11/13. Noncomm alg

Crystal basis.

$\mathfrak{g}$ – semisimple Lie algebra. $/\mathbb{C}$.

M – f.d. $\mathfrak{g}$-mod. $M = \bigoplus M_\lambda$. $\dim M_\lambda = 1$.

$U_q(\mathfrak{g})$

K – a field. $q \in K$. $q^n \neq 1$ $(n \geq 1)$

Defn. $U_q(sl_2)$ is the K-alg generated by e, f, t, t^{-1} s.t.

$$tet^{-1} = q^2 e \qquad tft^{-1} = q^{-2} f \qquad [e, f] = \frac{t - t^{-1}}{q - q^{-1}}$$

Hopf structure. $\Delta : U_q(sl_2) \longrightarrow U_q(sl_2) \otimes U_q(sl_2)$

$$\Delta(t) = t \otimes t \qquad \Delta(e) = e \otimes t^{-1} + 1 \otimes e$$

$$\Delta(f) = f \otimes 1 + t \otimes f$$

M_1 M_2 two rep, using Δ, we can make $M_1 \otimes M_2$ as a repn.

Reps: 2 – 1-dim rep. $K = K \cdot 1$ $\quad V_- = K \cdot 1_-$

$$t \cdot 1 = 1 \quad e \cdot 1 = f \cdot 1 = 0$$

$$t \cdot 1_- = -1_- \quad e \cdot 1_- = f \cdot 1_- = 0$$

f.d. Reps. $\ell \geq 0$. we have an $\ell+1$ dim module $V(\ell)$.

It has basis $\{u_k^{(\ell)}\}_{0 \leq k \leq \ell}$.

$$t \cdot u_k^{(\ell)} = q^{\ell - 2k} u_k^{(\ell)} \qquad e \cdot u_k^{(\ell)} = [\ell - k + 1] u_{k+1}^{(\ell-1)} \qquad f \cdot u_k^{(\ell)} = [k+1] u_{k+1}^{(\ell)}$$

$$u_k^{(\ell)} = f^{(k)} u_0^{(\ell)} = e^{(\ell - k)} u_\ell^{(\ell)}$$

Thm: Any f.d. $U_q(sl_2)$-module is completely irreducible. Any irr. is iso to $V(\ell)$ or $V(\ell) \otimes V_-$

The eigenvalues of t is q^ℓ or $-q^\ell$.

Cor. $M = \bigoplus_{k \in \mathbb{Z}} M_k$, $M_k = \{u \in M \mid t \cdot m = q^k m\}$.

M is a direct sum of $V(\ell)$'s.

QUE algebras

P: a free $\mathbb{Z}$-module (wt. Lattice)

I: index set (simple roots)

$h_i \in P^* = \mathrm{Hom}_{\mathbb{Z}}(P, \mathbb{Z})$ (simple coordinates) $\alpha_i \in P$ $i \in I$.

$(,) : P \times P \to \mathbb{Q}$ symm pair. $\langle , \rangle : P^* \times P \to \mathbb{Z}$

$(\alpha_i, \alpha_i) \in 2\mathbb{Z}_{>0}$ $(\alpha_i, \alpha_j) \leq 0$ $i \neq j$ $\langle h_i, \lambda \rangle = \dfrac{2(\alpha_i, \lambda)}{(\alpha_i, \alpha_i)}$.

Def: $U_q(\mathfrak{g})$ is the k-alg generated by $e_i \dots f_i$ $(i \in I)$

$q(h)$ $(h \in P^*)$

$q(0) = 1$ $q(h_1)q(h_2) = q(h_1 + h_2)$.

$q(h) e_i q(h)^{-1} = q^{\langle h, \alpha_i \rangle} e_i$ $q(h) f_i q(h)^{-1} = q^{-\langle h, \alpha_i \rangle} f_i$

$[e_i, f_j] = \delta_{ij} \dfrac{t_i - t_i^{-1}}{q_i - q_i^{-1}}$ $q_i = q^{\frac{(\alpha_i, \alpha_i)}{2}}$

$t_i = q\left(\dfrac{(\alpha_i, \alpha_i)}{2} h_i\right)$

+ Serre relations

This is also a Hopf alg.

