# DEFORMED PREPROJECTIVE ALGEBRAS AND NONCOMMUTATIVE DEFORMATIONS OF THE KLEINIAN SINGULARITIES

- $K$ algebraically closed field of characteristic $0$
- $\Gamma < SL(2,K)$ finite group.

We want to study the Kleinian singularity $K^2/\Gamma$

$$K^2/\Gamma \overset{(\text{Max})}{=} \operatorname{Spec} K[x,y]^\Gamma .$$

<u>We can consider the rings</u>

- $$S^\lambda = \frac{K\langle x,y\rangle * \Gamma}{(xy-yx-\lambda)}$$

$K\langle x,y\rangle = TK^2$ tensor algebra
noncomulative polynomials

where $\lambda \in Z(K\Gamma)$ center of the group algebra and $K\langle x,y\rangle * \Gamma$ is defined by relations

$$g p g^{-1} = g\cdot p \qquad g\in\Gamma \qquad p\in K\langle x,y\rangle \text{ non-commutative polynomial}$$

$\cdot$ = action of $\Gamma$ on $K\langle x,y\rangle$ induced by the action on $K^2$.

- and the rings

$$S^\lambda \supset \mathcal{O}^\lambda = e S^\lambda e \qquad e = \frac{1}{|\Gamma|}\sum_{g\in\Gamma} g$$

averaging idempotent

PROPERTIES OF $\mathcal{J}^\lambda$, $\mathcal{O}^\lambda$

$\mathcal{J}^0 = K[x,y] * \Gamma$ $\qquad$ $\mathcal{O}^0 = K[x,y]^\Gamma$

$\mathcal{J}^\lambda$, $\mathcal{O}^\lambda$ have increasing filtration obtained by assigning degrees $\deg g = 0$ $\forall g \in \Gamma$

$\deg x = \deg y = 1$

and

$gr\, \mathcal{J}^\lambda \cong K[x,y] * \Gamma \simeq \mathcal{J}^0$ $\qquad$ $gr\, \mathcal{O}^\lambda \cong K[x,y]^\Gamma \simeq \mathcal{O}^0$

→ so I am looking at some non-commutative deformations of $K^2/\Gamma$

We will use the theory of pre-projective algebras and deformed preprojective algebras to understand these rings and their representation theory.

I want to attach a deformed preprojective algebra to $\mathcal{O}^\lambda$. How to do this?

There are bijections

(I)

FINITE SUBGROUPS OF $SL_2(k)$ $\longleftrightarrow$ extended Dynkin diagrams

$\Gamma$ $\longleftrightarrow$ quiver $Q$
- vertices $I$ = irreducible representations $\{N_i\}_{i\in I}$ of $\Gamma$
- $\#$edges $i-j = \dim \operatorname{Hom}_\Gamma(N_i, k^2 \otimes N_j)$

$k^2$ = defining representation

(II)

$Z(k\Gamma) \longleftrightarrow k^I$

$\lambda \longleftrightarrow \{\lambda_i\} \quad \lambda_i = tr_{N_i}\lambda$

Theorem 0

- $\mathcal{O}^\lambda$ is Morita equivalent to $\Pi^\lambda(Q)$
- $\mathcal{O}^\lambda \cong e_0 \Pi^\lambda e_0$ $\quad$ $e_0$ = idempotent attached to the extending vertex

Recall $\Pi^\lambda(Q) = \dfrac{k\overline{Q}}{\langle \sum [a,a^*] - \sum_{i\in I} \lambda_i e_i \rangle}$

proof

$$\mathcal{S}^\lambda = \frac{k\langle x,y\rangle * \Gamma}{\langle xy-yx-\lambda\rangle} = \frac{C}{k\langle x,y\rangle * \Gamma} \qquad C \text{ is graded}$$

- $K\Gamma = \bigoplus_{i\in I} \operatorname{End} N_i$ take $f_i = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ \vdots & & & \\ 0 & 0 & \cdots & 0 \end{pmatrix} \in \operatorname{End} N_i$

  in particular $f_0 = \frac{1}{|\Gamma|} \sum_{g\in G} g = e$

- $f C_0 f = K^I$ take $f = \sum f_i$
- then take $f_i C_1 f_j \simeq \operatorname{Hom}_\Gamma(K\Gamma f_i, C_1 f_j) \simeq \operatorname{Hom}_\Gamma(K\Gamma f_i, k^2 \otimes K\Gamma f_j)$

  $C_1 = k^2 \otimes K\Gamma$ $\qquad \cong \operatorname{Hom}(N_i, V\otimes N_j)$

It is possible to choose $\theta_a \in f_i C_1 f_j$ $\phi_a \in f_j C_1 f_i$ s.t.

$$\sum_{\substack{a\in Q\\ h(a)=i}} \phi_a\theta_a - \sum_{\substack{a\in Q\\ t(a)=i}} \theta_a\phi_a = \delta_i f_i (xy-yx)$$

where $\delta = (\delta_i)$ is the minimal imaginary root $\delta_i = \dim N_i$.

