On the cohomology of simple Shimura varieties at non-quasi split primes

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Joint with Jingren Chi

I will report on joint work in progress with Jingren Chi.

Outline

Cohomology of Shimura varieties in the Langlands program

- A Shimura variety $\mathrm{Sh}_K := \mathrm{Sh}_K(\mathbb{G}, \mathbb{X})$ is a a variety defined in terms of a connected reductive group $\mathbb{G}$ over $\mathbb{Q}$ and a Hermitian symmetric domain $\mathbb{X}$ for $\mathbb{G}(\mathbb{R})$, satisfying certain axioms of Deligne, together with a sufficiently small compact open subgroup $K \subset \mathbb{G}(\mathbb{A}_f)$.
- The tower $\{\mathrm{Sh}_K(\mathbb{G}, \mathbb{X})\}_K$ consists of quasi-projective varieties all defined over a number field $\mathbb{E}$, called the reflex field. *For simplicity below, assume they are* **projective** *over* $\mathbb{E}$.
- Fix any prime number ℓ and ξ an algebraic representation of $\mathbb{G}$ giving ℓ-adic local systems $\mathcal{F}_{\xi,K}$ on Sh_K (e.g. $\mathcal{F}_{\xi,K} = \overline{\mathbb{Q}}_\ell$).
- The ℓ-adic étale cohomology groups

$$H^i_\xi := \varinjlim_K H^i_{\acute{e}t}(\mathrm{Sh}_K \otimes_{\mathbb{E}} \overline{\mathbb{Q}}\,,\, \mathcal{F}_{\xi,K}).$$

  carry actions of $\mathbb{G}(\mathbb{A}_f) \times \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{E})$.
- Overarching Goal: describe **explicitly** the virtual $\mathbb{G}(\mathbb{A}_f) \times \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{E})$-module $H^*_\xi := \sum_i (-1)^i H^i_\xi$.

Relation to global Langlands correspondence

- We can write $H^*_\xi = \bigoplus_{\pi_f} \pi_f \otimes \sigma_\xi(\pi_f)$.
- Here π_f ranges over irreducible admissible representations of $\mathbb{G}(\mathbb{A}_f)$ and $\sigma_\xi(\pi_f)$ is a virtual finite dimensional representation of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{E})$ or, looking locally, of $W_{\mathbb{E}_\mathfrak{p}}$.
- In principle this should give a description of each local factor of the Hasse-Weil zeta function $\zeta_\mathfrak{p}(\mathrm{Sh}_K, s)$ in terms of local factors of automorphic L-functions $L(s, \pi_p, r_\mathfrak{p})$.
- The map $\pi_f \mapsto \sigma_\xi(\pi_f)$ is supposed to reflect aspects of the **global Langlands correspondence** for the group $\mathbb{G}$.
- Expect local-global compatibility: **roughly**, if $\pi_f = \pi^p \otimes \pi_p$ for $p \neq \ell$, then $\pi_p \in \mathrm{Irrep}(\mathbb{G}(\mathbb{Q}_p))$ should belong to the **local L-packet** given by $\pm\sigma_\xi(\pi_f)|_{W_{\mathbb{E}_\mathfrak{p}}}$ where $\mathfrak{p} \in \mathrm{Spec}(\mathcal{O}_\mathbb{E})$ has $\mathfrak{p}|p$.
- First step: understand H^*_ξ as virtual $\mathbb{G}(\mathbb{A}_f) \times W_{\mathbb{E}_\mathfrak{p}}$-module, for the local Weil group $W_{\mathbb{E}_\mathfrak{p}} \subset \mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{E}_\mathfrak{p})$.

Example: Modular curves

- The **modular curves** are attached to $\mathbb{G} = \mathrm{GL}_2$ and $\mathbb{X} = \mathfrak{H}^{\pm}$.
- Here the Sh_K are (finite unions of) open Riemann surfaces.
- If for example $K = K(N) \subset \mathrm{GL}_2(\widehat{\mathbb{Z}})$ are the principal congruence subgroups for $N \geq 3$, then

$$\mathrm{Sh}_{K(N)}(\mathbb{C}) = \coprod_{\varphi(N)} \Gamma(N)\backslash\mathfrak{H}^{+}.$$

- They are moduli spaces of "elliptic curves + full level-N structure".
- $\mathbb{GL}_2(\mathbb{A}_f)$-action is given by classical Hecke operators.
- The study of their H^*_ξ and the associated modularity of the Hasse-Weil zeta functions was essentially completed by Eichler-Shimura in the 1950's.

