Notations

Γ = finite, connected graph, Γ_0 = vertices, Γ_1 = edges
Λ = orientation of Γ, $\alpha(\ell)$ = start $\beta(\ell)$ = end

$$\overset{\alpha(\ell)}{\bullet} \longrightarrow \overset{\beta(\ell)}{\bullet}$$

Representation category $\mathcal{L}(\Gamma, \Lambda)$.

Objects (V, f) = Assignments of vector space V_α to each vertex $\alpha \in \Gamma_0$, and $f_\ell : V_{\alpha(\ell)} \longrightarrow V_{\beta(\ell)}$ for each edge in Γ_1

Morphisms: Maps $\varphi_\alpha : V_\alpha \longrightarrow W_\alpha$ commuting w/ f_ℓ:

$$\begin{tikzcd}
V_{\alpha(\ell)} \arrow[r, "f_\ell"] \arrow[d, "\varphi_{\alpha(\ell)}"'] \arrow[dr, phantom, "\circlearrowleft"] & V_{\beta(\ell)} \arrow[d, "\varphi_{\beta(\ell)}"] \\
W_{\alpha(\ell)} \arrow[r, "g_\ell"'] & W_{\beta(\ell)}
\end{tikzcd}$$

Direct sum:

$$(V, f) \oplus (W, g) = (X, h)$$

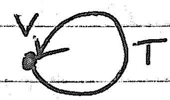
where $X_\alpha = V_\alpha \oplus W_\alpha$
$h_\ell = f_\ell \oplus g_\ell$

(V, f) is indecomposable if $(V, f) = (U, g) \oplus (W, h)$
$\Rightarrow$ One or the other is zero.

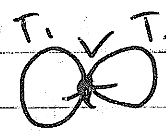
Gabriel's probem: Find all oriented graphs (Γ, Λ) w/ only finitely many indecomposable rep^n's. Classify them. (Called finite type)

Related: Find all graphs w/ ∞ indecomposables, but arranged into finitely many 1-d families. (Called tame)

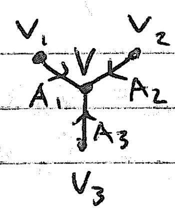
E.G. Jordan Quiver / $\mathbb{C}$ Tame.

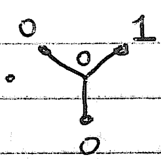
$J = \tilde{A}_0 =$ 

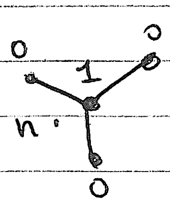
Indecomposables correspond to Jordan blocks. Thus they have a $\mathbb{N}$ parameter for the size of the Jordan block, and a $\mathbb{C}^\times$ parameter for the eigenvalue. Thus, they are tame, not finite type. We'll see that ~~[illegible]~~ existence of cycles precludes finite type, and existence of 2 cycles precludes tameness

E.G. Double loop 

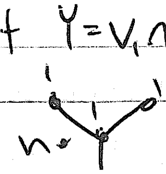
Nazarova Already a wild quiver. Can choose a basis for V to put T_1 into Jordan form, but then have no freedom to put T_2 into Jordan form. A complete classification here would have as a subproblem a description of the commuting variety $\{[T_1, T_2] = 0\}$ which is a known hard problem.

Eg. $G = D_4$: 

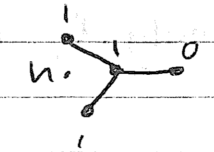
Step 1) Break off $\ker A_i = n \cdot$  to assume each A_i is injective



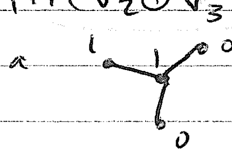
Step 2) Break off $V/(\mathrm{Im}A_1 + \mathrm{Im}A_2 + \mathrm{Im}A_3)$ to assume $V = V_1 + V_2 + V_3$

Step 3) Let $Y = V_1 \cap V_2 \cap V_3$. Choosing complements V', V_1', V_2', V_3' break off  to assume $\emptyset = V_1 \cap V_2 \cap V_3$.

Step 4) Let $Y = V_1 \cap V_2$. Let V_1', V_2', V' be complements.

$\Rightarrow$ split of  , assume $V_1 \cap V_2 = 0$. Similarly get $V_2 \cap V_3 = V_1 \cap V_3 = 0$.