There is an isomorphism $\Pi^\lambda(Q) \longrightarrow f \mathcal{S}^\lambda f$

$$e_i \longrightarrow f_i$$

$a: i \to j \leftarrow a \longrightarrow \phi_a \in f_j C f_i$

arrow with tail $i$ and head $j$ $\qquad a^* \longrightarrow \theta_a \in f_i C f_j$

Moreover $\mathcal{S}^\lambda f \mathcal{S}^\lambda = \mathcal{S}^\lambda$ so $\mathcal{S}^\lambda \overset{\text{Morita}}{\simeq} f\mathcal{S}^\lambda f$

Thus $\mathcal{S}^\lambda \overset{\text{Morita}}{\simeq} f\mathcal{S}^\lambda f \simeq \Pi^\lambda \Rightarrow \mathcal{S}^\lambda \overset{\text{Morita}}{\simeq} \Pi^\lambda$

$$\begin{aligned} \mathrm{Mod}(\mathcal{S}^\lambda) &\xrightarrow{\simeq} \mathrm{Mod}(\Pi^\lambda) \\ M &\longrightarrow fM \end{aligned}$$

equivalence

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Recall from last time

$\mathrm{Rep}(\Pi^\lambda, \alpha) = \mu_\alpha^{-1}(\lambda)$

where $\mu_\alpha : \mathrm{Rep}(\bar{Q}, \alpha) = T^*\mathrm{Rep}(Q\, \alpha) \longrightarrow \mathrm{End}_0(\alpha)$

is the moment map for the action of $G(\alpha) = \dfrac{\prod_i GL(\alpha_i; k)}{\mathbb{C}^*}$

Want to classify irreducible representation of $\mathcal{S}^\lambda$ using $\mathcal{S}^\lambda \overset{\text{Morita}}{\sim} \Pi^\lambda$ and the machinery developed last time for deformed preprojective algebras and some properties of root systems attached to Dynkin and extended Dynkin quivers.

Let's start by proving some statements true for any quiver.

<u>Observation</u>

If $\alpha$ is the dimension vector of a representation of $\Pi^\lambda$ then $\lambda\cdot\alpha = \sum \lambda_i \alpha_i = 0$

~~pf~~ compute the trace of $\sum_{a\in Q}[a, a^*] - \sum_{i\in I} \lambda_i e_i$

We say $\lambda$ is on a wall if $\lambda\cdot\alpha = 0$ for some root $\alpha$

<u>Theorem 1</u>

$\Pi^\lambda$ has a representation of dimension vector $\epsilon_i = (0, \dots, \underset{i}{1}, \dots 0)$ iff $\lambda_i = 0$. It is unique if $i$ is loop free

<u>Theorem 2</u>

Suppose $\lambda \in K^I$

1) If $\lambda$ is on a wall, $\Pi^\lambda$ has a simple representation of dimension vector a real root

2) If $\lambda$ is not on a wall then the dimension vector of any simple representation must be an imaginary root.

proof

- If $\lambda$ is on a wall $\exists$ $w$ s.t. $\mu = w\lambda$ has $\mu_i = 0$ for some $i$ loop free vertex. So $\Pi^\mu$ has a simple representation of dim vector $\varepsilon_i$. Thus $\Pi^\lambda$ has a representation of dimension vector $w'\varepsilon_i$ ($w'$ has minimal length s.t. $\mu = w'\lambda$).
- If $\lambda$ is not on a wall apply any sequence of reflection functors (there is never any problem) then if we consider an indecomposable representation then its dimension vector $\alpha$ has connected support. So if $\alpha$ not in the fundamental region ($(\alpha, \varepsilon_i) \leq 0$ $\forall i$ loop free) i.e. we have $(\alpha, \varepsilon_i) > 0$ for some loop free vertex $i$ apply reflection at $i$. Get a representation of dimension vector $s_i\alpha = \alpha - (\alpha, \varepsilon_i)\varepsilon_i < \alpha$. Can't go on forever so eventually get $\alpha$ is an imaginary root.

