Hall algebras

A small abelian category $\mathcal{A}$ is called finitary if:

1. $\forall M, N \in Ob(\mathcal{A})$ $\quad |Hom(M,N)| < \infty$
2. $\forall M, N \in Ob(\mathcal{A})$ $\quad |Ext^1(M,N)| < \infty$.

usually in examples $\mathcal{A}$ is linear over $\mathbb{F}_q$ and $\dim_{\mathbb{F}_q} Hom(M,N) < \infty$ $\quad \dim_{\mathbb{F}_q} Ext^1(M,N) < \infty$.

Example: ① Let A be a f.d. $\mathbb{F}_q$-alg. Mod A is finitary.

② $Rep_{\mathbb{F}_q} Q$ $\quad Q$ a quiver.

③ $Coh(X)$. where X is a proj. variety over $\mathbb{F}_q$.

$K(\mathcal{A})$ is the Grothendieck group over $\mathbb{Z}$.
Many times this will be a free $\mathbb{Z}$-module.
If any $M \in \mathcal{A}$ has finite length then $K(\mathcal{A})$ is freely gen. by the classes of the simple objects.

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Suppose $\mathcal{A}$ is finitary and $gldim(\mathcal{A}) < \infty$ and $|Ext^i(M,N)| < \infty$. Define

$$\langle M,N \rangle_m = \left(\prod_{i=1} |Ext^i(M,N)|^{(-1)^i}\right)^{\frac{1}{2}} \qquad Ext^i(M,N) = \{0\} \ i \gg 0$$

Ex: $\langle M,N \rangle$ depends only on iso. class, defines a form $\langle , \rangle : K(\mathcal{A}) \times K(\mathcal{A}) \to \mathbb{C}$.

this is the mult. Euler form.

$(M,N)_m = \langle M,N \rangle_m \langle N,M \rangle_m$ mult sym. Euler form.

When A is k-linear, we have additive versions:

$\langle M, N\rangle_a = \sum (-1)^i \dim \mathrm{Ext}^i(M,N)$

$(M,N)_a = \langle M,N\rangle_a + \langle N,M\rangle_a$.

Let $X = \mathrm{Ob}(A)/\sim$ be the set of iso classes of objects.

$$H_A := \bigoplus_{M \in X} \mathbb{C}[M]$$

Given M, N, R let $\mathcal{P}^R_{M,N}$ denote the set of short exact seq. $0 \to N \to R \to M \to 0$

$P^R_{M,N} = |\mathcal{P}^R_{M,N}|$. $\quad a_P = |\mathrm{Aut}(P)|$

Prop (Ringel) $\quad [M]\cdot[N] = \langle M,N\rangle_m \sum \frac{1}{a_M a_N} P^R_{M,N}[R]$ defines a structure of an ass. alg. on H_A. $[0]$ is the unit.

X is 'moduli space of objects'

$$H_A = \{ f : X \to \mathbb{C} \mid \mathrm{supp}(f) \text{ is finite}\}$$

$$(f\cdot g)(R) = \sum_{Q \subseteq R} \langle R/Q, Q\rangle_m f(R/Q)\, g(Q)$$

Lemma: $\frac{1}{a_M a_N} P^R_{M,N} = |\{ L \subset R \mid L \simeq N \quad R/L \simeq M\}|$.

proof of ass: $\quad (f\cdot(g\cdot h))(M) =$

$$\sum_{N \subset M} \langle M/N, N\rangle_m f(M/N)\,(g\cdot h)(N) =$$

$$= \sum_{N \subset M}\sum_{L \subseteq N} \langle M/N, N\rangle_m \langle N/L, L\rangle_m f(M/N)\, g(N/L)\, h(L)$$

$$= \sum_{L \subseteq N \subseteq M} \langle M/N, L\rangle_m \langle M/N, N/L\rangle_m \langle N/L, L\rangle_m f(M/N)\, g(N/L)\, h(L)$$

$$((f \cdot g) \cdot h)(M) = \sum_{R \subseteq M} \langle M/R, R \rangle_h \, (f \cdot g)(M/R) \, h(R)$$

$$= \sum_{R \subseteq M} \sum_{S \subseteq M/R} \langle M/R, R \rangle_h \langle (M/R)/S, S \rangle_h \, f((M/R)/S) \, g(S) \, h(R)$$

$$= \sum_{R \subseteq S' \subseteq M} \langle M/S', R \rangle_h \langle S'/R, R \rangle_h \langle M/S', S'/R \rangle_h \, f(M/S') \, g(S'/R) \, h(R)$$

$$\{S \mid S \subseteq M/R\} \rightleftarrows \{S' \mid R \subseteq S' \subseteq M\}$$

$$S = S'/R \qquad (M/R)/S \cong M/S'$$

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① H_A is graded by $K(A)$

$$H_A = \bigoplus_{\alpha \in K(A)} H_A[\alpha] \qquad H_A[\alpha] = \bigoplus_{\substack{M = \alpha \\ \text{in } K(A)}} \mathbb{C}[M]$$