Step 5) Let $Y = V_1 \cap (V_2 \oplus V_3)$. Let V_1' complement in V_1

Split off a  . Similarly for V_2, V_3

$\Rightarrow$ Assume
$$V_1 \subseteq V_2 \oplus V_3$$
$$V_2 \subseteq V_1 \oplus V_3$$
$$V_3 \subseteq V_1 \oplus V_2$$

$$\Rightarrow V_1 \oplus V_2 = V_1 \oplus V_3 = V_2 \oplus V_3 = V$$

$$\Rightarrow \dim V_1 = \dim V_2 = \dim V_3 := n \Rightarrow \dim V = 2n.$$

Step 6) Since $V_3 \subseteq V_1 \oplus V_2$, $\exists$ projection map $V_3 \to V_1$, $V_3 \to V_2$

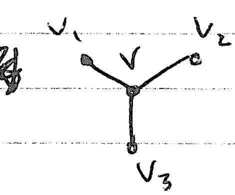
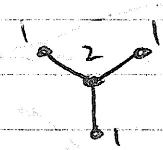
They are $\hookrightarrow$ b/c $V_3 \cap V_2 = V_3 \cap V_1 = 0$. $\Rightarrow \cong$ by dimension.

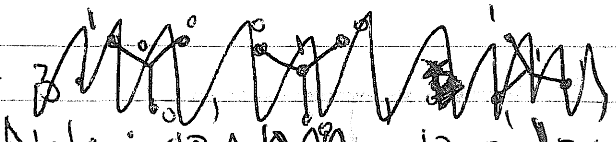
$B_1 : V_3 \hookrightarrow V_1 \oplus V_2 \to V_1$, $B_2 : V_3 \hookrightarrow V_1 \oplus V_2 \to V_2$

Def$^=$ $A = B_2 \circ B_1^{-1} : V_1 \to V_2$

Thus V_3 = graph of the $\cong$ A.

Choose a basis $\{e_i\}$ for V_1. Then $\{Ae_i\}$ is a basis for V_2. $\Rightarrow$ $\{e_i \oplus Ae_i\}$ is a basis for $V_3 \hookrightarrow V = V_1 \oplus V_2$

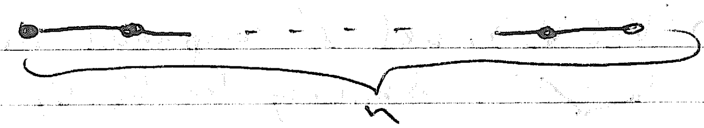
$\Rightarrow$  $= n \cdot$ 

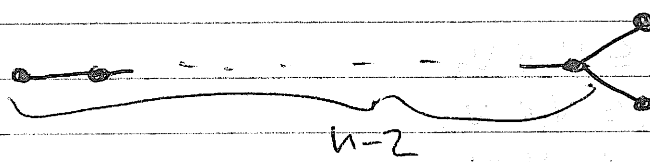
$=$  $\Rightarrow$ There are 12 indecomposables

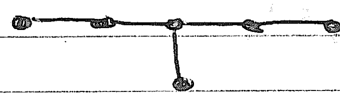
Note: 12 = |pos roots of D_4|

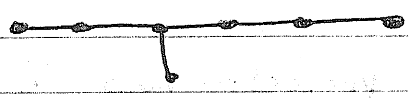
Ex: Do this for A_1, A_2.

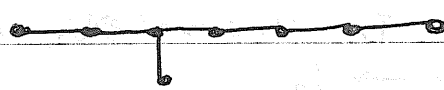
Gabriel's thm. ~~The category $\mathcal{L}(\Gamma, \Lambda)$ is~~
(Γ, Λ) is finite type if, and only if, Γ is a Dynkin diagram of type

A_n 

D_n 

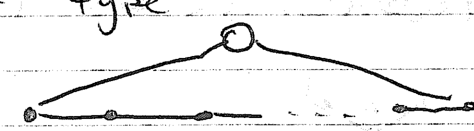
$E_6 =$ 

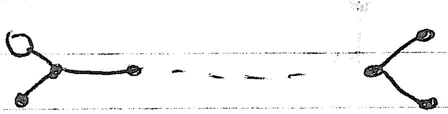
$E_7 =$ 

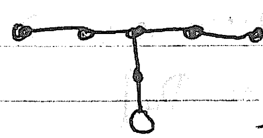
$E_8 =$ 

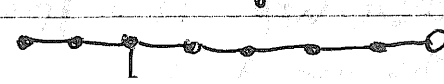
In this case, the indecomposables are in bijection w/ the positive roots of Γ.

