10/23.

Koszul duality in deformation Quantization.

1. Hochschild cohomology
2. Diff. graded category and their Hochs.
3. B. Keller's d.g. category Keller's condition
4. Koszul duality
5. Main theorem
6. Application
7. Pf of 5.

V — finite dim vector space $/\mathbb{C}$. $A = \cancel{S(\mathbb{C}} \, S(V^*)$ $B = \Lambda(V)$.

~~Consider polyvector field on V:~~ ~~section of $\Lambda^k(TM)$ — k-polyvector field on M.~~

Polyvector fields on $V \simeq$ Poly v. on $V^*[1]$

Functions on $V \simeq S(V^*)$

Functions on $V^*[1] \simeq \Lambda(V)$.

Poly v fields on $V \overset{D}{\simeq} S(V^*) \otimes_{\mathbb{C}} \Lambda^{\cdot} V = \mathrm{Fun}(V) \otimes \mathrm{Fun}(V^*[1])$

D: k-linear j-vector field on $V \xrightarrow{D}$ j-linear k-vector field on $V^*[1]$.

Quadratic bivector fields: $\alpha = \sum a_{ij}^{kl} x_i x_j \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j}$

$[\alpha, \alpha] = 0$ i.e. α is a Poisson bivector fields on V,

$\Rightarrow$ $f * g = fg + \hbar \cdot \alpha(df \wedge dg) + \hbar^2 \cdots\cdots$ s.t.

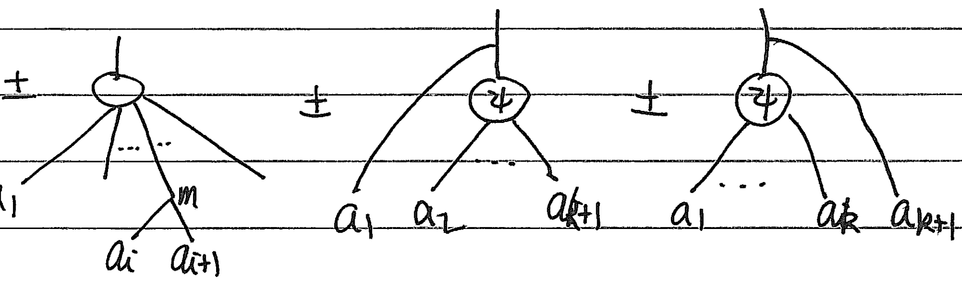
$(f*g)*h = f*(g*h)$

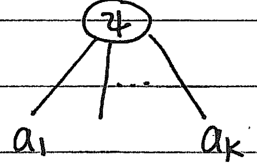
we can def. $d_{\hbar}$ of $S(V^*)$ and $\Lambda(V)$: $S(V^*)_\alpha$ $\Lambda(V)_{D(\alpha)}$

1.

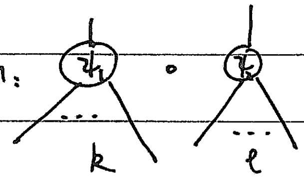
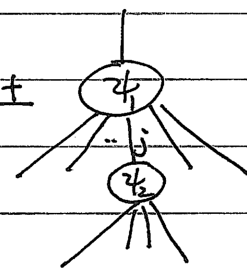
$Hoch^k(A) = Hom_{\mathbb{C}}(A^{\otimes k}, A)$

Differential of deg +1 and Lie bracket:

$d\psi = \sum \pm$ 



Ass. of $A \Leftrightarrow d^2 = 0$.

Composition:  $=\sum_j \pm$ 

$[\psi_1, \psi_2] = c\left(\psi_1 \circ \psi_2 - (-1)^{(k-1)(\ell-1)} \psi_2 \psi_1\right)$.

Ex. $\psi : A^{\otimes 2} \longrightarrow A$ $\qquad a * b = a \cdot b + \psi(a, b)$

Ass. $(a*b)*c = a*(b*c)$

$\Longleftrightarrow \quad d\psi + \frac{1}{2}[\psi, \psi] = 0$

Hoch - K - R Theorem: $A = SV^*$, $Hoch(SV^*)$

1) $HH^{\cdot}(S(V^*)) = T_{poly}(V)$

2) The induced Lie alg on $HH^{\cdot}(S(V^*)) \simeq T_{poly}(V)$

Def: $\varphi_{HKR} : T_{poly} \longrightarrow HH(A)$ $\quad (A = C^\infty(M))$

γ — k-poly vector fields.

