Field $K$ of dim 2 over alg closed $k$. $K = \mathrm{frac}(A)$ $A$ has GK dim 2

Thm: $\dim K = 1 \Rightarrow K$ comm.

List: $\dim K = 2$

- $[K : Z] < \infty$
- $Z = k$

quantum rational, Weyl alg, $q$-plane, Sklyanin

$yx = xy + 1$ , $yx = qxy$

Differential operators, $L$ – function fields of alg curves

$K = L(\partial)$.

- elliptic translation

$L$ = fn field ~~of~~ on elliptic curve $Y$, $\alpha$ trans. on $Y$

$K = L(t, \alpha)$. $t \cdot f = f^{\alpha} \cdot t$

Quantization:

start: $X = X_0$ a proj. variety over $k$. str sheaf: $\mathcal{O}_0$.

$R_1 = k[t]/(t^2)$.

Deformation of $X$: $\mathcal{O}_1$ sheaf of $R_1$-alg ① flat over $R_1$ ② $\mathcal{O}_1 \otimes_{R_1} k = \mathcal{O}_0$

noncomm str. of $\mathcal{O}_1$ is govered by commutator

$[u, v] \in t\mathcal{O}_1 \simeq \mathcal{O}_0$

Skew symmetric, derivation in each var.

Bracket: Sections of $\Lambda^2 T_X$

Case $X = \mathbb{P}^2$. $\quad \Lambda^2 T_{\mathbb{P}^2} \simeq (\Omega^2)^\vee$ where $\Omega^2 = \mathcal{O}_0(K)$.

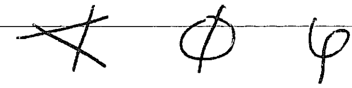
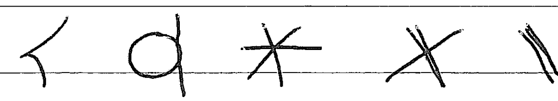
$$\Lambda^2 T_{\mathbb{P}^2} = \mathcal{O}(-K)$$

Surfaces with nonzero sections in $H^0(X, \mathcal{O}(-K))$.

- rational
- ruled
- $K_3$. $\quad K = 0$
- Abelian. $\quad K = 0$

Case $\mathbb{P}^2$: $\mathcal{O}(-K) = \mathcal{O}(3)$. cubic forms.

So a section vanishes on a cubic curve. : $Y$.

- smooth cubic. $\quad$ K = Sklyanian.
- singular cubic. $\quad$ $q$-plane 
- else $\quad$ Weyl. 

$X$ — proj scheme with ample inv. sheaf on $X$, $L$.

$$A_n = H^0(X, L^{\otimes n})$$

Find $L_1$, put $(A_1)_n = H^0(X, L^{\otimes n})$. say ${}_{\mathcal{O}_1}L_1$ what is $L_1 \otimes L_1$

Need $L_1$ to be a bimodule. ${}_{\mathcal{O}_1}L_1{}_{\mathcal{O}_1}$

as left module, $L_1$ loc generated freely for 1 gen.

Then let $\mathcal{O}_1' = \mathrm{End}({}_{\mathcal{O}_1}L_1)$ $\quad \mathcal{O}_1'$ is locally iso. $\mathcal{O}_1$.

$L_1{}_{\mathcal{O}_1'}$ $\quad L_1$ can be a bimodule only if $\mathcal{O}_1 \simeq \mathcal{O}_1'$.

Does not exist for $K_3$, Ab.

## Valuation

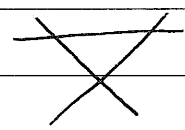
all fields on list have valuation (at least 1)

all valuation discrete. with residue fields in 1-var.

Valuation ring R. M max. $R/M = L$. func. field of curve.

Sklyanin: ex 1 value res. field is ell curve.

q-plane: values class by $\mathbb{Q}$.



$$yx = qxy$$
$$zy = yz$$
$$xz = zx.$$

Weyl. more complicated.

## Small conjecture

A domain. f. gen, and GK-dim $= 2$.

- dim of f.d. irr. rep are bounded.
- A not PI (Fract(A) not fin/$\mathbb{Z}$). only finite # irr. rep of dim $> 1$.
- A primitive ($\exists$ faithful irr. rep).

## How to make a quadratic field extension $K'/K$

$\dim_K K' = 2$. left or right.

Basis $(1, x)$ (L & R).

$\Rightarrow$ $x^2 = bx + c$

$xa = (\sigma a)x + (\delta a)$ $a \in K$. $\sigma$ - automomorphism of $K$.