(Γ, Λ) is tame iff Γ is of extended Dynkin diagram of type

$\tilde{A}_n =$ 

$\tilde{D}_n =$ 

$\tilde{E}_6 =$  $\hat{E}_7 =$ 

$\tilde{E}_8 =$ 

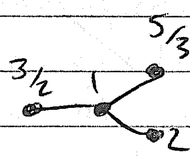
In this case, there is a classification but it is not discussed here.

In general, if we def$^{\underline{n}}$ dimension vectors $\dim(V)$ as usual, Kac has shown that only dimension vectors allowed are the positive roots. (as in Gabriel's th$^{\underline{m}}$. However, there is no complete classification of the representations at each dimension.)

Before Pizza: Part 1 of Gabriel's theorem.
After pizza: Part 2 of Gabriel's Theorem.

Let Γ be a finite connected graph.

Def$^{\underline{n}}$ $E_\Gamma = \{ v : \Gamma_0 \to \mathbb{Q} \}$ (e.g.



v is integral if $v_\alpha \in \mathbb{Z}$ $\forall \alpha \in \Gamma_0$

v is positive $(v > 0)$ if $v_\alpha > 0$ $\forall \alpha \in \Gamma_0$.

Obviously $\dim(V)$ is a positive integral element of E_Γ for any representation $(\neq 0)$.

Def$^{\underline{n}}$ A quadratic form on E_Γ by

$$B(v) = \langle v, v \rangle = \sum_{\alpha \in \Gamma_0} v_\alpha^2 - \sum_{\ell \in \Gamma_1} v_{\alpha(\ell)} v_{\beta(\ell)}$$

Ringel Form $\langle v, w \rangle = \sum_{\alpha \in \Gamma_0} v_\alpha w_\alpha - \sum_{\ell \in \Gamma_1} v_{\alpha(\ell)} w_{\beta(\ell)}$

Cartan Form $(v, w) = \langle v, w \rangle + \langle w, v \rangle$

Key observation: For each $\alpha \in \Gamma_0$, let $\bar{\alpha}$ denote the function $\bar{\alpha}(\beta) = \delta_{\alpha\beta}$.

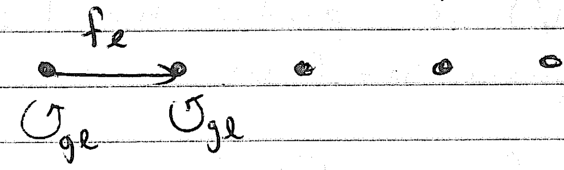
Then in this basis the Cartan form is exactly given by the Cartan Matrix $2\cdot Id - Adj(\Gamma)$ of the graph Γ. Thus, $B(x)$ is positive definite iff Γ is a Dynkin diagram, w/ simple edges.
Non-negative definiteness $\Leftrightarrow$ Extended Dynkin diagram.

Suppose $\mathcal{L}(\Gamma, \Lambda)$ has finitely many indecomposable elements up to isomorphism. Want to show that in this case B is positive definite.

~~For each $(m_\alpha) = \dim(V)$, V not necessarily indecomposable, there can be only finitely many iso-classes of that~~

Then for each positive integral vector, (m_α) there can be only finitely many iso-classes of rep$^{\underline{n}}$s.

Let's construct a moduli space for rep$^{\underline{n}}$s of dimension $= m_\alpha$.



1) Choose bases in each v.s.

2) Assign maps (as matrices to each arrow)

$$M = \prod Mat(m_{\alpha(e)}, m_{\beta(e)}),\ \text{an affine space}$$

3) Quotient out by ~~gl_α action~~ $G = \prod gl_\alpha$ action by conjugation in each vector space. Stabilizer is the 1-dimensional group of scalars.

$$\Rightarrow \dim(\text{Moduli}(m_\alpha)) = \sum_{e\in\Gamma_1} m_{\alpha(e)}\cdot m_{\beta(e)} - \sum_{\alpha\in\Gamma_0} m_\alpha^2 + 1.$$

(*)

Want $\quad 0 \geq \sum_{\ell \in \Gamma_1} m_{\alpha(\ell)} m_{\beta(\ell)} - \sum_{\alpha \in \Gamma_1} m_\alpha^2 + 1$

$\Leftrightarrow$ $B(m) \geq 1$. Since this is required for all m,
$\Leftrightarrow$ Γ is Dynkin type.

Note For tameness, get $B(m) \geq 0$, so extended Diagrams are allowed.

Pizza!

Recall: the Weyl group.