$A^{\otimes k} \longrightarrow A \qquad \varphi(\gamma)(f_1 \otimes \cdots \otimes f_k) = \frac{1}{k!} \gamma(df_1 \wedge \cdots \wedge df_k)$

$\varphi_{HKR} : T_{poly}(M) \longrightarrow Hoch^{\cdot}(C^\infty(M))$ is not a map of Lie algebras.

But on the level of coho, it is. i.e.

$$\varphi([\gamma_1, \gamma_2]_S) = [\varphi(\gamma_1), \varphi(\gamma_2)] + d_{Hoch} u_2(\gamma_1, \gamma_2)$$

where $u_2(\gamma_1, \gamma_2) : \Lambda^2 T_{poly} \longrightarrow Hoch^{\bullet}[-1]$.

We define $u_1 = \varphi_{HKR}$ u_2 as above. is there $u_3 \dots u_k \dots$

$$u_k : \Lambda^k T_{poly} \longrightarrow Hoch^{\bullet}[1-k]$$

L_∞-morphism: $u_1, \dots\dots u_k \dots\dots$

Ex: $\mathfrak{g}_1$ $\mathfrak{g}_2$ two dg. Lie alg, $u : \mathfrak{g}_1 \longrightarrow \mathfrak{g}_2$ — L_∞ morphism.

$$\tilde{\gamma} = u_1(\gamma) + \frac{1}{2} u_2(\gamma, \gamma) + \frac{1}{6} u_3(\gamma, \gamma, \gamma) + \dots + \frac{1}{k!} u_k(\gamma \dots \gamma) + \dots$$

each term has deg 1. γ s.t. the MC eqn in $\mathfrak{g}_1$.

Then $d_2 \tilde{\gamma} + \frac{1}{2}[\tilde{\gamma}, \tilde{\gamma}] = 0$

2.3 ~~Cat : A . B obj.~~

$Cat(A^{\bullet}, B^{\bullet}, K)$ A, B are dga K - B-A bimodule.

Obj: a, b, $Morph(a,a) = A$ $Morph(b,b) = B$ $Morph(b,a) = K$ $Morph(a,b) = 0$

In general, $X_1, \dots, X_m$ — objects.

$Morph(X_i, X_j)$ — complexes of vector space over $\mathbb{C}$.

$Morph(X_i, X_j) \otimes Morph(X_j, X_k) \longrightarrow Morph(X_i, X_k)$

a map of complexes.

Ex: one obj. $Mor(a,a) \otimes Mor(a,a) \longrightarrow Mor(a,a)$.

$A = Mor(a,a)$ A is a dga.

Keller's condition (A, B, K).

$B \to \mathrm{Hom}_{mod\text{-}A}(K, K)$ $\qquad$ $B \xrightarrow{qui} R\mathrm{Hom}_{mod\text{-}A}(K, K)$

$A^{op} \to \mathrm{Hom}_{B\text{-}mod}(K, K)$ $\qquad$ $A^{op} \xrightarrow{qui} R\mathrm{Hom}_{B\text{-}mod}(K, K)$.

① Suppose $B \xrightarrow[\varphi]{qui} A$ $\quad K = A$.

② A - Koszul quadratic alg, $\quad B$ — Koszul dual. $\quad K$ - Koszul complex.

$B \xrightarrow{quis.} A$ $\qquad HH^{\cdot}(B) \xrightarrow{\sim} HH^{\cdot}(A)$ $\qquad$ Thm: Keller's condition implies P_A P_B are quis.

$$\begin{tikzcd} & \mathrm{Hoch}^{\cdot}(\mathrm{cat}) \arrow[dl, "P_A"'] \arrow[dr, "P_B"] & \\ \mathrm{Hoch}^{\cdot}(A) & & \mathrm{Hoch}^{\cdot}(B) \end{tikzcd}$$

$\mathrm{Hoch}^{\cdot}(\mathrm{cat})$

$\oplus A^{\otimes k} \to A$ $\qquad d_H^A$

$\oplus B^{\otimes m} \to B$ $\qquad d_H^B$

$\oplus B^{\otimes l_1} \otimes K \otimes A^{\otimes l_2} \to K$ $\qquad \mathrm{Hoch}(B, K, A)$.