About the Kottwitz simple Shimura varieties

These have the following properties:

- They are moduli spaces of abelian varieties with PEL structure, initially defined over a number field (sometimes over a ring of integers.)
- As Shimura varieties, they are attached to a "fake" unitary group (more later).
- They are compact (more precisely, projective over the number field of definition).
- They have "no global endoscopy" – a technical result which means that the isogeny classes of their points modulo p have the simplest possible group-theoretic interpretation via Honda-Tate theory – almost as if they were modular curves.

More Shimura data and notation

- $(\mathbb{G}, \mathbb{X})$: $\mathbb{G}$ connected reductive $\mathbb{Q}$-group, $\mathbb{X} = \mathbb{G}(\mathbb{R})/K_\infty$ Hermitian symmetric space; these are required to satisfy the axioms of Deligne.
- In particular, $\mathbb{X} \ni h : \mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathbb{G}_{m,\mathbb{C}} \to \mathbb{G}_\mathbb{R}$ gives rise to a $\mathbb{G}(\mathbb{C})$-conjugacy class $\{\mu\}$ of *minuscule* cocharacters $\mathbb{G}_{m,\mathbb{C}} \to \mathbb{G}_\mathbb{C}$ (the Shimura cocharacters).
- Example: $\mathbb{G} = \mathbb{GL}_2$, $\mathbb{X} = \mathfrak{H}^\pm = \mathbb{C}\backslash\mathbb{R}$, and $\mu(z) = \mathrm{diag}(z, 1)$, i.e., $\mu = (1, 0)$.
- Let $\mathbb{E} \subset \mathbb{C}$ be the field of definition of $\{\mu\}$. Then for $K \subset \mathbb{G}(\mathbb{A}_f)$ ranging over compact open subgroups,

$$\mathrm{Sh}(\mathbb{G}, \mathbb{X})(\mathbb{C}) := \varprojlim_K \mathrm{Sh}_K(\mathbb{G}, \mathbb{X})(\mathbb{C}) := \varprojlim_K \mathbb{G}(\mathbb{Q})\backslash[\mathbb{X} \times \mathbb{G}(\mathbb{A}_f)/K],$$

  is the $\mathbb{C}$-points of inverse limit of quasi-projective varieties over $\mathbb{E}$.
- Abbreviate $\mathrm{Sh}_K := \mathrm{Sh}_K(\mathbb{G}, \mathbb{X})$. Fix primes $p \neq \ell$.

Statement of the problem

- Write $G = \mathbb{G}_{\mathbb{Q}_p}$, $E = \mathbb{E}_{\mathfrak{p}}$, ${}^L G = \widehat{G}(\bar{\mathbb{Q}}_\ell) \rtimes W_{\mathbb{Q}_p}$, and ${}^L G_E = \widehat{G}(\bar{\mathbb{Q}}_\ell) \rtimes W_E$.
- The Shimura cocharacter $\mu : \mathbb{G}_{m, \bar{\mathbb{Q}}_p} \to G_{\bar{\mathbb{Q}}_p}$ gives rise to an algebraic representation $r_{-\mu} : {}^L G \to \mathrm{Aut}(V_{-\mu})$ on the highest weight $\widehat{G}$-rep. $V_{-\mu}$.

Theorem (Target Theorem)

For "nice" unitary Shimura varieties (e.g., simple Kottwitz type, i.e.,compact, no global endoscopy), we have an isomorphism of virtual $\mathbb{G}(\mathbb{A}_f^p) \times K_p^0 \times W_{\mathbb{E}_\mathfrak{p}}$-modules

$$H^*_\xi \cong \sum_{\pi_f = \pi^p \otimes \pi_p} a(\pi_f)\, \pi_f^p \otimes \pi_p \otimes (r_{-\mu} \circ \varphi_{\pi_p}|_{W_{\mathbb{E}_\mathfrak{p}}}) | \cdot |_{\mathbb{E}_\mathfrak{p}}^{-\dim \mathrm{Sh}/2}.$$

for certain "virtual multiplicities" $a(\pi_f) = a_\xi(\pi_f) \in \mathbb{Z}$ defined by Kottwitz.