For each i, we have a map $U_{q_i}(sl_2) \longrightarrow U_q(\mathfrak{g})$

$$e, t, f \longmapsto e_i, t_i, f_i$$
$$q_i \longmapsto q.$$

Defn: a $U_q(\mathfrak{g})$-module is called integrable if.

① $M = \bigoplus_{\lambda \in P} M_\lambda \qquad M_\lambda = \{ m \in M \mid q(h)m = q^{\langle h, \lambda \rangle} m \}$.

② For any $i \in I$, M is a union of finite dimensional $U_q(\mathfrak{g})_i$-submodule

Defn: $\mathcal{O}_{int}(\mathfrak{g})$ is the category of int. $U_q(\mathfrak{g})$-module M s.t.

$$\forall\ u \in M,\ \exists\ \ell \geq 1 \text{ s.t. } e_{i_1} e_{i_2} \cdots e_{i_\ell} u = 0 \text{ for any } i_1 \cdots i_\ell \in I.$$

$P_+ = \{ \lambda \in P \mid \langle h_i, \lambda \rangle \geq 0\ \forall i \}$. dominant wts.

$\lambda \in P_+ \quad V(\lambda)$ is the mod generated by u_λ s.t. $q(h) u_\lambda = q^{\langle h, \lambda \rangle} u_\lambda$.

$$e_i u_\lambda = 0 \qquad f_i^{1+\langle h_i, \lambda \rangle} u_\lambda = 0$$

Thm: Assume $\operatorname{ch} K = 0$. the $V(\lambda)$ is irreducible in $\mathcal{O}_{int}(\mathfrak{g})$.

and $\mathcal{O}_{int}(\mathfrak{g})$ is semisimple. Any irreducible in $\mathcal{O}_{int}(\mathfrak{g})$ is iso to $V(\lambda)$ for some $\lambda \in P_+$.

Prob: Let M be an int. $U_q(\mathfrak{g})$-mod. need good basis for M

Know as $U_q(\mathfrak{g})_i$ $M = \bigoplus V(\ell_j)$ and $V(\ell_j)$ has a basis $\{ u_k^{(\ell_j)} \}_{0 \leq k \leq \ell_j}$.

Is there a basis B of the K-vec M s.t. for any i, $\exists$ $U_q(\mathfrak{g})_i$ iso $M \simeq \bigoplus V(\ell_j)$ by which B is sent to $\{ u_k^{(\ell_j)} \}$ in $\bigoplus V(\ell_j)$

Answer: No unless $q=0$.

Local bases

Let k be the field and $K = k(q)$ rational functions on q.

V — K-vector space. For $C \subset K$ subalg, ~~called~~ A C-lattice is a C-submodule
$L \subset V$ s.t. $V \simeq K \otimes_C L$.

$K \supset A := \{f(q) \mid \text{regular at } 0\}$. $\quad A/qA \xrightarrow{\sim} k$.
$f(q) \longmapsto f(0)$

Def: Let V be a K-v.s. A local basis of V at $q=0$ is a pair.
(L, B) L is an A-Lattice of V which is a free A-module.
B - is a basis of the k-v.s. L/qL.

Example: If B is a basis of V as a K-v.s. We can ass. (L, B)
where L is the A-mod generated by B and B is the image of
B under $L \to L/qL$.

If $\{V_j\}$ family of K-v.s. (L_j, B_j) $\quad L = \oplus L_j \subset \oplus V_j$

$B = \oplus B_j \subset \oplus L_j/qL_j$ is a local basis for $\oplus V_j$

V_1, V_2, (L_1, B_1) (L_2, B_2)

$L = L_1 \otimes_A L_2 \subset V_1 \otimes_K V_2$ $\quad B = B_1 \otimes B_2 = \{b_1 \otimes b_2 \mid b_i \in B_i\} \subset L_1/qL_1 \otimes L_2/qL_2$

is a local basis for $V_1 \otimes_? V_2$

Global basis $\quad K = k(z)$

$A = \{f(z) \text{ reg at } 0\}$. $\quad A_\infty = \{f(z) \text{ reg at } \infty\}$ $\quad V - K$ v.s.