$\psi(b_1, \dots, b_{l_1}, K, a_1, \dots, a_{l_2}) \in K$.

$d_B \psi(b_1, \dots, b_{l_1+1}, K, a_1, \dots, a_{l_2})$

$d_A \psi(b_1, \dots, b_{l_1}, K, a_1, \dots, a_{l_2+1})$

$$R_{-1} = \bigoplus_{j=1}^{n} \mathrm{Mor}(X_i, X_i)$$

$$R_K = \bigoplus_{\substack{\cancel{X_{i_1} \to X_{i_2} \to \dots \to X_{i_{k+1}}} \\ (i_1, \dots, i_{k+1})}} \mathrm{Hom}_{\mathbb{C}}\Big(\big(\mathrm{Mor}(X_{i_1}, X_{i_2}) \otimes \cdots \otimes \mathrm{Mor}(X_{i_k}, X_{i_{k+1}})\big), \mathrm{Mor}(X_1, X_{i_{k+1}})\Big)$$

$X_{i_1} \to X_{i_2} \to \dots \to X_{i_{k+1}}$ $\quad$ consider:

$$\begin{tikzcd} X_{i_1} \arrow[r] & X_{i_2} \arrow[r] \arrow[dr] & X_{i_3} \arrow[r] & \cdots \arrow[r] & X_{i_{k+1}} \\ & & X_{\alpha} \arrow[u] & & \end{tikzcd}$$

$$\sum_s d_s(\psi)\left(Mor(X_s, X_{i_1}), \dots, Mor(X_{i_k}, X_{i_{k+1}})\right) = Mor(X_s, X_{i_1})\circ\psi.$$

$$\mathrm{Hoch}(\mathrm{Cat}) = \begin{cases} \mathrm{Hoch}(A) \circlearrowleft \\ \mathrm{Hoch}(B) \circlearrowleft \\ \mathrm{Hoch}(B, K, A) \circlearrowleft \end{cases} \Big\} \text{ subcomplexes} \qquad \text{subcomplexes}$$

$$\mathrm{Hoch}(\mathrm{Cat}) \xrightarrow{p_A} \mathrm{Hoch}(A) \qquad \mathrm{Hoch}(B,K,A) \xrightarrow{p_B} \mathrm{Hoch}(B)$$

Consider $\varphi = p_A : \mathrm{Hoch}(\mathrm{Cat}\,A) \longrightarrow \mathrm{Hoch}(A)$.

$$\mathrm{Cone}(\varphi) = \mathrm{Hoch}(\mathrm{Cat}\,A)[1] \oplus \mathrm{Hoch}(A).$$

$$E_1 : \quad \cancel{\mathrm{Hoch}(\mathrm{Cat}\,A)} \quad \begin{array}{c} \mathrm{Hoch}(B,K,A) \circlearrowleft \\ \uparrow \\ \mathrm{Hoch}(B). \circlearrowleft \end{array}$$

$$\mathrm{Hom}_{\mathbb{C}}(B^{\otimes \ell_1} \otimes K \otimes A^{\otimes \ell_2}, K) = \mathrm{Hom}_{\mathbb{C}}(B^{\otimes \ell_1}, \mathrm{Hom}(K \otimes A^{\ell_2}, K))$$

$$\mathrm{Hom}_{\mathbb{C}}(K \otimes T(A), K) = \mathrm{Hom}_{\text{mod-}A}(\mathrm{Bar}_{\text{mod-}A}(K), K)$$

as left B-module $\longleftarrow$ $B = R\mathrm{Hom}_{\text{mod-}A}(K, K)$

$$\begin{tikzcd} & \mathrm{Hoch}(\mathrm{Cat}) \arrow[dl, "p_A"'] \arrow[dr, "p_B"] & \\ \mathrm{Hoch}(A) & & \mathrm{Hoch}(B) \end{tikzcd}$$

5. $A = SV^*$ $\quad B = \Lambda V.$ $\quad K =$ Koszul complex.

$$\begin{tikzcd} & \mathrm{Hoch}(SV^*) & \\ T_{poly}(V) \arrow[ur, "u^S"] \arrow[dr, "u^\Lambda"'] & & \mathrm{Hoch}(\mathrm{Cat}) \arrow[ul] \arrow[dl] \\ & \mathrm{Hoch}(\Lambda V) & \end{tikzcd}$$

$$T_{poly}(V) = T_{poly}(V^*[1])$$

A — $\mathbb{Z}_+$-graded alg over $\mathbb{C}$, $A = \bigoplus_{i=0}^{\infty} A[i]$ and $A[0] = \mathbb{C}$ $\dim A[i] < \infty$.

