Haiman's Conjecture and the Springer Correspondence

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https://mqtrinh.github.io/math/research/haiman_26-04.pdf

1 Combinatorics

Fix a graph with vertex set Γ.

A *(proper) coloring* of Γ is a function

$$\mathbf{c}\colon \Gamma \to \mathbf{Z}_+ \quad \text{s.t. } \mathbf{c}(i) \neq \mathbf{c}(j) \text{ for all edges } i\text{—}j.$$

(Birkhoff 1912) The function

$$\chi_\Gamma(m) = |\{\mathbf{c} \mid \mathbf{c}(\Gamma) \subseteq \{1, 2, \ldots, m\}\}|$$

is a polynomial in m.

(Stanley 1995) The *chromatic symmetric function*

$$X_\Gamma[x_1, x_2, \ldots] = \sum_{\mathbf{c}} x_{\mathbf{c}}, \quad \text{where } x_{\mathbf{c}} = \prod_{i \in \Gamma} x_{\mathbf{c}(i)},$$

also interpolates the values $\chi_\Gamma(m)$.

Henceforth, $\Gamma = \{1, 2, \ldots, n\}$.

(Shareshian–Wachs 2016) A *descent* of $\mathbf{c}$ is

$$\text{an edge } i \text{—} j \quad \text{s.t. } i < j \text{ and } \mathbf{c}(i) > \mathbf{c}(j).$$

Let $\mathrm{des}(\mathbf{c})$ count the descents of $\mathbf{c}$.

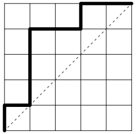
The *chromatic quasisymmetric function*

$$X_\Gamma(q)[x_1, x_2, \ldots] = \sum_{\mathbf{c}} q^{\mathrm{des}(\mathbf{c})} x_{\mathbf{c}}$$

need not be a symmetric function.

Yet it is for certain graphs arising from Dyck paths, or equivalently, Hessenberg sequences.

A 5×5 Dyck path:



Its Hessenberg sequence $\mathbf{m} = (\mathbf{m}_1, \mathbf{m}_2, \mathbf{m}_3, \mathbf{m}_4, \mathbf{m}_5)$:

$$(1, 4, 4, 5, 5).$$

Its *unit interval graph* $\Gamma(\mathbf{m})$:

$$i \text{—} j \text{ is an edge} \iff i < j \leq \mathbf{m}_i.$$

Frobenius characteristic:

$$\begin{aligned} \mathrm{K}_0(S_n) &\xrightarrow{\sim} \{\text{degree-}n \text{ symmetric functions}\} \\ \chi^\lambda &\mapsto \text{Schur } s_\lambda \\ \mathrm{Ind}_{S_\lambda}^{S_n}(1) &\mapsto \text{complete homogeneous } h_\lambda \\ \mathrm{Ind}_{S_\lambda}^{S_n}(\mathsf{sgn}) &\mapsto \text{elementary } e_\lambda \end{aligned}$$

Conj (Stanley–Stembridge 1993)

$$X_{\Gamma(\mathbf{m})}(1) \in \mathbf{Z}_{\geq 0}\langle e_\lambda \mid \lambda \vdash n\rangle.$$

Conj (Shareshian–Wachs 2016) Expand

$$X_{\Gamma(\mathbf{m})}(q) = \sum_{\lambda \vdash n} A_\lambda^{\mathbf{m}}(q) e_\lambda.$$

Then for each λ, the nonzero coefficients of $A_\lambda^{\mathbf{m}}(q)$ are positive and unimodal.

(Shareshian–Wachs 2016) Schur positivity holds = positivity holds with s_λ in place of e_λ.

The coefficients even have a combinatorial meaning, in terms of inversions in Young tableaux.

Example If $\mathbf{m} = (1, 4, 4, 5, 5)$, then

$$\begin{aligned} X_{\Gamma(\mathbf{m})}(q) &= (1 + 3q + 4q^2 + 3q^3 + q^4)\, s_{11111} \\ &\quad + (1+q)^4\, s_{2111} \\ &\quad + (1+q)^2\, s_{221} \\ &\quad + (1+q)^2\, s_{311} \\ &= (1+q)(1+q+q^3)\, e_{41} \\ &\quad + q(1+q)^2\, e_{311}. \end{aligned}$$

(Brosnan–Chow 2017) Schur unimodality holds.

What Brosnan–Chow actually proved was a geometric interpretation of $X_{\Gamma(\mathbf{m})}(q)$.