(L, B) — gives a vector bundle around 0 with a basis of the fiber at 0.

We want to extend the v.b. around 0 to a trivial v.b. on $\mathbb{P}^1$.

$$\begin{array}{ccccc} A_0 & \{0\} \subset & \mathbb{P}^1 & \supset \{\infty\} & A_\infty \\ & & \cup & & \\ & & \mathbb{P}^1 \setminus \{0, \infty\} & & \\ & & k[z, z^{-1}]. & & \end{array}$$

Let L be an A-Lattice of V.

L_∞ be a A_∞-Lattice of V

$V_{k[z,z^{-1}]}$ be a $k[z,z^{-1}]$-Lattice of V.

$E = L \cap L_\infty \cap V_{k[z,z^{-1}]}$. global sections of V.

Prop: The following are equivalent. ⓪ the vector bd is trivial

① $E \xrightarrow{\sim} L/zL$

② $E \xrightarrow{\sim} L_\infty / z^{-1} L_\infty$

③ $(L \cap V_{k[z,z^{-1}]}) \oplus (z^{-1} L_\infty \cap V_{k[z,z^{-1}]}) \xrightarrow{\sim} V_{k[z,z^{-1}]}$ $\left(\begin{array}{c} \text{ker coker} \\ H^*(V \otimes \mathcal{O}(1)) \end{array} \right.$

④ $A \otimes_k E \xrightarrow{\sim} L \qquad A_\infty \otimes_k E \xrightarrow{\sim} L_\infty$

$k[z, z^{-1}] \otimes_k E \xrightarrow{\sim} V_{k[z,z^{-1}]}$.

Def: if $(L, L_\infty, V_{k[z,z^{-1}]})$ satisfies any of the above we call it balanced.

Let $G : L/zL \xrightarrow{\sim} E$ if B is a local basis for L/zL, $G(B)$ is a global basis of E.

Crystal basis:

M int. $U_q(\mathfrak{g})$-module. $M = \bigoplus_{\lambda \in P} M_\lambda$.

Def. The Crystal basis of M is a local basis (L, B) of the K-v.s. s.t.

① $\exists$ local basis (L_λ, B_λ) of M_λ s.t. $(L, B) \in \bigoplus_{\lambda \in P} (L_\lambda, B_\lambda)$.

② For any $i \in I$, $\exists$ a $U_q(\mathfrak{g})_i$-linear iso $M \xrightarrow{\sim} \bigoplus V(\ell_j)$ by which (L, B) is sent to the local basis $\{u_k^{(\ell_j)}\}_{0 \le k \le \ell_j}$ of $\bigoplus_j V(\ell_j)$

Let (L, B) a crystal basis of M. Let $i \in I$, we can use ②:

for $b \in B$, $b \longrightarrow u_k^{(\ell_j)}$

Let $\tilde{f}_i b$ be the element of B s.t. $\tilde{f}_i b \longmapsto u_{k+1}^{(\ell_j)}$. $\tilde{f}_i b = 0$ if $k = \ell_j$

define $\tilde{e}_i b$. Then we get

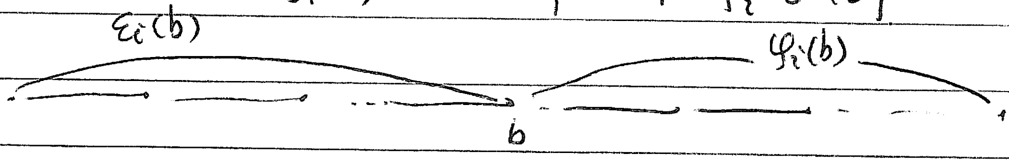
$$\begin{matrix}\tilde{f}_i \\ \tilde{e}_i\end{matrix} : B \longrightarrow B \cup \{0\}$$

for $i \in I$, join $b, b' \in B$ by an arrow labeled by i if $b' = \tilde{f}_i b \Leftrightarrow b = \tilde{e}_i b'$

Then we got a colored oriented graph. — Crystal gph.

