k algebraically closed field

Q quiver

I set of vertices

$i \xrightarrow{a} j$ $\quad a =$ arrow $\quad j = h(a) =$ head $\quad i = t(a) =$ tail

Representation of Q of dimension vector $\alpha \in \mathbb{N}^I$

$\alpha = \{\alpha_i\}_i \quad \alpha_i := \dim_k V_i$ where $V_i :=$ vector space attached to the vertex $i \in I$

$$\mathrm{Rep}(Q,\alpha) := \bigoplus_{a \in Q} \mathrm{Mat}(\alpha_{h(a)} \times \alpha_{t(a)}, k)$$

Isomorphism classes of representations

they correspond to the orbits of the group

$$G(\alpha) = \prod_{i \in I} GL(\alpha_i, k) \Big/ k^*$$

acting by conjugation.

If $x \in \mathrm{Rep}(Q,\alpha)$ $\quad x = \{x_a\}_{a \in Q}$ $\quad x_a \in \mathrm{Mat}(\alpha_{h(a)} \times \alpha_{t(a)}, k)$

and $g = \{g_i\} \in G(\alpha)$ $\quad g \cdot x = \{g_{h(a)} x_a g_{t(a)}^{-1}\}$

Double quiver

The double quiver of Q is $\overline{Q}$ s.t.:

- $\overline{Q}$ has same vertices as Q
- the set of arrows of $\overline{Q}$ is obtained by adjoining a reversed arrow $a^*: j \to i$ for any $Q \ni a : i \to j$.

Q
$$\begin{tikzcd} \bullet \arrow[r, "a"] & \bullet \arrow[r, "b"] & \bullet \end{tikzcd}$$

$\overline{Q}$
$$\begin{tikzcd} \bullet \arrow[r, "a"] & \bullet \arrow[r, "b"] \arrow[l, bend left, "a^*"] & \bullet \arrow[l, bend left, "b^*"] \end{tikzcd}$$

What is $Rep(\overline{Q}, \alpha)$?

For any arrow $a \in Q$ consider the trace pairing

$$Mat(\alpha_{h(a)} \times \alpha_{t(a)}, k) \times Mat(\alpha_{t(a)} \times \alpha_{h(a)}, k) \xrightarrow{tr} \mathbb{C}$$

$$(A, B) \longmapsto tr(AB)$$

since $Mat(\alpha_{t(a)} \times \alpha_{h(a)}, k) = Mat(\alpha_{h(a^*)} \times \alpha_{t(a^*)}, k)$ this gives an identification $\left(\bigoplus_{a\in Q} Mat(\alpha_{t(a)} \times \alpha_{h(a)}, k)\right)^* \simeq \bigoplus_{a \in Q} Mat(\alpha_{t(a^*)} \times \alpha_{h(a^*)}, k)$

and $\boxed{T^* Rep(Q, \alpha) \cong Rep(\overline{Q}, \alpha)}$

Moment map

Consider $\mu_\alpha : Rep(\bar{Q}, \alpha) \longrightarrow End(\alpha) = \bigoplus_{i \in I} Mat(\alpha_i; k)$

For $Rep(\bar{Q}, \alpha) \ni X = \{X_b\}_{b \in \bar{Q}}$ $\quad X_b \in Mat(X_{h(b)} \times X_{t(b)}, k)$

$$\mu_\alpha(x)_i := \sum_{\substack{a \in Q \\ h(a)=i}} X_a X_{a^*} - \sum_{\substack{a \in Q \\ t(a)=i}} X_{a^*} X_a$$

Define $End(\alpha)_0 = \{(\theta_i) \mid \sum_{i \in I} tr(\theta_i) = 0\} \subset End(\alpha)$. Then:

$$\mu_\alpha : Rep(\bar{Q}, \alpha) \longrightarrow End(\alpha)_0$$

Why moment map?

$Rep(\bar{Q}, \alpha) = T^* Rep(Q, \alpha)$ is a symplectic manifold.
$G(\alpha)$ acts symplectically on $T^* Rep(Q, \alpha)$ with the action induced by the action on $Rep(Q, \alpha)$
μ_α is a moment map for this action.

In fact:

(I) One can identify $End(\alpha)_0 \simeq (Lie\, G(\alpha))^*$ using trace pairing
Why? $End(\alpha) \times End(\alpha) \xrightarrow{tr} \mathbb{C}$

$$(\theta_i, \psi_i) \xrightarrow{tr} \sum_i tr(\theta_i \psi_i)$$

and $End(\alpha)_0$ is the annihilator of $K \subset End(\alpha)$.