② if A is semisimple with simple objects $S = \{S_i\}_{i \in I}$ then

$$[S_i] \cdot [S_j] = [S_i \oplus S_j] = [S_j][S_i]$$

$$[S_i]^2 = (1 + |\mathrm{End}(S_i)|)\,(S_i \oplus S_i)$$

free comm. on gen. $\{[S_i]\}_{i \in I}$.
usually noncommutative.

$$[M_1] \cdots [M_k] = \sum_R \left(\prod_{i<j} \langle M_i, M_j \rangle_h \,\middle|\, \{ L_k \subset \cdots \subset L_1 = R \} \right) [R]$$
$$L_i / L_{i+1} \cong M_i$$

Green's coproduct

Prop (Green) $\Delta : H_A \to H_A \otimes H_A$ is coasso. top. coalg

$$\Delta(R) = \sum_{M,N} \langle M, N \rangle_h \frac{1}{a_R} R^R_{M,N} [M] \otimes [N]$$

$$\varepsilon : H_A \to \mathbb{C} \qquad \varepsilon([M]) = \delta_{M,0}$$

(ass.) well defined because finitary.

proof: need to check that

$$\sum_S \langle S,Q\rangle_m \langle M,N\rangle_m \frac{1}{a_S a_R} P^S_{M,N} P^R_{S,Q} =$$
$$\sum_T \langle N,Q\rangle_m \langle M,T\rangle_m \frac{1}{a_T a_R} P^T_{N,Q} P^R_{M,T}$$

$$\Leftrightarrow \sum_S \frac{1}{a_S} P^S_{M,N} P^R_{S,Q} = \sum_T \frac{1}{a_T} P^T_{N,Q} P^R_{M,T}$$

$\Leftrightarrow$ ass. of mult. up to mult by $\langle M,N\rangle_m \langle M,Q\rangle \langle N,Q\rangle_m \frac{1}{a_M a_N a_Q}$

left hand side is coefficient of $[R]$ in $([M][N])\cdot[Q]$ right hand side coefficient of $[R]$ in $[M]\cdot([N]\cdot[Q])$

Prop: Assume $\overline{gldim}(A) \leq 1$.

$$\Delta(f)(M,N) = \frac{1}{|Ext^1(M,N)|} \sum_{\alpha \in Ext^1(M,N)} f(X_\alpha)$$

X_α is the middle term of the extension of M by N ass. to α.

proof: need to show

$$\frac{1}{|Ext^1(M,N)|} \left|\{\alpha \in Ext^1(M,N) \mid X_\alpha = R\}\right| = \langle M,N\rangle_m \frac{1}{a_R} P^R_{M,N}$$

this would follow from

equiv. classes of s.e.s $0 \to N \to R \to M \to 0$ $= \left|\{\alpha \in Ext^1(M,N) \mid X_\alpha = R\}\right| = |P^R_{M,N}| \frac{|Hom(M,N)|}{a_R}$

follows from action of $Aut(R)$ on such s.e.s with stabilizer $Hom(M,N)$

Δ respect the $K(A)$ grading

$$\Delta(H_A[\gamma]) \subset \prod_{\alpha+\beta=\gamma} H_A[\alpha] \otimes H_A[\beta]$$

If S is simple $\Delta[S] = [S]\otimes 1 + 1\otimes[S]$

Δ takes values in $H_A \otimes H_A$ if any fixed object has only fin. many subobj.

true for quivers not for curves.

for $\gamma \in K(A)$ let $1_\gamma = \sum_{\bar{M}=\gamma} [M]$

(might belong to a completion).

<u>Lemma:</u> $\Delta(1_\gamma) = \sum_{\alpha+\beta=\gamma} \langle \alpha,\beta\rangle_m^{-1}\, 1_\alpha \otimes 1_\beta$

The Hall bialg.

Assume A is hereditary ($gd(A) \leq 1$).