$W = \langle s_\alpha \mid \alpha$ is a simple root $\rangle$ (refl. through orthogonal hyperplane.)

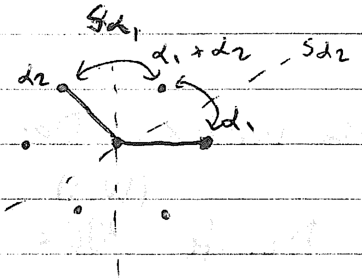
~~$s_\alpha(\beta)$~~ $s_\alpha(x) = x - 2\langle \bar{\alpha}, x \rangle \bar{\alpha}$. ~~In coordinates~~

In coordinates $(s_\alpha(x))_\beta = \begin{cases} x_\beta & \text{if } \alpha \neq \beta \\ -x_\alpha + \sum_{\ell \in \Gamma^\alpha} x_{\gamma(\ell)} \end{cases}$

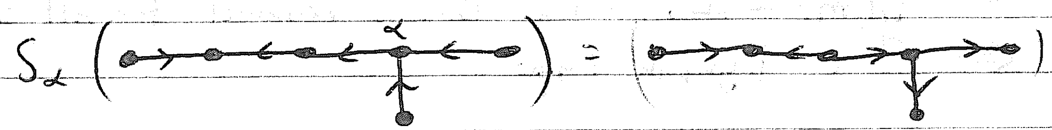
where γ is the other vertex of the edge ℓ, and Γ^α = star at α.

Facts 1) W preserves the integral lattice, and B.
2) If B is positive definite, then W is finite
~~3) $s_\alpha(\{\Pi^+ \setminus \{\alpha\}\}) = \Pi^+ \setminus \{\alpha\}$~~
3) $s_\alpha(\Pi^+ \setminus \{\alpha\}) = \Pi^+ \setminus \{\alpha\}$.

e.g. A_2



Reflection of quivers S_α



$$\Lambda \rightsquigarrow S_\alpha \Lambda.$$

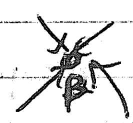
Want to define a functor

$$F_\alpha^\pm : \mathcal{L}(\Gamma,\Lambda) \longrightarrow \mathcal{L}(\Gamma, s_\alpha\Lambda)$$
s.t.
$$\dim(F_\alpha^\pm V) = s_\alpha \dim(V).$$

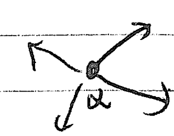
Image functors:

~~Let~~ Fix (Γ,Λ). ~~A vertex~~

Def$^{\underline{n}}$ $\beta \in \Gamma_0$ is (+) accessible if no ~~all~~ edges in Γ^α start there.

 (+) accessible

$\alpha \in \Gamma_0$ is (−) accessible —— " —— end there.



Let β be a (+) accessible vertex. Def$^{\underline{n}}$ $F_\beta^+(V)$ as follows. Denote $(W,g) = F_\beta^+(V,f)$.

Jackson!
Paul + ??

Vola?
Catahoula

$W_\gamma = V_\gamma$ if $\gamma \neq \beta$.

~~$f_\alpha =$~~

$g_\ell = f_\ell$ if $\ell \notin \Gamma^\beta$

~~Def^n W_α~~

~~For W_α, set Cons:~~

Let $W_\beta = \operatorname{Ker} h$, $h: \bigoplus_{\ell \in \Gamma^\beta} V_{\alpha(\ell)} \longrightarrow V_\beta$

$(v_1, \ldots, v_k) \longmapsto \sum f_i(v_i)$

Def^n $g_\ell : V_\beta \longrightarrow V_\alpha$ by projection onto the α^{th} factor.

Note: We reversed all the arrows @ β, so $F_\beta^+(V) \in \mathcal{L}(\Gamma, S_\beta \Lambda)$.

Similarly def^n, for α (-)-accessible

$\tilde{h} : V_\alpha \longrightarrow \bigoplus_{\ell \in \Gamma_1} V_{\beta(\ell)} \quad v \longmapsto (f_1(v), \ldots, f_k(v))$

$W_\alpha = \operatorname{Coker} \tilde{h}$.