- Here $K_p^0 \subset G(\mathbb{Q}_p)$ is a compact open subgroup (for us, a parahoric), and $\varphi_{\pi_p} : W_{\mathbb{Q}_p} \to {}^L G$ is the **semisimple** local Langlands parameter attached to π_p (exists in general by Fargues-Scholze).

Consequence for local Hasse-Weil zeta functions

We now take $\mathcal{F}_{\xi,K} = \bar{\mathbb{Q}}_\ell$.

Corollary

In the above situation, let $K \subset \mathbb{G}(\mathbb{A}_f)$ be any sufficiently small compact open subgroup contained in K_p^0. Then the semisimple local Hasse-Weil factor of Sh_K at the place $\mathfrak{p}$ of $\mathbb{E}$ is given by

$$\zeta_{\mathfrak{p}}^{\mathrm{ss}}(\mathrm{Sh}_K, s) = \prod_{\pi_f} L^{\mathrm{ss}}(s - \frac{\dim \mathrm{Sh}_K}{2}, \pi_p, r_{\mathfrak{p}})^{a(\pi_f) \dim \pi_f^K}$$

Here

$$r_{\mathfrak{p}} = \mathrm{Ind}_{\hat{G} \rtimes W_{\mathbb{E}_{\mathfrak{p}}}}^{\hat{G} \rtimes W_{\mathbb{Q}_p}} r_{-\mu}.$$

Our target case: Kottwitz simple Shimura varieties

- The Kottwitz simple Shimura varieties: let $\mathbb{F} \supset \mathbb{F}_0 \supset \mathbb{Q}$ be a CM field, let $(\mathbb{D}, *)$ be a central division algebra over $\mathbb{F}$ of rank n^2, with involution $*$ of 2nd type, let $\mathbb{G} = \mathbb{GU}(\mathbb{D}, *)$.
- We get a "fake" unitary group Shimura variety, which is projective, and has no global endoscopy.
- We do not assume G quasi-split over $\mathbb{Q}_p$.
- We **do** assume p is unramified in $\mathbb{F}_0$ and splits in $\mathbb{F}$.
- Assume $K = K^p \times K_p$.
- It is PEL type. Also, for K_p parahoric $\mathrm{Sh}_{K^pK_p}$ has integral $\mathcal{O}_E$-model the moduli space of abelian schemes with additional structure $(A_\bullet, \lambda, \iota, \eta^p)$. (Rapoport-Zink parahoric integral model).
- The endomorphism structure $\iota : \mathcal{O}_{\mathbb{D}^{\mathrm{op}}} \otimes \mathbb{Z}_{(p)} \to \mathrm{End}(A) \otimes \mathbb{Z}_{(p)}$ must satisfy Kottwitz' determinant condition, and η^p is a K^p-level structure on A.

Some History: Target Theorem proved by

- Kottwitz: in good reduction situation, where $G/\mathbb{Q}_p$ is unramified, $K_p^0 = \mathcal{G}(\mathbb{Z}_p)$ for a hyperspecial maximal parahoric group scheme $\mathcal{G}$. All π_p appearing are unramified.
- Harris-Taylor: in the Harris-Taylor cases. Among other things, essentially $\mathbb{G}_\mathbb{R} = \mathrm{GU}(1, n-1) \times \mathrm{GU}(0, n)^{[\mathbb{F}_0:\mathbb{Q}]-1}$. That is, essentially $\mu = (1, 0^{n-1})$. And $G = \prod_i \mathrm{GL}_{n_i} \times \prod_j D_j^\times$.
- Scholze-Shin: like Kottwitz (any signature at ∞, any μ), but require $G = \mathbb{G}_{\mathbb{Q}_p}$ a product of Weil-restrictions of GL_n's.
- Xu Shen: like Harris-Taylor, essentially $\mu = (1, 0^{n-1})$, but G required to be a certain product of units of division algebras with invariants $\pm 1/n_j$ (p-adic uniformization situation).