(II) Identifying $Rep(\bar{Q},\alpha)$ with its tangent space at any point the natural symplectic form on the cotangent bundle corresponds to

$$\omega(X,y) = \sum_{a\in Q}\left(tr(X_a y_{a^*}) - tr(X_{a^*} y_a)\right)$$

(III) Now for any $\theta \in End(\alpha)$ define the function $f : Rep(\bar{Q},\alpha) \longrightarrow k$ by

$$\boxed{f(X) = \sum_i tr(\theta_i \cdot \mu_\alpha(X)_i) = \langle \theta, \mu_\alpha(X) \rangle}$$

pairing between $Lie(G(\alpha))$, $Lie(G(\alpha))$

$$\boxed{df_X(y) = \omega([\theta,X],y) \quad \forall\, X,y \in Rep(\bar{Q},\alpha)}$$

to see it compute $\frac{d}{dt} f(X+ty)\Big|_{t=0}$

$$[\theta,X] = \{[\theta,X]_a\}_{a\in\bar{Q}} \quad [\theta,X]_a = \theta_{h(a)} X_a - X_a \theta_{t(a)} \quad \forall a \in \bar{Q}$$

$$\boxed{[\theta,X] = \theta^{\#}_X = \text{vector field generated by the action of } G(\alpha).}$$

$$\frac{d}{dt}\left((\exp t\theta)X\right)\Big|_{t=0} = \frac{d}{dt}\left(\{e^{t\theta_{h(a)}} X_a e^{-t\theta_{t(a)}}\}_a\right)\Big|_{t=0}$$

Thus

$$\left(d_X\langle\theta,\mu_\alpha(-)\rangle\right)(y) = \omega(\theta^{\#}_X, y)$$

as should be for moment maps.

Also satisfies $G(\alpha)$ equivariance $\mu_\alpha(g\cdot X) = Ad^*_g\, \mu_\alpha(X)$.

Want to consider Marsden-Weinstein reductions

$\mu_\alpha^{-1}(\theta) /\!\!/ G(\alpha)$ for some $\theta \in \mathrm{End}(\alpha)_0$ that are fixed points under $G(\alpha)$ action.

What are fixed points?

Are $\{\theta_i\}_{i \in I}$ s.t. $\theta_i = \lambda_i \cdot$ is a scalar matrix $\forall i$:

So are in bijection with $\lambda \in K^I$ s.t. $\sum_{i \in I} \lambda_i \cdot \alpha_i = 0$.

Define $p(\alpha) = 1 + \sum_{a \in Q} \alpha_{t(a)} \alpha_{h(a)} - \alpha \cdot \alpha$ $\qquad \alpha \cdot \alpha = \sum \alpha_i^2$

THEOREM 1 (CRAWLEY-BOEVEY)

If $\alpha \in \mathbb{N}^I$ T.F.A.E:

(1) μ_α is a flat morphism (of schemes)

(2) $\mu_\alpha^{-1}(0)$ has dimension $\alpha \cdot \alpha - 1 + 2p(\alpha)$

(3) $p(\alpha) \geq \sum_{t=1}^{r} p(\beta^{(t)})$ for any decomposition $\alpha = \beta^{(1)} + \beta^{(2)}$ s.t. $\beta^{(t)}$ are positive roots

(4) $p(\alpha) \geq \sum_{t=1}^{r} p(\beta^{(t)})$ for any decomposition $\alpha = \beta^{(1)} + \beta^{(2)}$ $\beta^{(t)}$ non zero in $\mathbb{N}^I$.

Deformed preprojective algebras (Do not depend on orientation)

$$\Pi^\lambda(Q) : \frac{K\overline{Q}}{\left(\sum_{a\in Q}[a,a^*] - \sum_{i\in I}\lambda_i e_i\right)}$$

e_i := idempotent at vertex i
= fixed path at vertex i

same as ideal generated by $\left(\sum_{\substack{a\in Q\\ h(a)=i}} aa^* - \sum_{\substack{a\in Q\\ t(a)=i}} a^*a - \lambda_i e_i\right)_i$.

So if $\lambda\in K^I$ and $\lambda\cdot\alpha=0$ then $\mu_\alpha^{-1}(\lambda)$ is the space of representations of Π^λ of dim vector α.

Π^λ does not depend on orientation (if the orientation of a is reverted then send $a\to a^*$, $a^*\to -a$ & get isomorphism)

Theorem 2

For $\lambda\in K^I$ and $\alpha\in\mathbb{N}^I$ T.F.A.E

(1) There is a simple Π^λ-module of dim. vector α

(2) α is positive root $\lambda\cdot\alpha=0$ and $p(\alpha)>\sum_{t=1}^{r} p(\beta^{(t)})$ for any decomposition $\alpha=\beta^{(1)}+ \;\; +\beta^{(r)}$ $r\geq 2$, $\beta^{(t)}$ positive s.t. $\lambda\cdot\beta^{(t)}=0$ $\forall t$.

In this case $\lambda\cdot\alpha=0$ is a reduced and irreducible complete intersection of dim. $\alpha\cdot\alpha-1+2p(\alpha)$ and the generic element $\mu_\alpha^{-1}(\lambda)$ is a simple module.