Let $G = \mathrm{GL}_n$, so that $\mathfrak{g} = \mathfrak{gl}_n$.
Let B be upper-triangular, so that $\mathfrak{b} = \bigoplus_{i \leq j} \mathfrak{g}_{i,j}$.

Fix $g \in G(\mathbf{C})$. Using the *Hessenberg space*

$$\mathfrak{H}_{\mathbf{m}} = \mathfrak{b} \oplus \bigoplus_{i < j \leq \mathbf{m}_i} \mathfrak{g}_{j,i},$$

form the *Hessenberg variety*

$$\mathrm{Hess}_{\mathbf{m},g} = \{xB \in G/B \mid x^{-1}gx \in \mathfrak{H}_{\mathbf{m}}\}.$$

For regular semisimple g, the action of $\pi_1(G^{rs}, g)$ on its cohomology factors through the Weyl group S_n.

Let ω be the involution $h_\lambda \leftrightarrow e_\lambda$.

(Brosnan–Chow 2017) For any $g \in G^{rs}(\mathbf{C})$,

$$\omega X_{\Gamma(\mathbf{m})}(q) = \sum_i q^i \, \mathrm{FrobChar}\left[\mathrm{H}^i(\mathrm{Hess}_{\mathbf{m},g})\right].$$

Schur unimodality (and palindromicity) now follow from hard Lefschetz.

More recently, by pure algebra:

(Hikita, Griffin–Mellit–Romero–Weigl–Wen 2025)

$$X_{\Gamma(\mathbf{m})}(1) \in \mathbf{Z}_{\geq 0}\langle e_\lambda \mid \lambda \vdash n\rangle$$

= the Stanley–Stembridge conjecture is true.

Methods do not extend to $X_{\Gamma(\mathbf{m})}(q)$.

2 Characters

The e-positivity and -unimodality of $X_{\Gamma(\mathbf{m})}(q)$ are subsumed by a more general conjecture.

Let H_n be the Hecke algebra of S_n over $\mathbf{Z}[\mathsf{v}^{\pm 1}]$.

Let $\{C'_w\}_w \subseteq H_n$ be Kazhdan–Lusztig's basis.

$$\chi^\lambda : S_n \to \mathbf{Z} \quad \text{deforms to} \quad \chi^\lambda_{\mathsf{v}} : H_n \to \mathbf{Z}[\mathsf{v}^{\pm 1}].$$

The nonzero coefficients of $\chi^\lambda_{\mathsf{v}}(C'_w)$ are palindromic.

They are also positive and unimodal due to the irreducibility of left cells in S_n.

Conj (Haiman 1993) For any $z \in S_n$, expand

$$\sum_{\lambda \vdash n} \chi^\lambda_{\mathsf{v}}(C'_z) s_\lambda = \sum_{\mu \vdash n} \alpha^z_\mu(\mathsf{v}) h_\mu.$$

Then for each μ, the nonzero coefficients of $\alpha^z_\mu(\mathsf{v})$ are positive and unimodal.

Haiman checked this up through $n = 7$.

Let $s_i = (i, i+1)$ in S_n.

(Clearman–Hyatt–Shelton–Skandera 2016) For any $\mathbf{m}$,

$$\omega X_{\Gamma(\mathbf{m})}(\mathsf{v}^2) = \mathsf{v}^{\ell(z)} \sum_{\lambda \vdash n} \chi^\lambda_{\mathsf{v}}(C'_{z_\mathbf{m}}) s_\lambda$$

where $z_\mathbf{m} = (s_{\mathbf{m}_1 - 1} \cdots s_1) \cdots (s_{\mathbf{m}_{n-1} - 1} \cdots s_{n-1})$.

Shareshian–Wachs	Haiman
$\mathbf{m}$	z
$\omega X_{\Gamma(\mathbf{m})}(\mathsf{v}^2)$	$\sum_\lambda \chi^\lambda_{\mathsf{v}}(C'_z) s_\lambda$

The $z_{\mathbf{m}}$'s are:

- smooth, in the sense of their Schubert varieties.
- precisely the 312-*avoiding permutations*.
- counted by $\frac{(2n)!}{(n+1)!n!} \sim \frac{1}{\sqrt{\pi}} n^{-3/2} 4^n$.

Example If $\mathbf{m} = (1, 4, 4, 5, 5)$, then $z_{\mathbf{m}} = s_3 s_2 s_3 s_4$.

What is the meaning of $\mathrm{Hess}_{\mathbf{m},g}$ and its cohomology in terms of $z_{\mathbf{m}}$?

Keeping $G = \mathrm{GL}_n$, form the *(closed) Lusztig variety*

$$\bar{Y}_{z,g} = \{xB \in G/B \mid x^{-1} g x \in \overline{BzB}\}.$$

For regular semisimple g, monodromy action on cohomology.