COROLLARY 3

If $Q$ Dynkin then $\Pi^\lambda(Q) \neq 0$ iff $\lambda$ on a wall.

Pf $\Pi^\lambda(Q)$ is finite dimensional since $\Pi^0(Q)$ is (Ringel) and $\Pi^0(Q) \twoheadrightarrow \mathrm{gr}\, \Pi^\lambda(Q)$ surjective homomorphism. Since Dynkin quivers have no imaginary root the dimension vector of any irreducible must be a real root so if $\lambda$ not on a wall $\Pi^\lambda(Q)$ has no fin. dim irreducible representations so it is 0!

COROLLARY 4

If $Q$ is any quiver $\lambda \in K^I$ $J \subset I$ is any subset of indices and if $Q'$ is the full subquiver of $Q$ on $J$ is a disjoint union of Dynkin quivers and $e = \sum_{i \notin J} e_i$ then $\Pi^\lambda(Q)$ is Morita equivalent to $e\Pi^\lambda(Q)e$ iff $\lambda'$ (restriction of $\lambda$ to $Q'$) is not on a wall

pf

$$\frac{\Pi^\lambda(Q)}{\Pi^\lambda(Q)e\Pi^\lambda(Q)} \cong \Pi^{\lambda'}(Q')$$

this is 0 iff $\lambda'$ is not on a wall.

## DOMINANCE

Define on $K$ a total ordering

(1) $a < b \Rightarrow a + c < b + c$

(2) $<$ coincides with usual order on $\mathbb{Z}$

(3) $\forall a \in K \ \exists m \in \mathbb{Z} \quad a < m$

($K = \mathbb{C}$ take lexicographic)

## DEFINITION

$\lambda$ is dominant if $\lambda_i \geq 0 \quad \forall i \in I$

For $\lambda$ dominant let $I_\lambda = \{ i \mid \lambda_i = 0 \}$

$Q_\lambda$ full subquiver of vertex set $I_\lambda$

## LEMMA (EASY!) 5

If $\lambda$ dominant any finite dimensional representation of $\Pi^\lambda$ is supported on $I_\lambda$ so it is a module for $\Pi^0(Q_\lambda)$. Thus

(1) If $Q$ is a disjoint union of Dynkin quivers

$$\Pi^\lambda(Q) \cong \Pi^0(Q_\lambda)$$

(2) If $Q_\lambda$ is a disjoint union of Dynkin quivers then $S_i$, $i \in I_\lambda$ are the only finite dim. simple modules

Now use reflection functors and properties of roots to extend those results for non dominant weights.

LEMMA $Q$ Dynkin or extended Dynkin and $\lambda \cdot \delta \neq 0$

(1) $Q$ Dynkin $\lambda$ is $W$-conjugate to a unique dominant weight

(2) $Q$ extended Dynkin $\frac{\lambda}{\lambda \cdot \delta}$ is conjugate to a unique dominant weight $\lambda^+$ and $\exists!\ w^+ \in W$ of minimal lenght s.t. $w^+\left(\frac{\lambda}{\lambda \cdot \delta}\right) = \lambda^+$

(3) $R_\lambda = \{\alpha \in \overset{\circ}{R} \mid \lambda \cdot \alpha = 0\}$ is the set of roots of a reduced system. $W_\lambda$ (generated by $s_\alpha$ $\alpha \in R_\lambda$) acts faithfully on $R_\lambda$ and it is its Weyl group

(4) $\exists\, \Sigma_\lambda \subset R_\lambda$ unique basis in $R^+_\lambda$ then $w^+ \Sigma_\lambda = \Sigma_{\lambda^+}$

(5) $W_\lambda$ stabilizer of $\lambda$ in $W$

(6) $\forall\ \lambda$ dominant $\Sigma_\lambda = \{\varepsilon_i \mid i \in I_\lambda\}$ and $Q_\lambda$ is a disjoint union of Dynkin quivers.

For extended Dynkin there will be essentially two cases $\lambda \cdot \delta \neq 0$ and $\lambda \cdot \delta = 0$

# MAIN THEOREM (6)

If $Q$ extended Dynkin and $\lambda \cdot \delta > 0$ then the isomorphism classes of fin dim simple modules are in bijection with $\Sigma_\lambda$.

proof

If $\lambda$ is dominant $\Sigma_\lambda = I_\lambda$ and $Q_\lambda$ is a disjoint union of Dynkin quivers. So since any representation of $\pi^\lambda(Q)$ is a representation of $\pi^0(Q_\lambda)$ so it is of the form $\varepsilon_i$!