Theorem (Chi-H.)

The Target theorem holds in the Scholze-Shin situation, except we may allow G to be any inner form of a product of Weil-restrictions of GL_n's.

- We thus generalize both Scholze-Shin and Shen, but we do not rely on their results (there is some overlap in method of course).

One caveat

- At the moment, Jingren and I cannot identify the π_p-factor as a representation of all of K_p^0, when $\mathbb{G}_{\mathbb{Q}_p}$ is not quasi-split.
- That is, we only understand Hecke operators at p given by functions $h \in C_c^\infty(K_p^0) = C_c^\infty(\mathcal{G}(\mathbb{Z}_p))$ which are **base change transfers** (more later) of elements $\tilde{h} \in C_c^\infty(G(\mathbb{Q}_{p^r}))$ for all sufficiently large r.
- This class includes the unit elements $h = 1_{K_p}$ for any congruence subgroup $K_p \subset K_p^0$.
- Hence, we have enough information to determine local Hasse-Weil zeta functions.

Counting Points

- We fix a parahoric $\mathcal{G}(\mathbb{Z}_p) = K_p^0$, and integral model $\mathcal{S}_{K_p^0} = \mathrm{Sh}_{K^p K_p^0}$ over $\mathcal{O}_E$. For $K_p \subset K_p^0$, we have finite étale morphisms $\pi_{K_p K_p^0} : \mathrm{Sh}_{K^p K_p} \to \mathrm{Sh}_{K^p K_p^0}$.
- From now on, fix sufficiently small $K = K^p K_p$ with $K_p \subset K_p^0$ arbitrarily deep. We drop ξ from the notation.
- Let $f^p \in C_c^\infty(\mathbb{G}(\mathbb{A}_f^p)//K^p)$, $h \in C_c^\infty(K_p^0//K_p)$, $\tau \in \mathrm{Frob}^j I_E$, $r = j[\kappa_E : \mathbb{F}_p]$ (sufficiently large).
- Then Grothendieck-Lefschetz trace formula applied to Frobenius-Hecke correspondence **should** give a formula (for *some* function $\phi_{\tau,h} \in C_c^\infty(G(\mathbb{Q}_{p^r}))$)

$$\mathrm{Tr}(\tau \times hf^p \,|\, H^*) = \sum_{(\gamma_0,\gamma,\delta)} c(\gamma_0,\gamma,\delta)\, O_\gamma(f^p)\, TO_{\delta\sigma}(\phi_{\tau,h}).$$

- Point: RHS resembles the geometric side of the Arthur-Selberg trace formula for $\mathbb{G}$. So after (pseudo)stabilization, this would connect H^* with automorphic representations.

- Much work has shown: if K_p parahoric (and $h = e_{K_p}$), ϕ_τ is determined by $\mathrm{Tr}(\tau | R\Psi^{\mathcal{M}^{\mathrm{loc}}_{K_p}}(\bar{\mathbb{Q}}_\ell))$ (in the **center** of a parahoric Hecke algebra).
- For deep K_p there is no integral model $\mathcal{S}_{K_p}$ and no local model $\mathcal{M}^{\mathrm{loc}}_{K_p}$, but nevertheless we still expect to be able to take a function in the **stable Bernstein center**.

Conjecture (The Test Function Conjecture, H.-Kottwitz)

For any Shimura variety, a formula like the above holds with $\phi_{\tau,h} = Z_{\tau,-\mu,r} \star \tilde{h}$. Here $\tilde{h} \in C_c^\infty(G(\mathbb{Q}_{p^r}))$ is any function with "base-change transfer" $h \in C_c^\infty(G(\mathbb{Q}_p))$.

- Here $Z_{\tau,-\mu,r}$ is an element in the usual Bernstein center $\mathfrak{Z}(G/\mathbb{Q}_{p^r})$ which is the image of an element $Z_{\tau,V_{-\mu,r}}$ in the stable Bernstein center $\mathfrak{Z}^{\mathrm{st}}(G/\mathbb{Q}_{p^r})$.
- "Base change transfer" is like the usual special case of twisted endoscopic transfer, except $G/\mathbb{Q}_p$ is not assumed quasi-split. So not every $\tilde{h}$ has a "base change transfer" h and not every given h comes from a $\tilde{h}$.