For unipotent u, Springer action on $\mathcal{B}_u := \bar{Y}_{e,u}$.

(Abreu–Nigro 2024) For any $g \in G^{rs}(\mathbf{C})$ and $\mathbf{m}$,

$$\bar{Y}_{z_{\mathbf{m}},g} \simeq \mathrm{Hess}_{\mathbf{m},g}.$$

The isomorphism on cohomology is S_n-equivariant.

Lusztig varieties exist for any reductive group G.

How much of the story generalizes beyond GL_n?

Let G be connected reductive with Weyl group W.

Fix a Borel $B \subseteq G$ and element $z \in W$.

(Brosnan–J. Hong–D. Lee 2025) Fix $g \in G(\mathbf{C})$. Using

$$\mathfrak{H}_z = \mathfrak{b} \oplus \bigoplus_{\substack{\alpha \in \Phi^+ \\ s_\alpha \leq z}} \mathfrak{g}_{-\alpha},$$

form $\mathrm{Hess}_{z,g} = \{xB \in G/B \mid x^{-1}gx \in \mathfrak{H}_z\}$.

If z is smooth and g is regular semisimple, then:

- $\mathrm{Hess}_{z,g}$ is smooth.
- There is a flat degeneration from $\bar{Y}_{z,g}$ to $\mathrm{Hess}_{z,g}$.
- The isomorphism on cohomology is W-equivariant.

3 Calculations

How to generalize Haiman's conjecture beyond GL_n?

Haiman's conjecture is about the Laurent polynomials

$$\alpha_\mu^z(\mathsf{v}) \in \mathbf{Z}[\mathsf{v}^{\pm 1}]$$

defined by

$$\sum_{\chi \in \mathrm{Irr}(S_n)} \chi_{\mathsf{v}}(C'_z)\chi = \sum_{\mu \vdash n} \alpha_\mu^z(\mathsf{v}) \operatorname{Ind}_{S_\mu}^{S_n}(1).$$

Problem 1 Beyond S_n, even the positivity of $\chi_{\mathsf{v}}(C'_z)$ for irreducible χ can fail.

Problem 2 Beyond S_n, not enough conjugacy classes of parabolic subgroups to replace the S_μ.

Solution to (1) (T., Brosnan–Hong–Lee–E. Lee 2026)

Replace the left-hand side with the (intersection) cohomology of $\bar{Y}_{z,g}$ for regular semisimple g.

Thm For any $g \in G^{rs}(\mathbf{C})$ and $w \in W$,

$$\sum_i \mathsf{v}^{2i}\,\mathrm{IH}^{2i}(\bar{Y}_{z,g}) = \mathsf{v}^{\ell(z)} \sum_{\chi,\psi} \{\chi,\psi\}\psi_{\mathsf{v}}(c_z)\chi$$

where $\{-,-\}$ is Lusztig's exotic Fourier transform truncated to $\mathrm{Irr}(W)$.

Mostly follows from (multiplicity formula of Lusztig) + (local constancy result of Abreu–Nigro).

Lets us compute tons of examples using CHEVIE.

Solution to (2) (T., Brosnan–Hong–Lee–E. Lee 2026)

Replace the $\mathrm{Ind}_{S_\mu}^{S_n}(\mathbf{1})$ with characters arising from the *Springer correspondence* for G.

Recall that this is an injective map

$$\mathrm{Irr}(W) \to \left\{ (u,\kappa) \;\middle|\; \begin{array}{l} u \in G(\mathbf{C}) \text{ unipotent}, \\ \kappa \in \mathrm{Irr}(\pi_0(G_u)) \end{array} \right\} \Big/ G.$$

If $\psi \in \mathrm{Irr}(W)$ is the character of $\mathrm{H}^{\mathrm{top}}(\mathcal{B}_u)_\kappa$ under the Springer action, then $\psi \mapsto [u,\kappa]$.

In this case, set $spr_\psi = \sum_j \mathrm{H}^j(\mathcal{B}_u)_\kappa$.

Example If $\psi = \chi^\mu \in \mathrm{Irr}(S_n)$, then $spr_\mu = \mathrm{Ind}_{S_\mu}^{S_n}(\mathbf{1})$.