Let $K = \mathbb{C}[K(A)]$ the group alg. Let

$\tilde{H}_A := H_A \underset{\mathbb{C}}{\otimes} K$ and give the structure of an alg (with H_A and K subalg.)

with relations $k_\alpha [M] k_\alpha^{-1} = (\alpha, M)_m [M]$

extend $\Delta: \tilde{H}_A \to \tilde{H}_A \otimes \tilde{H}_A$ by setting

$\Delta(k_\alpha) = k_\alpha \otimes k_\alpha$

$$\Delta([R]k_\alpha) = \sum_{M,N} \langle M,N\rangle_m \frac{1}{a_R} P^R_{M,N} [M] k_{\bar{N}+\alpha} \otimes [N]k_\alpha$$

<u>Thm:</u> $(\tilde{H}_A, m, \Delta)$ is a top bialg.

$$\varepsilon([M]k_\alpha) = \begin{cases} 0 & M \neq 0 \\ 1 & M = 0 \end{cases}$$

Can use old coproduct with twisted product on $H_A \otimes H_A$ coming from braiding $(\ ,\ )_m$

$$(x \otimes y) \cdot (z \otimes w) = (wt(y), wt(z))_m \, (xz \otimes yw)$$

<u>Scalar product</u>

<u>prop</u>: $(\ ,\ ) : \bar{H}_A \otimes \bar{H}_A \to \mathbb{C}$ ~~([M],[N])~~ sym. non deg.

$$([M] K_\alpha, [N] K_\beta) = \frac{\delta_{M,N}}{a_M} (\alpha, \beta)_m$$

is a Hopf pairing $(xy, z) = (x \otimes y, \Delta(z))$

Th: if the fin. subobj cond. is satisfied $\exists$ an antipode.

<u>Functoriality:</u>

$F : A \to B$ $\qquad f : X_A \to X_B$

$$f^* : H_B \to H_A^{coop} \qquad [M] \mapsto \sum_{\substack{R \in X_A \\ F(R) \simeq M}} [R]$$

$$f_* : H_A \to H_B \qquad [M] \to [F(M)]$$

not necessarily maps of alg. or coalg.

<u>prop</u>: Let A, B be finitary $F : A \to B$ full faithful.

Then f_* is an embedding of alg. and if A, B are hereditary f^* is a map of coalg.
If $F(A)$ is stable under subobjects then f_* is a morphism of coalg and f^* is a hom. of alg.

Hall algebras of quivers

Let $K = \mathbb{F}_q$ and $\vec{Q}$ a quiver we want to compute $H_{\vec{Q}}$ of $Rep_k \vec{Q}$ (finitary, hereditary).
Let $v = q^{\frac{1}{2}}$.

__Example:__ $\vec{Q} = \bullet$ single simple object S any object is iso. to $S^{\oplus j}$. $Hom(S,S) = k$

$Ext^1(S,S) = \{0\}$ $\quad \langle S, S \rangle_m = q^{\frac{1}{2}} = v$

$$[S^{\oplus i}] \cdot [S^{\oplus j}] = v^{ij} \, |Gr(j, i+j)| \, [S^{\oplus i+j}] =$$

$$v^{ij} \begin{bmatrix} i+j \\ j \end{bmatrix}_+ [S^{\oplus i+j}]$$

$$\Rightarrow [S]^n = v^{n(n-1)} (n)! \, [S^{\oplus n}]$$

__Example:__ $\overset{1}{\bullet} \longrightarrow \overset{2}{\bullet}$ $\quad S_1 \quad S_2$ simple modules.

$$Ext^1(S_1, S_2) = \{0\} \qquad Ext^1(S_2, S_1) = k$$

$\langle S_1, S_2 \rangle_m = q^{-\frac{1}{2}} = v^{-1}$. Let I_{12} be the unique nontrivial extension of S_2 by S_1.
$Hom(S_2, I_{12}) = k \Rightarrow$

$$[S_1][S_2] = v^{-1}\left([S_1 \oplus S_2] + [I_{12}]\right)$$

no nontriv extensions of S_1 by S_2 so $\langle S_2, S_1 \rangle_q = 1$
and

$$[S_2] \cdot [S_1] = [S_1 \oplus S_2]$$

$$[S_2] \cdot [S_1]^2 = v^{\frac{1}{2}}(v^2+1) \, [S_2][S_1^{\oplus 2}] = v^{\frac{1}{2}}(v^2+1) \, [S_1^{\oplus 2} \oplus S_2]$$

$$[S_1]^2 [S_2] = v^{\frac{1}{2}}(v^2+1) \, [S_1^{\oplus 2}][S_2] =$$

$$= v^{-\frac{1}{2}}(v^2+1)\left([S_1^{\oplus 2} \oplus S_2] + [S_1 \oplus I_{12}]\right)$$

$$[S_1][S_2][S_1] = [S_1][S_1 \oplus S_2] =$$

$$(v^2+1)[S_1^{\oplus 2} \oplus S_2] + [S_1 \oplus I_{12}]$$

since there are $q+1$ submod of $S_1^{\oplus 2} \oplus S_2$ iso to $S_1 \oplus S_2$ and only one sub mod of $I_{12} \oplus S_2$ iso to $S_1 \oplus S_2$.