(Stable) Bernstein center – motivated by Vogan

- Change notation: F non-arch. local, G connected reductive over F. Choose isom $\mathbb{C} \cong \overline{\mathbb{Q}}_\ell$.
- The Bernstein variety: a variety structure on the set of supercuspidal supports $(M(F), \sigma)_G$. (Act by twisting σ by elements of the torus $X^{\mathrm{un}}(M) = \mathrm{Hom}(M(F)/M(F)^1, \mathbb{C}^\times)$.)
- The Bernstein center is the ring of regular functions $\mathfrak{Z}(G/F)$.
- The stable Bernstein variety: a variety structure on $\widehat{G}$-conjugacy classes of semisimple Langlands parameters $\varphi : W_F \to {}^L G$.
- If $\varphi : W_F \to {}^L G$ **factors minimally** through ${}^L M \subset {}^L G$, we can twist it by any 1-cocycle in the torus $H^1(\mathrm{Frob}_F, Z(\widehat{M})^{I_F})^\circ \cong X^{\mathrm{un}}(M)$.
- Stabilizers are finite, so get infinite union of tori mod finite groups. The **stable Bernstein center** is the ring of regular functions $\mathfrak{Z}^{\mathrm{st}}(G/F)$.

Theorem (Fargues-Scholze)

For every G/F, there is a semisimple local Langlands correspondence $\pi \mapsto \varphi_\pi$, which is compatible with unramified twists, with normalized parabolic induction, and which is suitably functorial. Consequently, there is a natural homomorphism of commutative rings $\mathfrak{Z}^{\mathrm{st}}(G/F) \to \mathfrak{Z}(G/F)$.

- In particular, given $Z \in \mathfrak{Z}^{\mathrm{st}}(G/F)$, we can take its image Z in $\mathfrak{Z}(G/F)$ (thus view Z as a distribution on $G(F)$).
- Given any $(V, r) \in \mathrm{Rep}({}^L G)$, we get $Z_{\tau,V} \in \mathfrak{Z}^{\mathrm{st}}(G/F)$ by

$$Z_{\tau,V}(\varphi) := \mathrm{tr}(\tau \,|\, r \circ \varphi).$$

- **Upshot:** The Test Function Conjecture is now unconditional, thanks to Fargues-Scholze: they provided the missing homomorphism $\mathfrak{Z}^{\mathrm{st}}(G/F) \to \mathfrak{Z}(G/F)$.

- Scholze constructed functions $\phi_{\tau,h}$ which satisfy the point counting formula, using deformations of p-divisible groups.
- This applies to PEL Shimura varieties (Alex Youcis made progress extending this to abelian type Shimura varieties).
- Recall we don't have an integral model at level K_p. Scholze suggested pushing down to the integral model $\mathcal{S}_{K_p^0}$ and studying the nearby cycles of the sheaf $\pi_{K_p^0 K_p,*}(\mathcal{F}_\xi)$ on the generic fiber $\mathrm{Sh}_{K_p^0}$.
- But he still needed a purely local construction of the test function $\phi_{\tau,h}$, which he defined as follows.
- Setting: $\kappa \supset \kappa_E$ perfect field of characteristic p, endowed with $\mathcal{O}_E \to \kappa$. Let $H_\bullet$ chain of p-divisible groups over κ. (Really, consider those with (P)EL structure: action of $\mathcal{O}_B$, Kottwitz determinant condition, etc.).

Definition

The deformation space $\mathcal{X}_{H_\bullet}$ of $H_\bullet$ is the functor that associates to any Artin local $\mathcal{O}_E$-algebra R with residue field κ the set of isomorphism classes of p-divisible groups (with (P)EL)-structure) $\tilde{H}_\bullet$ over $\mathrm{Spec}(R)$, together with an isomorphism $\tilde{H}_\bullet \otimes_R \kappa \xrightarrow{\sim} H_\bullet$, i.e. an isomorphism of p-divisible $\mathcal{O}_B$-modules $\tilde{H}_\bullet \otimes_R \kappa \xrightarrow{\sim} H_\bullet$.