For $b \in B$, set $\varepsilon_i(b) = \max\{n \ge 0 \mid \tilde{e}_i^n b \ne 0\}$

$\varphi_i(b) = \max\{n \ge 0 \mid \tilde{f}_i^n b \ne 0\}$



$b \in B_\lambda$, $\mathrm{wt}(b) = \lambda$.

Propties:

① $\{M_j\}$ $\{(L_j, B_j)\}$, $\Rightarrow$ $\bigoplus M_j$ has a crystal basis given by $\bigoplus (L_j, B_j)$.

② Thm: M_j $j=1,2$. be an int $U_q(\mathfrak{g})$-mod and (L_j, B_j) crystal basis of M_j, $\cancel{j+}$ $j=1,2$. Then

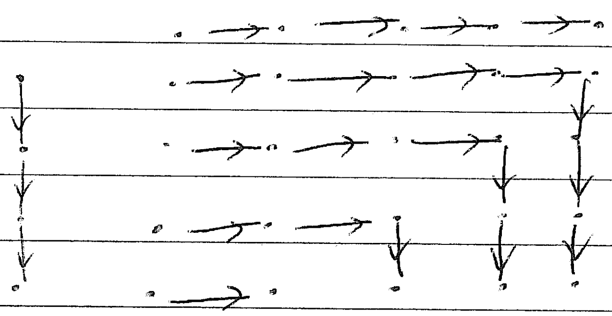
(i) $(L_1, B_1) \otimes (L_2, B_2)$ is a crystal basis of $M_1 \otimes_K M_2$

(ii) $b_j \in B_j$ and $i \in I$,

$$\tilde{e}_i (b_1 \otimes b_2) = \begin{cases} \tilde{e}_i b_1 \otimes b_2 & \varphi_i(b_1) \geq \varepsilon_i(b_2) \\ b_1 \otimes \tilde{e}_i b_2 & \varphi_i(b_1) < \varepsilon_i(b_2) \end{cases}$$

$$\tilde{f}_i (b_1 \otimes b_2) = \begin{cases} \tilde{f}_i b_1 \otimes b_2 & \varphi_i(b_1) > \varepsilon_i(b_2) \\ b_1 \otimes \tilde{f}_i b_2 & \varphi_i(b_1) \leq \varepsilon_i(b_2) \end{cases}$$

Ex:



③ Thm: ⓘ $V(\lambda)$ has a unique crystal basis $(L(\lambda), V(\lambda))$.

ⓘⓘ $B(\lambda) = \{ \tilde{f}_{i_1}^{a_1} \cdots \tilde{f}_{i_\ell}^{a_\ell} u_\lambda \mid \ell \geq 0 \;\; i_1 \ldots i_\ell \in I \;\; a_1 \ldots a_\ell > 0 \} - \{0\}$.

Thm: Let M be a $U_q(\mathfrak{g})$-mod in $\mathcal{O}_{int}(\mathfrak{g})$ and (L, B) be a crystal basis for M. $\exists !$ $U_q(\mathfrak{g})$-map $M \xrightarrow{\sim} \bigoplus V(\lambda_j)$ by which (L, B) is iso to $\bigoplus (L(\lambda_j), B(\lambda_j))$.

Call $b \in B$ a h.w. if $\forall i$, $\tilde{e}_i b = 0$.

decom of M into irr. rep $\longleftrightarrow$ decom of B into connected comp.

$\longleftrightarrow$ h.w. vector of B.

Lemma: $\tilde{e}_i(b_1 \otimes b_2) = 0 \Leftrightarrow \tilde{e}_i b_1 = 0$ and $\varepsilon_i(b_2) \leq \varphi_i(b_1)$.

Prop: $\lambda, \mu \in P_+$. $V(\lambda) \otimes V(\mu) \simeq \bigoplus V(\lambda + wt(b))$.

$b \in B(\mu)$ s.t. $\varepsilon_i(b) \leq \langle h_i, \lambda \rangle$ for all $i \in I$.