$\delta$ - $\sigma$-derivation $\delta(ab) = \sigma(a)\delta b + \delta a\, b$

$(xx)a = (bx+c)a = b\alpha(a)x + b(\delta a) + ca$

$x(xa) = x(\alpha(a)x + \delta(a)) = \alpha^2(a)x^2 + \delta(\alpha(a))x + (\alpha\delta a)x + \delta^2 a$

$= \alpha^2(a)bx + \alpha^2(a)c + \cdots$

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- G-K dimension
- Twisting comm scheme
- Proj. noncomm curves graded alg of GK dim 2.
- Proj A for A graded in general

A finitely generated alg over $k = \bar{k}$.

G-K-dim: Choose $V \subset A$ finite dimensional vector space

- $1 \in V$
- generate A as a k-alg.

$V^n = \mathrm{Span}(V \times V \times \cdots \times V)$.

$V^1 \subset V^2 \subset \cdots$ since $1 \in V$ and $A = \bigcup V^n$.

Let $a_n = \dim V^n$. $a_n \le n^{d+\varepsilon}$ $\varepsilon > 0$ $n \gg 0$

$GK(A) = d$

E.g. $k[x,y]$ has GK dim 2.

$V = \mathrm{Span}(1, x, y)$ $a_n = \frac{1}{2}(n+1)(n+2)$

Main results.

- $GK(A) = 0 \Rightarrow A$ is finite dimensional.
- $GK(A) > 0 \Rightarrow GK(A) \ge 1$.
- $GK(A) > 1 \Rightarrow GK(A) \ge 2$. (Bergman's Gap Thm)
- $\forall r \in \mathbb{R}$ $r \ge 2$ $\exists A$ $GK(A) = r$

· $A$ is a graded domain. $GK(A) > 2 \Rightarrow GK(A) \geq 3$

· $A$ a graded domain $GK(A) > 3 \Rightarrow$ ?

$X$ scheme proper over $k$. (proper $\Rightarrow H^0(L)$ – finitely dim).

$L$ – invertible sheaf.

$L$ – ample $\Leftrightarrow \forall$ Coherent sheaf $H^0(F \otimes L^{\otimes n})$ generates $F \otimes L^{\otimes n}$ $(n \geq n_0)$

Serre's thm: Suppose $L$ (very)? ample. Let $A_n = H^0(L^{\otimes n})$

$$A = \bigoplus_{n \geq 0} A_n \text{ graded ring finitely generated.}$$

$$X = \operatorname{Proj} A.$$

Then $\operatorname{Coh} X \sim \text{Grad-}A\text{-mod} / (\text{Torsions})$

Torsion means. $M_{-n} \oplus M_{-n+1} \oplus \cdots\cdots$ bounded on the right.

( f. module as $\mathcal{O}$-mod ).

If $A = A_0 \oplus A_1 \oplus \cdots$ is graded $k$-alg, not necessarily commutative, then

$$\operatorname{Proj} A \underset{\text{defn}}{=} \text{graded f.g. } A\text{-mod} / \text{torsion}$$

$$\operatorname{Proj} A = ( grA/tor , \text{ shift}, A)$$

Prop: Every inv. bimodule on $X$ has the form $L_\sigma$. $\sigma$ automorphism of $X$

as a left module, ${}_{\mathcal{O}}L_\sigma$ is invertible sheaf also as right module but shifted by $\sigma$, i.e. $s$ – section of $L$, $f$ function. $s \cdot f = f^\sigma s$

$$\begin{tikzcd} X \arrow[r, "\alpha"] \arrow[dr, "f^\alpha"'] & X \arrow[d, "f"] \\ & k \end{tikzcd}$$

Lemma: $L_\alpha \otimes M_\tau$ ($\alpha, \tau$ autos) $= (L \otimes \alpha^* M)_{\alpha\tau}$

(For any coherent sheaf) as left module.

Define graded ring $A_n = H^0(L_\alpha^{\otimes n}) = H^0(L \otimes L^{\alpha} \otimes \cdots \otimes L^{\alpha^{n-1}})$

$\alpha \in A_i$, $\beta \in A_j$.

$\alpha \cdot \beta = \alpha \otimes \beta^{\alpha^i}$

Defn: $L_\alpha$ ample if $\forall$ coherent sheaf $F$.

$H^0(F \otimes L_\alpha^{\otimes n})$ generates $F \otimes L_\alpha^{\otimes n}$ if $n >> 0$

Theorem: $L_\alpha$ is ample, then $\operatorname{Proj} A = \left( \operatorname{Gr} A / \operatorname{tor} \right) \simeq (\operatorname{Coh} X)$

and $A$ is Noetherian.