- Scholze proved: The functor $\mathcal{X}_{H_\bullet}$ is represented by a complete Noetherian local $\mathcal{O}_E$-algebra $R_{H_\bullet}$ with residue field κ.
- Let $X_{H_\bullet}$ denote the Raynaud generic fiber of the formal scheme $\mathrm{Spf}(R_{H_\bullet})$. This is a rigid analytic space over $k := W_{\mathcal{O}_E}(\kappa)[\frac{1}{p}]$.
- If chain $H_\bullet$ corresponds to parahoric $\mathcal{G}(\mathbb{Z}_p)$, any $K \subset \mathcal{G}(\mathbb{Z}_p)$ gives étale cover $\pi_K : X_{H_\bullet,K} \to X_{H_\bullet}$ (parametrize level-K structures on universal p-adic Tate module).
- Rational covariant Dieudonné module $M_\bullet \otimes \mathbb{Q}$ with σ-linear Frobenius F can be integrally rigidified:

$$(M_\bullet, F) \cong (\Lambda_\bullet, p\delta\sigma),$$

  for $\delta \in G(W_{\mathcal{O}_E}(\kappa)[\frac{1}{p}])$, well-defined up to $G(W_{\mathcal{O}_E}(\kappa))$-$\sigma$-conjugacy.
- The association $H_\bullet \mapsto \delta \in G(W_{\mathcal{O}_E}(\kappa)[\frac{1}{p}])/\sim$ is an injection.

End of Scholze construction

Definition

Let $\delta \in G(\mathbb{Q}_{p^r})$. If δ comes from an $H_\bullet$ over $\kappa = \mathbb{F}_{p^r}$ with controlled cohomology, then set

$$\phi_{\tau,h}(\delta) = \mathrm{tr}(\tau \times h \,|\, H^*(X_{H_\bullet,K} \otimes_k \hat{\bar{k}}, \mathbb{Q}_\ell)),$$

where K is any normal pro-p open compact subgroup such that h is K-biinvariant. If δ does not arise this way, set $\phi_{\tau,h}(\delta) = 0$.

- Scholze: the function $\phi_{\tau,h}$ is locally constant $\mathbb{Q}$-valued, compactly supported function on $G(\mathbb{Q}_{p^r})$. It is $\mathcal{G}(\mathbb{Z}_{p^r})$-$\sigma$-conjugacy invariant. It is independent of $\ell \neq p$.
- His main result: This function $\phi_{\tau,h}$ satisfies the point counting formula (PEL type cases):

$$\mathrm{tr}(\tau \times hf^p \,|\, H^*) = \sum_{(\gamma_0;\gamma,\delta)} c(\gamma_0,\gamma,\delta)\, O_\gamma(f^p)\, TO_{\delta\sigma}(\phi_{\tau,h}).$$

Why controlled cohomology?

This just means the cohomology of the rigid analytic space is the same as that of some quasi-compact admissible open subset, where finiteness theorems are known.

Local to global geometry in Scholze's method

- Let $\pi : Sh_{K_p K^p} \to Sh_{K_\mathcal{L} K^p}$ be the natural projection to the parahoric level $K_\mathcal{L}$, which has an integral model $\mathcal{S}_{K_\mathcal{L} K^p}$ over $\mathcal{O}_E$.
- By the proper base change theorem,

$$H^*(Sh_{K_p K^p} \otimes_\mathbb{E} \overline{\mathbb{E}}, \mathcal{F}_\xi) \cong H^*(Sh_{K_\mathcal{L} K^p} \otimes_\mathbb{E} \overline{\mathbb{E}}, \pi_* \mathcal{F}_\xi)$$

is

$$H^*(\mathcal{S}_{K_\mathcal{L} K^p} \otimes_{\kappa_E} \overline{\mathbb{F}}_p, R\psi \pi_* \mathcal{F}_\xi).$$

- Say $f^p = 1_{K^p g^p K^p}$. Then

$$\mathrm{tr}(\tau \times hf^p | H^*_\xi) = \sum_{x \in \mathrm{Fix}_{j,\mathcal{L}}(g^p)} \mathrm{tr}(\tau \times hf^p | (R\psi \pi_* \mathcal{F}_\xi)_x).$$

- Now use "Serre-Tate": deforming an abelian variety is the same as deforming its p-divisible group.