Tools

Roots, reflections, Weyl group

Q quiver

$(,)$ symmetric Ringel form

$q(\alpha) = \frac{1}{2}(\alpha, \alpha)$ Tits form

$p(\alpha) = 1 - q(\alpha)$

$\boxed{\text{simple roots}}$

$$\varepsilon_i := (0, \dots, \overset{i}{1}, \dots 0)$$

s.t. i is a loop free vertex

Π = set of simple roots

$\boxed{\text{simple reflections}}$

$$s_i : \mathbb{Z}^I \longrightarrow \mathbb{Z}^I$$

$$\alpha \longrightarrow s_i(\alpha) = \alpha - (\alpha, \varepsilon_i)\varepsilon_i$$

$r_i : \mathbb{K}^I \longrightarrow K^I$

dual reflection s.t.

$r_i \lambda \cdot \alpha = \lambda \cdot s_i \alpha \quad \forall \alpha \in \mathbb{Z}^I$

$\boxed{\text{Weyl group}}$

W generated by s_i

$\boxed{\text{fundamental region}}$

$$F = \{\alpha \in \mathbb{N}^I \mid \alpha \neq 0, \text{ support of } \alpha \text{ is connected}, (\alpha, \varepsilon_i) \leq 0 \ \forall i\}$$

$\boxed{\text{Root system}}$

$$\Delta^{re}(Q) = \bigcup_{w \in W} w(\Pi)$$

orbits of simple ~~roots~~ roots under Weyl group

$$\Delta(Q) = \Delta^{re}(Q) \cup \Delta^{im}(Q)$$

$$\Delta^{im}(Q) = \bigcup_{w \in W} w(F \cup -F)$$

orbits of $\pm$ elements in the fund. region

$\Delta = \Delta_+ \cup \Delta_-$

examples

- if Q Dynkin $\Delta(Q) = \Delta^{re}(Q)$ and roots are the usual ones
- if Q is extended Dynkin (tame)
  $\Delta^{im}(Q) = \mathbb{Z}\delta$ δ = positive minimal root spanning radical of $(,)$

if $\alpha \in \Delta^{re}(Q)$ $(\alpha,\alpha)=1$

if $\alpha \in \Delta^{im}(Q)$ $(\alpha,\alpha) \leq 0$

$\boxed{\text{TOOL 1}}$ $\boxed{\text{Kac's theorem}}$

Kac's theorem for representation of quivers when K is algebraically closed

(Q,Ω) quiver with orientation Ω

a) $\exists$ an indecomposable representation of dimension $\alpha \in \mathbb{Z}_+^n \setminus \{0\}$ iff $\alpha \in \Delta_+(\Gamma)$

b) $\exists!$ indecomposable representation of dimension α iff $\alpha \in \Delta_+^{re}(\Gamma)$

c) if $\alpha \in \Delta_+^{im}(\Gamma)$ then $\max_i \dim Z_i = 1 - \frac{(\alpha,\alpha)}{2} > 0$
where Z_i are irreducible components of $\operatorname{Rep}_{\text{ind}}(Q,\alpha) / G_\alpha$

$\hookrightarrow$ indecomposable are parametrized by a finite union of algebraic varieties

$\boxed{\text{TOOL II}}$ $\boxed{\text{Reflection functors for deformed preprojective algebras}}$

If i is a loop free vertex and $\lambda_i \neq 0$, then there is an equivalence

$$E_i : \Pi^\lambda(Q)\text{-mod} \longrightarrow \Pi^{r_i \lambda}(Q)\text{-mod}$$

acting as s_i on dimension vectors.

<u>RMK</u> this works since in this case we don't have the usual problem for reflection functors. The only positive root sent to negative by s_i is ε_i. But if ε_i is dim. vector for a representation of $\Pi^\lambda(Q)$ then $\lambda_i = 0$ since $\lambda \cdot \varepsilon_i = \lambda_i$!!

<u>DEFINITION</u>

suppose i is a sink. Take a module V of dim vector α

$\to \bullet \leftarrow$

- $V_\oplus = \bigoplus_{\substack{a \in Q \\ h(a)=i}} V_{t(a)}$
- $\begin{cases} \mu_a : V_{t(a)} \hookrightarrow V_\oplus \\ \pi_a : V_\oplus \twoheadrightarrow V_{t(a)} \end{cases}$
- $\pi = \frac{1}{\lambda_i} \sum_{\substack{a \in Q \\ h(a)=i}} V_a \pi_a \qquad \mu = \sum_{\substack{a \in Q \\ h(a)=i}} \mu_a V_{a^*}$

get this from relations of preprojective algebras

$\boxed{\pi\mu = 1_{V_i}} \Rightarrow \mu\pi$ idempotent in $\mathrm{End}(V_\oplus)$

take $V_j' = V_j$ if $j \neq i$ $\quad V_i' = \mathrm{Im}(1-\mu\pi) = \mathrm{Ker}\,\pi$

$V_a' = V_a \quad V_{a^*}' = V_{a^*}$ if $h(a) \neq i$, $\quad V_a' = -\lambda_i(1-\mu\pi)\mu_a \quad V_{a^*}' = \pi_a|_{V_i'}$ if $h(a) = i$

- s_i admissible for (λ, α) if $\lambda_i \neq 0$
- $\sim$ smallest equivalence relation such that $(\lambda, \alpha) \sim (r_i(\lambda), s_i(\alpha))$ when s_i admissible

How are fibers $\mu_\alpha^{-1}(\lambda)$, $\mu_{s_i(\alpha)}^{-1}(r_i(\lambda))$ related?