So beyond type A, study the nonzero coefficients of the Laurent polynomials

$$\alpha^z_{\psi,G}(\mathsf{v}) \in \mathbf{Z}[\mathsf{v}^{\pm 1}]$$

defined by

$$\sum_{\chi,\psi} \{\chi,\psi\}\psi_{\mathsf{v}}(c_z)\chi = \sum_{\psi \in \mathrm{Irr}(W)} \alpha^z_{\psi,G}(\mathsf{v}) spr_\psi$$

! Although positivity often holds, it can still fail, starting in type B_2 where there is a single failure.

! Even up to absolute value, unimodality can fail, starting in type C_4.

Example The $\alpha^z_{\psi,G}$ in type G_2.

	e	2	1	21, 12	212	121
$\phi_{1,0}$				(101)	(1020)	(1020)
$\phi'_{1,3}$						
$\phi_{2,1}$				-1	$-(10)$	$-(10)$
$\phi_{2,2}$		(10)		1		(10)
$\phi''_{1,3}$			(10)	1	(10)	
$\phi_{1,6}$	1					

	2121, 1212	21212	12121	$w_\circ$
$\phi_{1,0}$	(10202)	(102020)	(102020)	(1020202)
$\phi'_{1,3}$	1	(10)	(10)	
$\phi_{2,1}$	-1			
$\phi_{2,2}$			$-(10)$	
$\phi''_{1,3}$	1		(10)	
$\phi_{1,6}$				

Above, $(a_m \cdots a_1 a_0) := a_0 + \sum_{i=1}^m a_i(\mathsf{v}^i + \mathsf{v}^{-i})$.

What properties of z or ψ ensure that $\alpha^z_{\psi,G}$ has positive unimodal coefficients?

Induction Conj (T. 2026)

Suppose that $z \in W'$ for some parabolic $W' \subseteq W$, corresponding to a Levi $G' \subseteq G$. Then

$$\text{the } \alpha^z_{\psi,G} \text{ for } \psi \in \mathrm{Irr}(W)$$

are $\mathbf{Z}_{\geq 0}$-linear combinations of

$$\text{the } \alpha^z_{\psi',G'} \text{ for } \psi' \in \mathrm{Irr}(W').$$

Thm (T. 2026)

True when W' is a product of symmetric groups.

Proof

$\mathrm{IH}^{2i}(\bar{Y}_{z,g})$ and $\sum_j \mathrm{H}^j(\mathcal{B}_u)$ enjoy the same compatibility with induction from W' to W.

Inflation Conj (T. 2026)

Suppose that ψ is inflated from a product of symmetric groups.

Then for any $z \in W$, the nonzero coefficients of $\alpha^z_{\psi,G}$ all have the same sign and are unimodal up to sign.

If the inflation morphism does not descend to

$$W(G_2) \to W(A_2) = S_3,$$

then the sign is positive.

I checked this

- up through rank 5,
- in rank 6 for rationally smooth z,
- in rank 7 for rationally smooth z of length ≤ 24.

Example Type F_4.

		unimodal	positive	inflated
$\phi_{1,0}$	F_4			✓
$\phi'_{2,4}$	$F_4(a_1)(11)$			✓
$\phi_{4,1}$	$F_4(a_1)$	×	×	
$\phi''_{2,4}$	$F_4(a_2)(11)$			✓
$\phi_{9,2}$	$F_4(a_2)$	×	×	
$\phi'_{8,3}$	C_3			
$\phi''_{8,3}$	B_3			
$\phi'_{1,12}$	$F_4(a_3)(211)$			✓
$\phi''_{6,6}$	$F_4(a_3)(22)$			
$\phi'_{9,6}$	$F_4(a_3)(31)$		×	
$\phi_{12,4}$	$F_4(a_3)$		×	
$\phi'_{4,7}$	$C_3(a_1)(11)$		×	
$\phi_{16,5}$	$C_3(a_1)$		×	
$\phi'_{6,6}$	$\tilde{A}_2 + A_1$		×	
$\phi_{4,8}$	$B_2(11)$			✓
$\phi''_{9,6}$	B_2		×	
$\phi''_{4,7}$	$A_2 + \tilde{A}_1$		×	
$\phi'_{8,9}$	$\tilde{A}_2$			
$\phi''_{1,12}$	$A_2(11)$			✓
$\phi''_{8,9}$	A_2			
$\phi_{9,10}$	$A_1 + \tilde{A}_1$			
$\phi'_{2,16}$	$\tilde{A}_1(11)$			✓
$\phi_{4,13}$	$\tilde{A}_1$		×	
$\phi''_{2,16}$	A_1			✓
$\phi_{1,24}$	1			✓

Webpage with data and accompanying code: https://mqtrinh.github.io/math/research/code/chevie/

Claude Sonnet 4.6 helped me write the regex scripts.

Thank you for listening.