$\varepsilon: A \to \mathbb{C}$. $A_+ = \bigoplus_{i \geq 1} A[i]$

$\mathrm{Ext}^i_A(\mathbb{C}, \mathbb{C})$ — graded vector space and degree $\leq -i$. i.e. $\mathrm{Ext}^i_A(\mathbb{C},\mathbb{C})$ lives in deg $\leq -i$.

$\mathrm{Ext}^1_A(\mathbb{C},\mathbb{C}) = \left(A_+ / A_+^2 \right)^*$ — dual space of generator.

Def: A is Koszul if $\mathrm{Ext}^i_A(\mathbb{C},\mathbb{C})$ lives exactly in deg $-i$. $\forall i \geq 1$.

Remark: Koszul condition for $i=1$, A is generated by $A[1] = V$, $A = TV/I$ I homm idea

$i=2$, this is the dual space of relations. $(\mathrm{Ext}^2_A(\mathbb{C},\mathbb{C})) = L^*$

conditions says that $L \subset V \otimes V$, relations are quadratic.

Kozul condition for $i \leq 2 \Rightarrow$ we say A is a quadratic alg.

Suppose A is quadratic, then we can define.

$$A^! = TV^* / (L^\perp) \quad \text{where } L^\perp \in V^* \otimes V^* \text{ (also quadratic)}.$$

Thm: $\mathrm{Ext}^i_A(\mathbb{C},\mathbb{C})[-i] = A^![i]$

If A is Koszul, $\mathrm{Ext}^i_A(\mathbb{C},\mathbb{C}) = A^![i]$ and $A^! \cong \bigoplus \mathrm{Ext}^i_A(\mathbb{C},\mathbb{C})$ as an algebra,

- $(A^!)^! = A$

A - any quadratic alg, we can define Koszul complex of A.

Ex: 1) $A = S(V)$ 2) $A = \Lambda(V)$ 3) $A = TV$ 4) $A = \underset{0}{\mathbb{C}} \oplus \underset{1}{V}$

1) $(SV)^! = \Lambda(V)$ 2) $\Lambda(V)^! = S(V)$ 3) $(TV)^! = \mathbb{C} \oplus V^*$.

Koszul complex: $SV^* \otimes \Lambda V^*$ $SV^* \to \mathbb{C}$

$\to SV^* \otimes \Lambda^2 V^* \to SV^* \otimes V^* \to SV^* \to \mathbb{C}$

Def: The Koszul complex of A is $A \otimes (A^!)^* = K$ $\quad$ $*$ – restrict dual.

$d(a \otimes b) = \sum a v_i \otimes b v_i^*$ $\quad$ v_i basis of V $\quad$ v_i^* – dual basis of V^*.

$d^2(a \otimes b) = a v_i v_j \otimes b v_i^* v_j^*$ $\quad$ $*: \left(V^* \otimes A^![i] \to A^![i+1]\right) \Rightarrow V \otimes A^![i+1] \to A^![i]$.

$\left(U \supset L \quad U^* \supset L^\perp \quad c \in U \otimes U^* \to 0 \in U/L \otimes U^*/L^\perp \right)$

$\sum v_i v_j \otimes v_i^* v_j^* \to 0$ in $A \otimes (A^!)^*$

Thm: A Koszul $\Leftrightarrow$ Koszul complex is exact.

Cor: A Koszul $\Leftrightarrow$ $A^!$ is Koszul

pf: $\operatorname{Kosz}(A) \cong \operatorname{Kosz}(A^!)^*$.

$\Rightarrow$ if $h_A(t) = \sum \dim A[i] t^i$

$h_A(t)\, h_{A^!}(-t) = 1$ $\quad$ (Euler–Poincaré)