~~$g_\ell\, g_\ell\, V_\beta$~~ $g_\ell : V_\beta \longrightarrow V_\alpha$

$$\begin{tikzcd} V_\beta \arrow[r, "g_\ell"] \arrow[dr] & V_\alpha \\ & \bigoplus_{\ell \in \Gamma_1} V_{\beta(\ell)} \arrow[u, "/\operatorname{Im} \tilde{h}"'] \end{tikzcd}$$

$Th^{\underline{m}}$ 1.1) ~~Either~~ Let β be (+)-accessible. Either

1) $V = L_\beta$, then $F_\beta^+(L_\beta) = 0$
2) $F_\beta^+(V)$ is indecomposable, $F_\beta^- F_\beta^+(V) \cong V$ and $\dim F_\beta^+(V) = S_\beta(\dim(V))$.

Likewise, let α be (−)-accessible. Either

1) $V = L_\alpha$, $F_\alpha^-(L_\alpha) = 0$
2) $F_\alpha^-(V)$ is indecomposable $F_\alpha^+ F_\alpha^-(V) \cong V$ and $\dim F_\alpha^-(V) = S_\alpha(\dim(V))$.

optional

$F_+^\beta(V) \xrightarrow{\tilde{h}} \bigoplus_{\ell \in \Gamma^\beta} V_{\alpha(\ell)} \xrightarrow{h} V_\beta$, then $\operatorname{Ker} h = \operatorname{Im} \tilde{h}$ by construction.

Thus $F_\beta^- F_\beta^+(V)_\beta = \bigoplus_{\ell \in \Gamma^\beta} V_{\alpha(\ell)} / \operatorname{Im} \tilde{h} = \bigoplus_{\ell \in \Gamma} V_{\alpha(\ell)} / \operatorname{Ker} h \to V_\beta$ defines $i_V^\beta : F_\beta^- F_\beta^+(V) \longrightarrow V$, Injective by $\text{def}^{\underline{n}}$.

Similarly $\text{def}^{\underline{n}}$ p_V^α an epimorphism

$V_\alpha \xrightarrow{\tilde{h}} \bigoplus_{\ell \in \Gamma^\beta} V_{\beta(\ell)} \xrightarrow{h} F_\alpha^-(V)$, then $\operatorname{Im} \tilde{h} = \operatorname{Ker} h$

$V_\alpha \xrightarrow{\tilde{h}} \bigoplus_{\ell \in \Gamma^\beta} V_{\beta(\ell)} / \operatorname{Ker} h = \bigoplus_{\ell \in \Gamma_\beta} V_{b(\ell)} / \operatorname{Im} \tilde{h} = F_\alpha^+ F_\alpha^-(V)$

Lemma 1) $F_\alpha^\pm(V_1 \oplus V_2) \simeq F_\alpha^\pm(V_1) \oplus F_\alpha^\pm(V_2)$

2) p_V^α is epi, i_V^β is mono.
3) If either is iso, dimensions are reflections
4) $\operatorname{Ker} p_V^\alpha$ is concentrated @ α
   $\operatorname{Coker} i_V^\beta$ — " — β
5) $V \simeq F_\alpha^+ F_\alpha^-(V) \oplus \operatorname{Ker} p_V^\alpha$
   $V \simeq F_\beta^- F_\beta^+(V) \oplus \operatorname{Coker} i_\beta^V$

Def$^{\underline{n}}$: A sequence $\alpha_1, ---, \alpha_k$ is (+) accessible w/ respect to Λ if α_1 is (+) accessible for Λ, α_2 is (+) accessible for $\sigma_{\alpha_1}\Lambda$, α_3 is (+)-acc for $\sigma_{\alpha_2}\sigma_{\alpha_1}\Lambda$, etc. Similarly for (-) accessible.

Back to root systems. Let $\alpha_1, ---, \alpha_n$ be an arbitrary ordering of the simple roots. Fix it. Def$^{\underline{n}}$. $c = s_{\alpha_n} - - - - s_{\alpha_1}$ (depends on ordering)

Lemma: Supposing Γ = Dynkin, so B is non-deg.

1) c has no non-zero invariant vectors.
   (Each σ_{α_i} changes only the i^{th} coordinate.
   $\Rightarrow v$ c invariant then v is σ_{α_i} invariant $\forall i$
   $\sigma_{\alpha_i}(v) = v - 2\langle\alpha_i, v\rangle\alpha_i \Rightarrow \langle\alpha_i, v\rangle = 0 \Rightarrow v = 0$ by non-deg.

2) $\forall x$, $\exists i$ s.t. $c^i x \not> 0$.
   Pf: $y = x + cx + c^2x + \cdots + c^{h-1}x$ is invariant, where $h = |c|$. So one must be $\not> 0$.