LEMMA 1

$\exists$ scheme T s.t.

$$\mu_\alpha^{-1}(\lambda) \xleftarrow{f} T \xrightarrow{g} \mu_{s_i(\alpha)}^{-1}(r_i(\lambda))$$

f is a principal $GL(s_i(\alpha)_i, K)$-bundle

g " " , $GL(\alpha_i, K)$-bundle

so fibers have same number of irreducible components and

$$\dim \mu_{s_i(\alpha)}^{-1}(r_i(\lambda)) - s_i(\alpha) \cdot s_i(\alpha) = \dim \mu_\alpha^{-1}(\lambda) - \alpha \cdot \alpha$$

proof

FACT $S = \{ (X\ X^*\ Y\ Y^*) \mid XX^* = v1 \quad YY^* = -v1 \quad X^*X - Y^*Y = v1 \}$

$\subset Mat(n \times (n+m)) \times Mat((n+m) \times n) \times Mat(m, (n+m)) \times Mat((n+m) \times m)$

$$\big\downarrow \pi$$

$X = \{ (X, X^*) \mid XX^* = v \}$ is a principal $GL(m)$-bun

$\subset Mat(n \times (n+m)) \times Mat((n+m) \times n)$

Now suppose i = sink. Take Q' quiver obtained by removing all arrows with head at i.

Take $n = \alpha_i$. $m = s_i(\alpha)_i = -\alpha_i + \sum_{a \mid h(a)=i} \alpha_{t(a)}$

Then

$$\operatorname{Rep}(\bar{Q}, \alpha) = \operatorname{Rep}(\bar{Q}', \alpha) \times \operatorname{Mat}(n \times (n+m)) \times \operatorname{Mat}(n+m, n)$$

$$\ni x = (x', X, X^*)$$

$$\mu_\alpha(x)_i = \sum_a x_a x_{a^*} = X X^*$$

$$\operatorname{Rep}(\bar{Q}, s_i(\alpha)) = \operatorname{Rep}(\bar{Q}', \alpha) \times \operatorname{Mat}(m \times (n+m)) \times \operatorname{Mat}(n+m, m)$$

$$\ni y = (x', Y, Y^*)$$

$$\mu_{s_i(\alpha)}(y)_i = \sum_{a \in H} y_a y_{a^*} = Y Y^*$$

take $v = \lambda_i$ get $GL(s_i(\alpha)_i, K) / GL(\alpha_i, K)$ bundles

$$f' : \operatorname{Rep}(\bar{Q}', \alpha) \times S \longrightarrow \operatorname{Rep}(\bar{Q}', \alpha) \times X = \{x \in \operatorname{Rep}(\bar{Q}, \alpha) \mid \mu_\alpha(x)_i = \lambda_i\}$$

$$g' : \operatorname{Rep}(\bar{Q}', \alpha) \times S \longrightarrow \operatorname{Rep}(\bar{Q}', \alpha) \times \tilde{X} = \{y \in \operatorname{Rep}(\bar{Q}, \alpha) \mid \mu_{s_i(\alpha)}(y)_i = ?\}$$

prove f' g' restrict to a scheme T as in theorem by proving

$$\mu_\alpha(f'(z))_j - \lambda_j = \mu_{s_i(\alpha)}(g'(z))_j - r_i(\lambda)_j$$

in $\operatorname{Mat}(\alpha_j)$ (use $X^*X - Y^*Y = \lambda_i$).

$\boxed{\text{TOOL III}}$ Exact sequence

If $X \in Rep(Q,\alpha)$ $X = \{X_a\}_{a\in Q}$ then there is an exact sequence ($*$ = dual vector space).

$$\star \quad 0 \to Ext^1(X,X)^* \to Rep(Q^{op},\alpha) \xrightarrow{c} End(\alpha) \xrightarrow{\epsilon} End(X)^* \to 0$$

where $c : \underset{Rep(Q^{op},\alpha)}{\{y_{a^*}\}} \longrightarrow \sum_{a\in Q} [X_a\ y_{a^*}]$

$\epsilon : (\theta_i) \longrightarrow (\phi_i) \longrightarrow \sum_i tr(\theta_i \phi_i)$

proof

$\star$ comes from the dual sequence

$$0 \to End(X) \to End(\alpha) \xrightarrow{f} Rep(Q,\alpha) \to Ext^1(X,X) \to 0$$

where f $\{\theta_i\}_i \longrightarrow \left(\theta_{h(a)} X_a - X_a \theta_{t(a)}\right)_a$.

