflection setting.

So we still have the PBW theorem for $H_{1,c}$.

and we have the tensor product decomposition.

$$H_{1,c} = \mathbb{C}[\mathfrak{h}] \otimes \mathbb{C}W \otimes \mathbb{C}[\mathfrak{h}^*] \quad \text{as vector space}$$

Representation theory for Ration Cherednik algebra when $t \neq 0$

As we can see, we can suppose $t=1$. Now we consider the repn theory of $H_{1,c}$.

As we show, $H_{1,c}$ is an ass. alg with triangular decomposition

$$\mathbb{C}[\mathfrak{h}] \otimes \mathbb{C}W \otimes \mathbb{C}[\mathfrak{h}^*] \quad \text{or} \quad S(\mathfrak{h}^*) \otimes \mathbb{C}W \otimes S(\mathfrak{h}) \qquad S(\mathfrak{h}) - \text{Symmetric product}$$

We begin with Verma modules of $H_{1,c}$.

Recall for a simple lie alg, $\mathfrak{g}$, $U(\mathfrak{g}) = U(n_-) \otimes U(\mathfrak{h}) \otimes U(n_+)$.

the lowest highest weight vector is a eigen-vector for $\mathfrak{h}$ s.t. it is killed by $\cancel{U(n_-)}$ $U(n_+)$. and $M(\tau) = \mathrm{Ind}_{U(n_+)\otimes U(\mathfrak{h})}^{U(\mathfrak{g})} \mathbb{1}_\tau$.

Now let we make a correspondence: $U(n_-) \longleftrightarrow \mathbb{C}[\mathfrak{h}] = S(\mathfrak{h}^*)$ $U(\mathfrak{h}) \longleftrightarrow \mathbb{C}W$

$U(n_+) \longleftrightarrow \mathbb{C}[\mathfrak{h}^*] = S(\mathfrak{h})$

then the ~~lowest~~ highest weight (vector) should correspond to irreducible representations of W.

Let $\tau \in \mathrm{Irr}(W)$. and realized τ as a repn of $\mathbb{C}W \ltimes \underset{\substack{\| \\ S(\mathfrak{h})}}{\mathbb{C}[\mathfrak{h}^*]}$ by letting $y \in \mathfrak{h}$ acts by 0.

Then we define $M_c(\tau) = \mathrm{Ind}_{\mathbb{C}W \ltimes \mathbb{C}[\mathfrak{h}^*]}^{H_{1,c}} (\tau) = \mathbb{C}[\mathfrak{h}] \otimes \tau$. (as a vector space)

We name $M_c(\tau)$ the standard module / Verma module.

Example. (polynomial representation).

Choose τ to be the trivial module. Then $M_c(\mathbb{C}) = \mathbb{C}[\mathfrak{h}]$.

W acts on this space by natural action $\mathfrak{h}^*$ we know ~~is~~ obviously way.

$y \in \mathfrak{h}$ acts by Dunkl operators

The next thing is we want to find the maximal proper submodule of the Verma module.

Element h.

Let $\{y_i\}$ b a basis of $\mathfrak{h}$, $\{x_i\}$ be the dual basis.

Define $h = \sum_i x_i y_i + \frac{\dim \mathfrak{h}}{2} - \sum_{s\in S} \frac{2c_s}{1-\lambda_s} \cdot s$. $\left(h = \frac{1}{2}\sum (x_i y_i + y_i x_i) \right)$

By direct computation, we can see that

① h is W-invariant. ② $[h, x] = x$ $[h, y] = -y$.
or $[h, w] = 0$.

The under the action of ad_h, we can give $H_{,c}$ a decomposition as eigen space of h. also, it puts a grading: W - deg $=0$ V — deg -1 V^* — deg $=1$.
"Canonical grading"

Now consider a module $M_c(\tau)$, then the lowest eigenvalue of h in $M_c(\tau)$ is
$$h(\tau) = \ell/2 - \sum_s \frac{2c_s}{1-\lambda_s} s|_\tau$$

(pf: it is easy to see that $\frac{\ell}{2} - \sum_s \frac{2c_s}{1-\lambda_s} s$ acts on $M_c(\tau)$ is a scalar.
which is $\frac{\ell}{2} - \sum_s \frac{2c_s}{1-\lambda_s} s|_\tau = c$

We only need to consider the action of $\sum x_i y_i$

Any element in $M_c(\tau)$ has the form $f(x).v$.

$$\sum x_i y_i \cdot f(x) v = \sum x_i [y_i, f(x)] \cdot v$$

$$h \cdot f(x) v = [h, f(x)] v + f(x) \cdot h v = [h, f(x)] v + c v.$$

but $f(x)$ is a polynomial, so $[h, f(x)] v = n f(x) v$. $\Rightarrow$ done.)

Thus. we have the following Corollary.