Wrinkle

$$\mathrm{Tr}(\tau \times hf^p \,|\, H^*) = \sum_{(\gamma_0;\gamma,\delta)} c(\gamma_0, \gamma, \delta)\, O_\gamma(f^p)\, TO_{\delta\sigma}(\phi_{\tau,h}).$$

- Wrinkle: Scholze requires $G/\mathbb{Q}_p$ *unramified*, in particular, quasi-split.
- In general, there is an obstruction to constructing the triple $(\gamma_0; \gamma, \delta)$ from a Frobenius-Hecke fixed point: if G is **not quasi-split**, then there is no reason that $N_r(\delta) := \delta\sigma(\delta)\cdots\sigma^{r-1}(\delta)$ should be stably conjugate to an element in $G(\mathbb{Q}_p)$.
- This is the main source of difficulty we encounter for non-quasi-split groups.
- To make things work, we need to know the **vanishing property**: $TO_{\delta\sigma}(\phi_{\tau,h}) = 0$ if $N_r(\delta)$ is not stably conjuate to $G(\mathbb{Q}_p)$.
- Also needed for spectral reasons: How is Scholze's $\phi_{\tau,h}$ related to the conjectural test functions $Z_{\tau,-\mu,r} \star \tilde{h}$?

Relation between test functions

- How is Scholze's $\phi_{\tau,h}$ related to the conjectural test function $Z_{\tau,-\mu,r} \star \tilde{h}$ and their transfers to $G^*(\mathbb{Q}_p)$, where G^* is the quasi-split inner form of $G/\mathbb{Q}_p$?

Theorem (Chi-H.)

In the various situations of the Target Theorem, the two test functions essentially agree, in the sense that for any $\delta \in G(\mathbb{Q}_{p^r})$ with $N_r(\delta)$ semisimple, we have $TO_{\delta\sigma}(\phi_{\tau,h}) = TO_{\delta\sigma}(Z_{\tau,-\mu,r} \star \tilde{h})$.

- Thus, in light of Scholze point-counting formula, the Test Function Conjecture holds here.

- Our main local results are the following.

Theorem (Chi-H.)

In the situation of the Target Theorem,

(1) *The Scholze function $\phi_{\tau,h}$ satisfies the vanishing property. Thus it has a base-change to a function $b(\phi_{\tau,h}) \in C_c^\infty(G(\mathbb{Q}_p))$.*

(2) *If h is the base-change transfer of a function $\tilde{h} \in C_c^\infty(G(\mathbb{Q}_{p^r}))$, then we have $b(\phi_{\tau,h}) = Z_{\tau,-\mu,1} \star h$.*

- The unusual notion of base-change in (1) appears because we need to use **pseudo-stabilization** to rewrite the point-counting formula in terms of automorphic representations for $\mathbb{G}$.
- Global method: (1) is proved first: by embedding the local situation into that attached to a certain global inner form $\mathbb{G}''$ with $\mathbb{G}_{\mathbb{Q}_p} = \mathbb{G}''_{\mathbb{Q}_p}$. Using Harris-Taylor and Fargues methods, we can arrange

$$\mathrm{tr}(\tau \times hf^p | H^*_{\mathbb{G}''}) = \sum_\pi m(\pi)\, \mathrm{tr}(f_{\tau,h} f^p f_\infty^{\mathbb{G}''} \,|\, \pi)$$

where $f_{\tau,h} := Z_{\tau,-\mu,1} \star h$ and the sum ranges over automorphic representations of $\mathbb{G}''$.