In fact the kernel of f is $End(X)$ ($\{\theta_i\}$ "commutes" with X_a!)

The cokernel has same dimension of $Ext^1(X,X)$ since

$\boxed{\text{Ringel formula}}$ $\dim End(X) - \dim Ext^1(X,X) = \langle\alpha,\alpha\rangle = \dim End(\alpha) - \dim Rep(Q,\alpha)$

Get $\star$ by applying trace pairing

$$\underset{\theta}{Hom(k^a, k^b)} \times \underset{\phi}{Hom(k^b, k^a)} \xrightarrow[tr(\theta\phi)]{} k$$

to identify $Rep(Q^{op},\alpha) \simeq Rep(Q,\alpha)^*$ $\quad End(\alpha) \simeq End(\alpha)^*$

LEMMA 2

If $\lambda \in K^I$ and X is a repr of Q lifting to Π^λ then

$$\sum \lambda_i \operatorname{tr}(\theta_i) = 0 \quad \forall \{\theta\} \in \operatorname{End}(X)$$

pf

since X lifts λ is in the image of c so in the kernel of t

THEOREM 3 Let $\pi : \mu_\alpha^{-1}(\lambda) \longrightarrow \operatorname{Rep}(Q, \alpha)$

If $\lambda \in K^I$ a representation X of Q lifts to a representation of Π^λ iff for any β which is the dimension vector of a direct summand $\lambda \cdot \beta = 0$.

If X lifts then $\pi^{-1}(X) \cong \operatorname{Ext}^1(X, X)^*$

pf

If X lifts and there's a summand with dim. β then take $\theta =$ projection and apply LEMMA 2 above

Suppose w.l.o.g. that X is indecomposable and $\lambda \cdot \alpha = 0$ then since any endomorphism θ of X is the sum of a nilpotent matrix and a scalar one (the semisimple part is scalar by indecomposability) we have $\sum \lambda_i \operatorname{tr}(\theta_i) = 0$

$\qquad\qquad\qquad\qquad\qquad\quad \parallel$

constant $\to c \sum \lambda_i \alpha_i$

DEFINITION

If X is a scheme with an action of an algebraic group G and U is a constructible subset of X which is G-stable we define the NUMBER OF PARAMETERS or MODULARITY of G on U by

$$\dim_G U = \max_d \left(\dim(U \cap X_d) + d - \dim G\right)$$

where X_d is the locally closed subset of X of points with stabilizer of dimension d so with orbits of dimension $\dim G - d$.

THEOREM 4 $\quad \pi : \mu_\alpha^{-1}(\lambda) \longrightarrow \operatorname{Rep}(Q,\alpha)$

If U is a $G(\alpha)$-stable constructible subset of $\operatorname{Rep}(Q,\alpha)$ contained in $\operatorname{Im}\pi$ then

$$\dim \pi^{-1}(U) = \dim_{G(\alpha)} U + \alpha\cdot\alpha - q(\alpha)$$

If U is a $G(\alpha)$ orbit $\pi^{-1}(U)$ is irreducible of dimension $\alpha\cdot\alpha - q(\alpha)$

proof partitioning U we can suppose all representations $x \in U$ have endomorphism ring of same dimension e.

If $x \in U$ then $\pi^{-1}(x) \leftrightarrows \operatorname{Ext}^1(x,x)^*$ so $\dim \pi^{-1}(x) = e - q(\alpha)$

So $\dim \pi^{-1}(U) = \dim U + e - q(\alpha)$.

On the other hand each $G(\alpha)$ orbit on U has dimension $\dim G(\alpha) + 1 - e$ so $\dim_{G(\alpha)} U = \dim U - 1 + e - \dim G(\alpha)$.

and formula follows.

Now suppose $U = G(\alpha) x$. Since $\dim_{G(\alpha)} U = 0$

$\pi^{-1}(U)$ has dimension $\alpha \cdot \alpha - q(\alpha)$. Now observe that $G(\alpha)$ acts on $\mu_\alpha^{-1}(\lambda)$ and it is π equivariant

If $\pi^{-1}(U)$ not irreducible one can find Z_1, Z_2 non empty disjoint $G(\alpha)$-stable open sets.

But $\pi(Z_i) = U$ so $\pi^{-1}(x) \cap Z_i$ are non empty disjoint open subset of $\pi^{-1}(x)$! Impossible since $\pi^{-1}(x) = \mathrm{Ext}^1(x, x)^*$ is irreducible!

SKETCH OF PROOF OF THEOREM 1

SOME MORE PRELIMINARIES

if $\alpha = \beta^{(1)} + \quad + \beta^{(r)}$ $\beta^{(t)}$ positive roots define $I(\beta^{(1)}, \ , \beta^{(r)})$ the subset of $\mathrm{Rep}(Q, \alpha)$ of representations having the corresponding decomposition (into indecomposables).