Say $L = \mathcal{O}(D)$

$L^{\alpha} = \mathcal{O}(\alpha^{-1}(D))$ $\qquad L_\alpha^{\otimes n} = \mathcal{O}(D + \alpha^{-1}D + \cdots + \alpha^{-n+1}D)$

Call $D$ - $\alpha$-ample if $\mathcal{O}(D)_\alpha = L_\alpha$ is ample.

Thm: $\alpha$ auto. of $X$ $\quad N = NS$ group. of $X = \mathrm{Div}/\text{numerical equi.}$

a f.g. abelian gp.

$\exists$ $\alpha$-ample divisor $D$ iff action of $\alpha$ on $N$ is quasi-unipotent

(eigenvalues roots of 1)

E.g. $X$ — elliptic surfaces over $\mathbb{P}^1$.

$$\begin{array}{c} X \\ \downarrow \\ \mathbb{P}^1 \end{array}$$

generic fiber is elliptic curve. all the fibers are irr.

$\exists$ a section $Z^\circ$.

Say $N$ has $\mathbb{Z}$-basis $F, Z_0, Z_1, Z_2$.

Let $\sigma$ be translation by $Z_1$, New basis, $F, Y = Z_1 - Z_0, Z_0, Z_2$

$$\sigma F = F, \quad \sigma Y = Y + *F \quad \sigma Z_0 = Z_0 + Y + * \quad \sigma Z_2 = Z_2 + Y + *F$$

$\sigma$ - unipotent operators

Take $D$ $\sigma$-ample. $GK(A) = 5$ $\quad A = \bigoplus H^\circ(L_\sigma^{\otimes n})$.

$A$ domain $GK(A) = 1 \Rightarrow A$ commutative

Take $A$ graded domain, $GK(A) = 2$.

Theorem: Assume $A_0 = k$ & $A$ is generated by $A_1$. Then there exists proj alg curve $Y$ auto. $\sigma$. invertible sheaf $L$. and $A_n \approx H^\circ(Y, L_\sigma^{\otimes n})$ if $n \gg 0$

Also $A$ is Noetherian.

Pf: Graded fractions of $A$. $\quad Q_n = \{ \text{fractions } \frac{a}{b} \mid a \in A_{m+n} \quad b \in A_m \}$

$$Q = \bigoplus Q_n.$$

Take $z \in A_1$. then $Q_n = Q_0 z^n$. $\quad Q_0$ is the skew-field. $K$.

$$Q = Q_0[z, z^{-1}] \quad \sigma \text{ auto. of } K.$$

$A$ has GK dim 2 $\Rightarrow$ GK tr deg of $K = 1$.

$\Rightarrow$ $K$ is comm and function field of some alg curve (smooth) $X$.

Define $Y$ write $A_n = \bar{A}_n z^n$. $\bar{A}_n \subset K$. finite-dim vector space.

$\bar{A}_n$ def map : $X \xrightarrow{\phi_n} \mathbb{P}^*$

Let $Y_n = \phi_n(X)$.

Thm : $Y_n$ stable for large $n$. This defines $Y$. $\square$

Say $A$ graded ring. category $\mathrm{Proj}\, A := \left( \text{grd}\, A\text{-mod} / \text{tor} \right)$ right.

autoequi: shift $s$

structure obj : $A_A$

$(\mathrm{Proj}\, A, s, A_A)$.

Thm : $\mathcal{C}$ - abelian category $k$- linear and exact direct limit. $s$ auto equi. $\mathcal{O}$ - object. $s$ ample, $H^0(M)$ finite dimensional, $\mathcal{O}$ is Noetherian.

$(\mathcal{C}, s, \mathcal{O}) \simeq (\mathrm{Proj}\, A, s, A_A)$ for $A = \bigoplus H^0(s^n \mathcal{O})$ right Noetherian.

( $s$ is ample if for all $M \twoheadrightarrow N$ in $\mathcal{C}$ then $H^0(s^n M) \to H^0(s^n N)$ surj for $n \gg 0$ )

$H^0(M) = \mathrm{Hom}(\mathcal{O}, M)$

$M, N \in grA$.

$\underline{\mathrm{Hom}}(M,N) = \bigoplus_n \mathrm{Hom}_{grA}(M, s^n N)$ $\quad \chi_1 = \mathrm{Ext}^1(A/A_{\geq n}, M)$ is right bounded if $n \gg 0$.

$\chi_1 \Rightarrow$ converse.