If $\lambda$ is not dominant choose $w^+ \lambda = \lambda^+$ so $\pi^\lambda \overset{\text{Morita}}{\sim} \pi^{\lambda^+}$ and equivalence acts as $w^+$ on dimension vector. Result follows since $w^+ \Sigma_\lambda = \Sigma_{\lambda^+} = I$

# What happens if $\lambda\cdot\delta=0$?

We will prove

**Theorem 7**

If $\lambda\cdot\delta=0$ $\quad \mathcal{O}^{\lambda}=e_0\Pi^{\lambda}e_0 \simeq K[\mathrm{Rep}(\Pi_{\lambda},\delta)]^{GL(\delta)}$

(recall $\mathrm{Rep}(\Pi^{\lambda},\delta)=\mu_{\delta}^{-1}(\lambda)$ fiber of moment map)

$\boxed{\text{STEP 0}}$

give alternative description of $K[\mathrm{Rep}(\Pi^{\lambda},\delta)]$

EASY FACT

$$K[\mathrm{Rep}(\overline{Q},\delta)]=K[s_{apq} \mid a\in\overline{Q} \quad 1\le p\le\delta_{h(a)},\ 1\le q\le\delta_{t(a)}]$$

take $J_{\lambda}$ = ideal generated by

$$\sum_{\substack{a\in Q\\ h(a)=i}}\sum_{r=1}^{\delta_{t(a)}} s_{apr}s_{a^*rq}-\sum_{\substack{a\in Q\\ t(a)=i}}\sum_{r=1}^{\delta_{h(a)}} s_{a^*pr}s_{arq}-\delta_{pq}\lambda_i \qquad \begin{matrix} i\in I\\ 1\le p,q\le\delta_i\end{matrix}$$

we have

$$K[\mathrm{Rep}(\Pi^{\lambda},\delta)]=\frac{K[s_{apq}]}{J_{\lambda}}$$

Let $\ell = \sum_i \delta_i$ then rows and columns of $Mat(\ell, K[s_{apq}])$ can be indexed by pairs

$i, p$ $\quad i \in I \quad 1 \le p \le \delta_i$

There is an isomorphism

$$K\bar{Q} \longrightarrow Mat(\ell, K[s_{apq}])$$

$$e_i \longrightarrow \begin{array}{c} \\ i1 \\ \vdots \\ i\delta_i \\ \\ \end{array} \left(\begin{array}{c|c|c} 0 & 0 & 0 \\ \hline 0 & \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix} & 0 \\ \hline 0 & 0 & 0 \end{array}\right)$$
$$\qquad\qquad\qquad\qquad i1 \quad \cdots \quad i\delta_i$$

$$a \longrightarrow \begin{array}{c} \\ h(a)1 \\ \vdots \\ h(a)\delta_{h(a)} \\ \\ \end{array} \left(\begin{array}{c|ccc|c} 0 & & 0 & & 0 \\ \hline 0 & s_{a11} & \cdots & s_{a1\delta_{t(a)}} & 0 \\ & \vdots & & \vdots & \\ & s_{a\delta_{h(a)}1} & \cdots & s_{a\delta_{h(a)}\delta_{t(a)}} & \\ \hline 0 & & 0 & & 0 \end{array}\right)$$
$$\qquad\qquad\qquad\qquad t(a)1 \quad \cdots \quad t(a)\delta_{t(a)}$$

This map descends to a map

$$\Pi^\lambda \longrightarrow \mathrm{Mat}(\ell, \mathbb{K}[\mathrm{Rep}(\Pi^\lambda, \delta)])$$

since $\delta_0 = 1$ (0 = extending vertex correspond to trivial representation) we have

$$e_0 \Pi^\lambda e_0 \longrightarrow \mathbb{K}[\mathrm{Rep}(\Pi^\lambda, \delta)] \quad \left( e_0 \to \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right)$$

The image is $GL(\delta)$ invariant so we get a map

$$\phi_\lambda : \mathcal{O}^\lambda \longrightarrow \mathbb{K}[\mathrm{Rep}(\Pi^\lambda, \delta)]^{GL(\delta)} \left( = \left( \frac{\mathbb{K}[S_{a\,pq}]}{J_\lambda} \right)^{GL(\delta)} \right)$$

CLAIM <u>this map is an isomorphism</u>

$\boxed{\text{STEP 1}}$ prove result for $\mathfrak{h}' = \{\lambda \mid \lambda \cdot \alpha \neq 0 \text{ for all } \alpha \text{ real root}\}$ (i.e. prove the result for generic $\lambda$).