LEMMA 3

$$\dim_{G(\alpha)} I(\beta^{(1)}, \quad , \beta^{(r)}) = \sum_{t=1}^{r} p(\beta^{(t)})$$

Pf

to prove this you "substitute" $I(\beta^{(1)}, \ , \beta^{(r)})$ with the set $I' \subset \mathrm{Rep}(Q, \beta^{(1)}) \times \quad \times \mathrm{Rep}(Q, \beta^{(r)})$ such that each representation of dim $\beta^{(t)}$ is indecomposable.

Theorem 5

given (λ, α) with $\lambda \cdot \alpha = 0$

$$\dim \mu_\alpha^{-1}(\lambda) = \alpha \cdot \alpha - q(\alpha) + m$$

$$m = \max \left\{ \sum_{t=1}^{r} p(\beta^{(t)}) \right\}_{r \geq 1} \quad \alpha = \beta^{(1)} + \quad + \beta^{(r)}, \ \lambda \cdot \beta^{(t)} = 0.$$

Pf decompose $\mathrm{Rep}(Q, \alpha)$ as a finite union of sets of the form $I(\beta^{(1)}, , \beta^{(r)})$. $\pi^{-1}(I(\beta^{(1)} \ \beta^{(r)}))$ is empty if $\beta^{(t)} \cdot \lambda \neq 0$ for some t otherwise has dimension $\sum_{t=1}^{r} p(\beta^{(t)}) + \alpha \cdot \alpha - q(\alpha) \rightsquigarrow$ result follows taking max!

Canonical decomposition. (Kac)

$\exists!$ decomposition $\alpha = \beta^{(1)} + \dots + \beta^{(r)}$ s.t. $I(\beta^{(1)}, \dots, \beta^{(r)})$ is open dense in $\mathrm{Rep}(Q, \alpha)$

Schur roots (Kac)

A Schur representation $x \in \mathrm{Rep}(Q, \alpha)$ is one such that $\mathrm{End}\, x = K$ (brick).

α is called a Schur root if $\mathrm{Rep}(Q, \alpha)$ contains a Schur representation. In this case the set of Schur representations is a dense open set.

$\alpha = \alpha$ is the canonical decomposition for a Schur root. Vice versa is also true: if $\exists$ dense open subset of $\mathrm{Rep}(Q, \alpha)$ consisting of indecomposables then (i.e. $\alpha = \alpha$ is canonical decomposition) then α is a Schur root.

LEMMA 4 If $\alpha \in \mathbb{N}^I$ has canonical decomposition $\alpha = \beta^{(1)} + \dots + \beta^{(r)}$, $r \geq 2$ then $p(\alpha) < \sum_t p(\beta^{(t)})$

Pf by Kac $(\beta^{(s)}, \beta^{(t)}) > 0$ for $s \neq t$

PROOF OF THEOREM 1 (recall $p(\alpha) = 1 + \sum_{a \in Q} \alpha_{t(a)} \alpha_{h(a)} - \alpha \cdot \alpha$)

$\mu_\alpha : \mathrm{Rep}(\bar{Q}, \alpha) \longrightarrow \mathrm{End}\,\alpha_0$

So the relative dimension of μ_α is:

$$d = 2 \sum_{a \in Q} \alpha_{t(a)} \alpha_{h(a)} - \alpha \cdot \alpha + 1 = 2p(\alpha) - \alpha \cdot \alpha + 1$$

(1)$\Rightarrow$(2) since μ_α is flat its image U is open in $\mathrm{End}(\alpha)_0$.
So we get a surjective flat morphism $\mathrm{Rep}(\bar{Q}, \alpha) \longrightarrow U$.
Since $0 \in U$ $\dim \mu_\alpha^{-1}(0) =$ relative dimension of $\mu_\alpha = d$.

(2)$\Rightarrow$(3) follows from THEOREM 5

(3)$\Rightarrow$(4) if $p(\alpha) < \sum_t p(\beta^{(t)})$ for some decomposition the inequality stays true when we replace each $\beta^{(t)}$ by all terms in its canonical decomposition. But now all terms are positive roots!

(4)$\Rightarrow$(1) if (4) holds then the canonical decomposition can only have 1 term i.e. α is a schur root.
So take $x \in \mathrm{Rep}(Q, \alpha)$ s.t. $\mathrm{End}(x) = k$. In this case the map c in the exact sequence has 1 dimensional cokernel. Since $\mathrm{im}(c)$ is contained in $\mathrm{End}\,\alpha_0$ (which has codimension 1) we have surjectivity i.e. $\mathrm{im}(c) = \mathrm{End}(\alpha)_0$.
So the moment map is surjective (each element of $\mathrm{End}(\alpha)_0$ is the image under μ_α of an element of $\mathrm{Rep}(\bar{Q}, \alpha)$ restricting to x as Q-representation)