Cor: The sum $J_c(\tau)$ of all proper submodules of $M_c(\tau)$ is proper submodule of $M_c(\tau)$ and the quotient $L_c(\tau) := M_c(\tau)/J_c(\tau)$ is irreducible

Pf: Consider the eigenvalue of h in $J_c(\tau)$.
every one should satisfying $\mu > h(\tau)$. so $J_c(\tau) \neq M_c(\tau)$. □

Category $\mathcal{O}$

Definition: The category $\mathcal{O} = \mathcal{O}_c$ of $H_{1,c}$-modules is the category of $H_{1,c}$-modules V, such that V is the direct sum of finite-dimensional generalized eigenspaces of h. and the real part of the spectrum of h is bounded below.

Example: $M_c(\tau)$, $L_c(\tau) \in \mathcal{O}_c$

Properties of category $\mathcal{O}$.

① $\mathcal{O}$ is an abelian subcategory of the category of all $H_{1,c}$-modules
it is closed under extensions.

② The set of isomorphism classes of simple obj is $\{L_c(\tau)\}_{\tau \in \mathrm{Irr} W}$
every obj of ~~$M_c(\tau)$ has finite length~~. $\mathcal{O}$ has a finite Jordan-Hölder series

③ For generic c, $M_c(\tau) = L_c(\tau)$, so $\mathcal{O}$ is semisimple.

Examples of the representations of $H_{1,t}$.

A_1-case

$W = S_1 = \langle s \rangle \quad \mathfrak{h} = \mathbb{C}y \quad \mathfrak{h}^* = \mathbb{C}x. \quad \langle x, y \rangle = 1.$

$H_{1,t}$ is defined to be the algebra generated by s, x, ξ with relations.

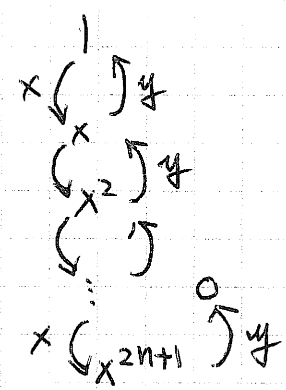
$s^2 = 1, \quad sx = -xs, \quad sy = -ys \quad [y, x] = 1 - 2cs$

① $M_c(\mathbb{C}) = \mathbb{C}[x]$ with the following action.

$x : x^i \longmapsto x^{i+1} \qquad y = \partial_x - \frac{c}{x}(1-s) : x^i \longmapsto \begin{cases} i x^{i-1} & i \text{ even} \\ (i-2c) x^{i-1} & i \text{ odd.} \end{cases}$

$s : x^i \longmapsto (-1)^i x^i$

$h(\mathbb{C}) = \frac{1}{2} - c.$



Then in this case.

when $c \notin \frac{1}{2} + \mathbb{Z}_{\geq 0}, \quad L_c(\mathbb{C}) = M_c(\mathbb{C}).$

when $c \in \frac{1}{2} + \mathbb{Z}_{\geq 0} \quad$ Let $c = \frac{1}{2} + n.$

we can see that $x^{2n+1}\mathbb{C}[x]$ is a submodule of $\mathbb{C}[x]$. and it is maximal.

So $J_c(\mathbb{C}) = x^{2n+1}\mathbb{C}[x] \qquad L_c(\mathbb{C}) = \mathbb{C}[x]/(x^{2n+1})$

② $M_c(\det) = \mathbb{C}[x]$ with the following action.

$x : x^i \longmapsto x^{i+1} \qquad s : x^i \longmapsto (-1)^{i+1} x^i$

$y : x^i \longmapsto \begin{cases} i x^{i-1}, & i \text{ even} \\ (i+2c) x^{i-1}, & i \text{ odd} \end{cases}$

$h(\det) = \frac{1}{2} + c.$

Then in this case,

when $c \notin -(\frac{1}{2} + \mathbb{Z}_{\geq 0}) \qquad M_c(\det) = L_c(\mathbb{C})$

when $c \in -(\frac{1}{2} + \mathbb{Z}_{\geq 0}) \quad c = -\frac{1}{2} - n \quad n \geq 0, \quad J_c(\det) = x^{2n+1}\mathbb{C}[x], \quad L_c(\det) = \mathbb{C}[x]/(x^{2n+1})$

Category $\mathcal{O}$

when $c \notin \frac{1}{2}+\mathbb{Z}$, generic, $\mathcal{O}$ is semisimple.

when $c \in \frac{1}{2}+\mathbb{Z}$, $\mathcal{O}$ is equivalent to the principal block of category $\mathcal{O}$ for $\mathfrak{sl}_2(\mathbb{C})$

$c \in -(\frac{1}{2}+\mathbb{Z}_{\geq 0})$ we have.

$$0 \to M_c(\mathbb{C}) \to M_c(\det) \to L_c(\det) \to 0$$

$$\cancel{0 \to M_c(\det) \to P(\mathbb{C}) \to M_c(\mathbb{C}) \to 0}$$

$c \in \frac{1}{2}+\mathbb{Z}_{\geq 0}$ we have.

$$0 \to M_c(\det) \to M_c(\mathbb{C}) \to L_c(\mathbb{C}) \to 0$$