<u>LEMMA 8</u>

$\mathrm{Rep}(\Pi^\lambda, \delta) /\!\!/ GL(\delta)$ has dimension at least 2 for $\lambda \in \mathfrak{h}'$

Pf since $\lambda \in \mathfrak{h}'$ any representation of $\Pi^\lambda$ of dimension vector $\delta$ is simple. In fact since $\delta_0 = 1$ a composition factor should be of dimension $\alpha$ with $\alpha_0 = 0$ so it would be a representation of $\Pi^{\lambda'}(Q')$ where $\lambda', Q'$ are obtained by deleting the vertex $o$. But this algebra is $0$! ($Q'$ is Dynkin and $\alpha\cdot\lambda' \neq 0$ $\forall \alpha$ Dynkin root!)

By the repr. theory of $Q$ $\exists$ a 1-dimensional subvariety $C$ of $\operatorname{Rep}(Q,\delta)$ consisting of non isomorphic representations with endomorphism algebra $K$ so it lifts to $\Pi^{\lambda}$.

So $\operatorname{Rep}(\Pi^{\lambda},\delta)$ contains a 2-dimensional subscheme of irreducible pairwise non isomorphic representations

$$\left(\begin{array}{l} 0 \to \operatorname{Ext}^1(X,X)^* \to \operatorname{Rep}(Q^{op},\delta) \to \operatorname{End}(\delta) \to \operatorname{End}(X) \to 0 \\ \text{and } \langle\delta,\delta\rangle = 0 \Rightarrow \dim \operatorname{Ext}^1(X,X) = \dim \operatorname{End}(X) = 1 \\ \text{by Ringel formula } \dim \operatorname{End}(X) - \dim \operatorname{Ext}^1(X,X) = \langle\delta,\delta\rangle \end{array}\right)$$

- Now consider $X = (X_a)_{a\in\bar{Q}} \in \operatorname{Rep}(\bar{Q},\delta)$ and $p = a_1 \ldots a_m$ a path starting and ending at the same point. The function $tr_p(X) = tr(X_{a_1} \ldots X_{a_m})$ is $GL(\delta)$-invariant. Such functions generate $K[\operatorname{Rep}(\Pi^{\lambda},\delta)]^{GL(\delta)}$ (non trivial but true by results of Marsei and Le Bruyn 1990)

• since $\lambda \in h'$ $\Pi^\lambda e_0 \Pi^\lambda = \Pi^\lambda$ (use again that

we have $\dfrac{\Pi^\lambda}{\Pi^\lambda e_0 \Pi^\lambda} \cong \Pi^{\lambda'}(Q')$ $\lambda', Q'$ obtained by

deleting vertex 0. $Q'$ Dynkin and $\lambda' \cdot \alpha \neq 0$ for any

root so $\Pi^{\lambda'}(Q') = 0$! )

Since $\Pi^\lambda e_0 \Pi^\lambda = \Pi^\lambda$ any element of $e_i \Pi^\lambda e_i$ can

be seen as a sum of paths passing through the

vertex 0.

A cyclic permutation of a path gives rise to the

same trace function. So $\mathbb{C}[\mathrm{Rep}(\Pi^\lambda, \delta)]^{GL(\delta)}$

is generated by $\mathrm{tr}_p$ where $\mathrm{start}\,p = \mathrm{end}\,p = 0$.

But this is exactly $\mathrm{Im}(\phi_\lambda)$: so $\phi_\lambda$ is surjective.

Since $O_\lambda$ has GK-dimension 2 any proper quotient

has GK dimension at most 1. But $\dim \mathbb{C}[\mathrm{Rep}(\Pi^\lambda, \delta)]^{GL(\delta)}$

has dimension at least 2! So $\phi_\lambda$ must be

surjective! So it is an isomorphism.

$\boxed{\text{STEP 2}}$ Use flatness of $\mu_\delta$ to extend the result to

any $\lambda$ s.t. $\lambda \cdot \delta = 0$ (to any fiber of $\mu_\delta$)