Consider the fibers $\mu_\alpha^{-1}(\phi)$ $\phi \in \operatorname{End}(\alpha)_0$ and (not necessarily product of scalar matrices)

consider the projection $\tilde{\pi} : \mu_\alpha^{-1}(\phi) \longrightarrow \operatorname{Rep}(Q,\alpha)$

if U is constructible then

$$\dim \tilde{\pi}^{-1}(U) \leq \underset{\substack{\| \\ p(\alpha) \text{ if take } U = I(\alpha)}}{\dim_{G(\alpha)}} U + \alpha\cdot\alpha - q(\alpha) \qquad (q(\alpha) = 1 - p(\alpha))$$

so by the hypothesis $\mu_\alpha^{-1}(\phi)$ has dimension at most d (use THEOREMS). Clearly it is equidimensional of dimension d (always true $\dim \mu_\alpha^{-1}(\phi) \geq d$ = relative dimension of μ_α)

Now use Lemma in EGA (use that $\operatorname{End}(\alpha)_0$ and $\operatorname{Rep}(\bar{Q},\alpha)$ are regular, the morphism is surjective on fibers have all same dimension equal to relative dimension).

Proof of theorem 2

2) $\Rightarrow$ 1)

It is enough to prove that

<u>theorem</u> 6

if (λ, α) is a pair with $\lambda \in \Sigma_\lambda$ then $\mu_\alpha^{-1}(\lambda)$ is a reduced & irreducible complete intersection of dimension $\alpha \cdot \alpha - 1 + 2p(\alpha)$ and the general element is simple

STEP 1

prove that if $\alpha \in \Sigma_\lambda$ then $\exists$ an equivalent pair $(\lambda', \alpha') \sim (\lambda, \alpha)$ such that α' is a coordinate vector at a loop free vertex or in the fundamental region (1st case if α real, 2nd case if α imaginary)

STEP 2

prove $\mu_\alpha^{-1}(\lambda)$ is irreducible of dimension $\alpha \cdot \alpha - 1 + 2p(\alpha)$ (so it is also complete intersection in this special situation)

Enough to prove it when α is coordinate vector (trivial!) or in fundamental region to do this we just need to prove irreducibility since

- we already proved $\mu_\alpha^{-1}(\lambda)$ is equidimensional of dimension $\alpha\cdot\alpha - 1 + 2p(\alpha)$.

- $\pi : \mu_\alpha^{-1}(\lambda) \to Rep(Q,\alpha)$ we know image of π by EXACT SEQUENCE and we know $\pi^{-1}(x) \simeq Ext^1(x,x)^*$ so it is irreducible.

Write $\mu_\alpha^{-1}(\lambda)$ as a union $\pi^{-1}(I(\beta^{(1)}, \dots, \beta^{(r)}))$ all except $\pi^{-1}(I(\alpha))$ have dimension smaller than d.

- suppose $q(\alpha) < 0$ then $B(\alpha) = \{$representations x in $Rep(Q,\alpha) \mid End(x) = k\}$ is dense in $Rep(Q,\alpha)$. $\pi^{-1}(I(\alpha) \setminus B(\alpha))$ has dimension less than d. Thus enough to prove $\pi^{-1}(B(\alpha))$ irreducible. Consider the map $\pi^{-1}(B(\alpha)) \to B(\alpha)$ is dominant, $\pi^{-1}(B(\alpha))$ is equidimensional, $B(\alpha)$ is irreducible all fibers are irreducible of same dimension (exact sequence)
$\Rightarrow$ $\pi^{-1}(B(\alpha))$ irreducible (just from algebraic geometry)

- if $q(\alpha) = 0$ then Q is extended Dynkin $\alpha = \delta$ or $\alpha = m\delta$.
If $\alpha = \delta$ $B(\alpha)$ is dense $\rightsquigarrow$ do as before

- if $\alpha = m\delta$ then it must be $\lambda \cdot \delta \neq 0$ $\Rightarrow$ k has characteristic p and $\alpha = p\delta$ (otherwise $\exists$ decomposition $m\delta = p\delta + \quad + p\delta$)

Now $\operatorname{Im}\pi \subset$ representations of Q with no summand of dim $n\delta$ $n < p$ $(n\delta \cdot \lambda \neq 0$ in this case!$)$

thus $\operatorname{Im}\pi = I(\alpha) +$ finitely many other orbits (result by Kac)

Now $\pi^{-1}(I(\alpha))$ is obtained from $\mu_\alpha^{-1}(\lambda)$ by removing inverse images of finitely many orbits of dimension strictly smaller than d. So $\pi^{-1}(I(\alpha))$ is equidimensional of dim d and by same argument as above is irreducible. So $\mu_\alpha^{-1}(\lambda)$ is irreducible too.

STEP 3
prove that the set of simple representations in $\mu_\alpha^{-1}(\lambda)$ is open. So since $\mu_\alpha^{-1}(\lambda)$ is irreducible to prove its general element is a simple representation it's enough to prove that there exists 1

STEP 4
prove existence of one irreducible representation case by case. If there is no simple rep since $\mu_\alpha^{-1}(\lambda)$ is irreducible $\exists$ β s.t. generic point has a subrep of dim β (the set of such points is closed).