Back to $\mathcal{L}(\Gamma, \Lambda)$

1) Assume Γ has no oriented cycles (true for A, D, E)
2) Choose an ordering of the vertices of Γ s.t. for any $\ell \in \Gamma_1$ $\alpha(\ell) < \beta(\ell)$.

Def$^{\underline{n}}$ $\varphi^+ = F^+_{\alpha_n} \cdots F^+_{\alpha_1}$, $\varphi^- = F^-_{\alpha_1} \cdots F^-_{\alpha_n}$
Call φ^+, φ^- coxeter functors
(Note $\varphi^+, \varphi^- : \mathcal{L}(\Gamma, \Lambda) \longrightarrow \mathcal{L}(\Gamma, \Lambda)$)

Prop: φ^+, φ^- independent of numbering.

Skippable

First, note that if γ_2, γ_1 are not connected by an edge then $F^{\pm}_{\gamma_1} F^{\pm}_{\gamma_2} = F^{\pm}_{\gamma_2} F^{\pm}_{\gamma_1}$

If $\alpha_1, \ldots, \alpha_n$ and $\alpha_1' \cdots \alpha_n'$ be two suitable numberings. Suppose $\alpha_1 = \alpha_m'$. Then by construction, $\alpha_1', \alpha_2', \ldots, \alpha_{m-1}'$ are not joined to α_1

$$\alpha_1 \rightleftarrows \alpha_i'$$

Thus $F^+_{\alpha_m'} \cdots F^m_{\alpha_1'} = F_{\alpha_{m-1}'} \cdots F_{\alpha_1'} F_{\alpha_1}$

$\vdots$

$F_{\alpha_m} \cdots F_{\alpha_1}$

Finally, The bijection.

By ~~cor~~ prev thm, $c^k(\dim V) \not> 0$, $\Rightarrow$ ~~some~~ Letting

$$\beta_1, \ldots, \beta_{nk} = (\underbrace{\alpha_1, \ldots, \alpha_n, \alpha_1, \ldots, \alpha_n, \alpha_1, \ldots \alpha_n}_{k \text{ times}})$$

Right before it drops of the map, we get

~~$V \approx \sigma$~~

$$V \approx F^-_{\beta_1} \cdot F^-_{\beta_2} \cdots F^-_{\beta_i}(L_{\beta_{i+1}}), \quad \dim V = \sigma_{\beta_1} \cdots \sigma_{\beta_i}(\beta_{i+1})$$

$\Rightarrow \dim V$ is a positive root, V determined by that root.

~~Alternatively~~, Conversely, Let x be a pos root. By lemma, $c^k x \not> 0$. Let i be the last index s.t. $\sigma_{\beta_i} \sigma_{\beta_{i-1}} \cdots \sigma_{\beta_1}(x) > 0$. Then $\sigma_{\beta_i} \cdots \sigma_{\beta_1}(x) = \beta_{i+1}$

Thus by cor 2, $V = F^-_{\beta_1} \cdots F^-_{\beta_i}(L_{\beta_{i+1}})$ is indecomposable and $\dim V = x$.

$Th^{\underline{m}}$: Let Γ be a graph w/o cycles, and let Λ + Λ' be two orientations on it. Then $\exists$ a (+)-accessible sequence (wrt Λ) s.t. $\Lambda \rightsquigarrow \Lambda'$

Pf: Suffices to consider case where Λ, Λ' differ by one edge l. Then $\Gamma \setminus l$ is disconnected. Pick Coxeter sequence on component containing $\beta(e)$. It switches all edges besides l twice, switches l once. QED.

Now let Λ, Λ' be two orientations on Γ, and let $\alpha_1, \ldots, \alpha_k$ be ~~(+) the sequence~~ a (+) accessible seq taking $\Lambda \rightsquigarrow \Lambda'$.

Let $\mathcal{M}$ = set of iso-classes of reps for (Γ, Λ),
$\tilde{\mathcal{M}}$ = set of $F^-_{\alpha_1} \cdots F^-_{\alpha_{i-1}}(L_{\alpha_i})$

$\mathcal{M}'$ = " ——————— " (Γ, Λ')
$\tilde{\mathcal{M}}'$ = set of — $F^+_{\alpha_k} \cdots F^+_{\alpha_{i+1}}(L_{\alpha_i})$

Then $F_{\alpha_k} \cdots F_{\alpha_1}$ gives a one-one correspondence between $\mathcal{M} \setminus \tilde{\mathcal{M}}$ and $\mathcal{M}' \setminus \tilde{\mathcal{M}}'$