- The stable Bernstein center at p does not change when $\mathbb{G}''$ is replaced by a global inner form $\mathbb{G}'$ which is isomorphic to $\mathbb{G}''$ outside p, ∞, but which is **quasi-split** at p.
- For a well-chosen $\mathbb{G}'$, we can count points using *generalized Kottwitz triples* $(\gamma_0'; \gamma, \delta)$ attached to $\mathbb{G}'$. Recall ${\mathbb{G}'}_{\mathbb{Q}_p}$ is quasi-split by assumption. We get

$$\mathrm{tr}(\tau \times hf^p \,|\, H^*_{\mathbb{G}''}) = \sum_{(\gamma_0';\gamma,\delta)} c(\gamma_0', \gamma, \delta)\, O_\gamma(f^p)\, TO_{\delta\sigma}(\phi_{\tau,h})$$

where $(\gamma_0'; \gamma, \delta)$ ranges over generalized Kottwitz triples such that $\gamma_0' \in \mathbb{G}'(\mathbb{Q})$ is transfered from $\delta \in \mathbb{G}(\mathbb{Q}_{p^r}) = G(\mathbb{Q}_{p^r})$.

- The RHS is $\sum_{\pi'} m(\pi')\mathrm{tr}\,\pi'(f^*_{\tau,h} f^p f^{\mathbb{G}'}_\infty)$, where the sum is over automorphic representations π' **of $\mathbb{G}'$(!)** and $f^*_{\tau,h} \in \mathbb{G}'(\mathbb{Q}_p)$ is the twisted endoscopic transfer of $\phi_{\tau,h} \in \mathbb{G}(\mathbb{Q}_{p^r})$.

- Then a comparison of global trace formulas for $\mathbb{G}'$ vs $\mathbb{G}''$ (namely global Jacquet-Langlands) allows us to prove that $z_{\tau,-\mu} * h$ is the Langlands-Jacquet transfer of $f^*_{\tau,h}$.
- We can then deduce (1,2) for the local data (G, μ).
- The Target Theorem follows from the local results, using standard pseudo-stabilization techniques of Kottwitz.
- One more interesting local ingredient, proved by a global method.

Twisted local Jacquet-Langlands transfer

- Notation: F p-adic field, F_r/F unramified extension of degree r, $\mathrm{Gal}(F_r/F) = \langle \sigma \rangle$.
- G^* a product of Weil restrictions of general linear groups over finite extensions of F, G any F-inner form of G^*.
- $G^*_r = \mathrm{Res}_{F_r/F} G^*_{F_r}$ with F-automorphism σ^*; Similarly (G_r, σ).
- A key result is: if $\phi \in \mathcal{H}(G_r)$ has the vanishing property for its $TO_{\delta\sigma}$, and $Z_r \in \mathcal{Z}(G_r)$, then $Z_r * \phi$ has the vanishing property.

The main tool in proving the above fact about the vanishing property is the following result.

Theorem

Let π^* be an irreducible tempered representation of G^* and let Π^* be the base change lift of π^*, which is by definition a σ-stable representation of G^*_r with canonical intertwining operator I^*_σ, see Arthur-Clozel.

1. *If Π^* has a Langlands-Jacquet transfer to an irreducible tempered representation Π of G_r, then Π is σ-stable and we can choose an intertwining operator I_σ on Π such that for any $\phi \in C_c^\infty(G_r)$ with stable base change transfer $\phi^* \in C_c^\infty(G^*)$ we have*

$$\mathrm{Tr}(\phi I_\sigma | \Pi) = e(\mathbf{G}_r)\mathrm{Tr}(\phi^* | \pi^*).$$

2. *If Π^* does not have a Langlands-Jacquet transfer to G_r, then for any $\phi \in C_c^\infty(G_r)$ with stable base change transfer $\phi^* \in C_c^\infty(G^*)$ we have*

$$\mathrm{Tr}(\phi^* | \pi^*) = 0.$$

The above gives rise to the following criterion for the vanishing property.

Proposition

Let $\phi \in C_c^\infty(G(F_r))$ and let $\phi^* \in C_c^\infty(G^*(F))$ be its stable base change transfer. Then ϕ has the vanishing property if and only if $\mathrm{Tr}(\phi^*|\pi^*) = 0$ for any irreducible tempered representation π^* of $G^*(F)$ that does not have a Langlands-Jacquet transfer to $G(F)$.