• if $q(\alpha)<0$ or $\alpha=\delta$ α is a Schur root thus $I(\alpha)$ is dense in $\mathrm{Rep}(Q,\alpha)$ and each $x \in I(\alpha)$ extends to a repr of $\Pi^{\lambda}(Q)$ so it has a subrepr of dimension β. Same is true for Q^{op} $\Rightarrow$ generic $x \in \mathrm{Rep}(Q,\alpha)$ has both a subrepr of dim β and $\alpha-\beta$ $\Rightarrow$ generic x decomposes into 2 representation of dim $\alpha-\beta$ and β
(by result of Schofield)
CONTRADICTION!

• if $\alpha = p\delta$ char $k = p$ $\lambda \cdot \delta = 0$

$\Rightarrow$ any $x \in I(p\delta)$ extends to Π^{λ} $\Rightarrow$ has a subrepr of dimension β and $\alpha-\beta$ $\Rightarrow \beta = n\delta$ $n<p$ but $n\delta \cdot \lambda = n(\delta\cdot\lambda) \neq 0$! So it is impossible.

It's left to prove fiber is reduced.

the generic element of a fiber is simple so it is abelian i.e. $\mathrm{End}\, x = k$. So x has trivial stabilizer for the action of $G(\alpha) = \dfrac{\prod_i G(\alpha_i)}{k^*}$ so it is a smooth point of the fiber of the moment map.

So fiber is generically smooth $\Rightarrow$ generically reduced. since it's complete intersection it is also Cohen Macaulay so it is reduced

1) $\Rightarrow$ 2) harder!! (we know that for $\alpha \in \Sigma_\lambda$ $\exists$ an irreducible repr.)

STEP 1

R^+_λ = positive roots s.t. $\alpha \cdot \lambda = 0$

define $F_\lambda = \{\alpha \in R^+_\lambda \mid (\alpha', \varepsilon_i) \le 0$ for any $(\lambda', \alpha') \sim (\lambda, \alpha)$ and i s.t. $\lambda'_i = 0\}$

STEP 2

prove that if $\exists$ simple repr. of Π^λ of dimension vector α then either (λ, α) equivalent to a pair (λ', α') with α' coordinate vector of a loop-free vertex or $\alpha \in F_\lambda$

2.1

prove if $\alpha \in F_\lambda$ $\exists$ $(\lambda', \alpha') \sim (\lambda, \alpha)$ α' in fundamental region $((\alpha', \varepsilon_i) \le 0$ $\forall i$ also when $\lambda_i \neq 0)$.

2.2

if Π^λ has a simple repr of dim α and $i \in I$ then either $\alpha = \varepsilon_i$, $\lambda_i \neq 0$, $(\alpha, \varepsilon_i) \le 0$.

Pf suppose i sink. $V_\oplus = \bigoplus_{\substack{a \in Q \\ h(a) = i}} V_{t(a)}$ then we

if $\lambda_i = 0$ let V be a simple repr of dim α.

have maps

$$V_i \underset{\phi}{\overset{\theta}{\rightleftarrows}} V_\oplus$$

$\phi\theta = 0$.

if $\ker(\theta) \neq 0$ V has a subrepr W $W_i := \ker(\theta)$ $W_j = 0$ $\forall j \neq i$

if $\mathrm{Im}(\phi) \neq V_i$ V .. ... W $W_i := \mathrm{Im}(\phi)$ $W_j = V_j$ $\forall j \neq i$

so if V simple and $\alpha \neq \varepsilon_i$ θ is injective and ϕ surjective.

Since $\phi\theta = 0$ ϕ induces a surjection

$$V_{\oplus}/\mathrm{Im}(\theta) \rightarrow V_i$$

So $\dim V_{\oplus} \geq 2 \dim V_i \Rightarrow (\alpha\, \varepsilon_i) = 2\dim V_i - \dim V_{\oplus} \leq 0$

2.3

prove if Π^{λ} has simple repr of dim α then $\exists\, (\lambda', \alpha') \sim (\lambda, \alpha)$ with α' coordinate vector on the fundamental region. So it is a root

2.4

if $\nexists\, (\lambda' \alpha') \sim (\lambda, \alpha)$ α' coord vector at loop free vertex then we show that $\alpha \in F_{\lambda}$. If $(\lambda', \alpha') \sim (\lambda, \alpha)$ then $\exists$ simple repr of $\Pi^{\lambda'}$ of dim vector α'. If i is a vertex with $\lambda_i' = 0$ and there is a loop at i $(\alpha', \varepsilon_i) \leq 0$ automatically if there is no loop at i $(\alpha', \varepsilon_i) \leq 0$ by 2.2.

get STEP 2

STEP 3

classify $F_{\lambda} \setminus \Sigma_{\lambda}$ all possibilities (for any Q and any λ)

STEP 4

prove that if $\alpha \in F_{\lambda} \setminus \Sigma_{\lambda}$ it can't be the dimension of a simple representation