$$\Rightarrow [S_1]^2[S_2] - (v+v^{-1})[S_1][S_2][S_1] + [S_2][S_1]^2$$
$$= 0$$

$$[S_2]^2[S_1] - (v+v^{-1})[S_2][S_1][S_2] + [S_1][S_2]^2 = 0$$

recall $\bar{A} = (a_{ij})_{i,j \in I}$ $a_{ij} = 2\delta_{ij} - c_{ij} - c_{ji}$

$$\langle S_i, S_j \rangle_a = \delta_{ij} - c_{ij}$$
$$\langle S_i, S_j \rangle_m = q^{\frac{1}{2}}(\delta_{ij} - c_{ij})$$

c_{ij} = number of arrows from i to j

$\{S_i \mid i \in I\}$ collection of simple objects.

$\mathrm{Rep}_{k}\vec{Q}$ hereditary $\dim \mathrm{Ext}^1(S_i, S_j) = c_{ij}$

$Q = \oplus \mathbb{Z}\alpha_i$ $\{\alpha_i \mid i \in I\}$ set of simple roots.

$$g : K(\mathrm{Rep}\,\vec{Q}) \to Q$$
$$V \mapsto \sum_i \dim(V_i)\alpha_i$$

$(\, , \,)_a \to$ Cartan form.

$U_v(b_+)$ gen by $E_i, K_i^{\pm 1}$ $i \in I$

relations: $K_i K_j = K_j K_i$ $K_i E_j K_i^{-1} = v^{a_{ij}} E_j$

$$\sum_{l=0}^{1-a_{ij}} (-1)^l \begin{bmatrix} 1-a_{ij} \\ l \end{bmatrix} E_i^{\,l} E_j E_i^{\,1-a_{ij}-l} = 0$$

$\Delta(K_i) = K_i \otimes K_i$ $\Delta(E_i) = E_i \otimes 1 + K_i \otimes E_i$

$S(K_i) = K_i^{-1}$ $S(E_i) = -K_i^{-1} E_i$.

Thm (Ringel, Green) $E_i \mapsto [S_i]$ $K_i \mapsto K_{S_i}$ extends to an embedding of Hopf alg.

$$\psi: U_v(b_+) \to \widetilde{H}_{\vec{Q}}$$

ψ is an iso. iff $\vec{Q}$ is of finite type.

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Assume $\vec{Q}$ is of finite type. Note that the set of objects of Rep $\vec{Q}$ is independent of the field and can be canonically identified with maps $\Delta^+ \to \mathbb{N}$. Denote the set of objects by $O_{\vec{Q}}$

Prop: Let $\vec{Q}$ be of finite type. Then

(i) For each object $L \in O_{\vec{Q}}$ $\exists!$ $f_L \in U_{v}^{res}(b_+)$ s.t. for any finite field $k = \mathbb{F}_q$ $(f_L)|_{v=v}$:

$\psi^{-1}{}_v([L])$.

(ii) Let M, N, R be obj. of $O_{\vec{Q}}$. $\exists!$ $P_{M,N}^R(t) \in \mathbb{Q}[t]$ $(\mathbb{Z}[t])$ s.t. for any $k = \mathbb{F}_q$

$\frac{1}{a_M a_N} b_{M,N}^R = P_{M,N}^R(q)$

proof by induction using

Lemma: Let $\vec{Q}$ be a finite quiver. $\exists$ total ordering of the set of indecom. (indep of k) s.t.

$$M < N \Rightarrow Hom(N, M) = Ext^1(M, N) = \{0\}$$

$$Ext^1(M, M) = (0) \qquad End(M) = k.$$

Def: The generic Hall alg is the alg

$$\underline{H}\vec{Q} = \bigoplus_{M \in \mathcal{O}_{\vec{Q}}} \mathbb{C}[t^{\frac{1}{2}}, t^{-\frac{1}{2}}] f_M$$

$$f_M \cdot f_N = \sum_R t^{\frac{1}{2}\langle M, N\rangle_a} P^R_{M,N}(t) f_R$$

Cor: $\varphi : U_v(n_+) \xrightarrow{\sim} \underline{H}\vec{Q}$

$v = t^{\frac{1}{2}}$