Symmetric subgroup schemes, Frobenius splittings, and quantum symmetric pairs.

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(Joint work with Jinfeng Song (NUS)).

§1 Quantum groups.

§2 Quantum symmetric pairs.

§1. Quantum groups.

§1.1. Chevalley group schemes.

§1.2. Quantum groups at $\sqrt[p]{1}$.

§1.3. Frobenius splitting of reductive groups.

§1.3 Frobenius splitting of flag varieties.

§1.1. Chevalley group schemes.

- $G_{\mathbb{C}}$ connected reductive group $/\mathbb{C}$. (or any $k = \bar{k}$).
- $T \subset B$, $X = \mathrm{Hom}(T, \mathbb{C}^*)$, $Y = \mathrm{Hom}(\mathbb{C}^*, T)$.
- I : the set of simple roots.

Example: $SL_2(\mathbb{C})$, $B = \begin{bmatrix} * & * \\ 0 & * \end{bmatrix}$, $T = \begin{bmatrix} * & 0 \\ 0 & * \end{bmatrix}$.

- Chevalley (1955) constructed a Chevalley group scheme $G_{\mathbb{Z}}$ associated to the root datum.

- Kostant's $\mathbb{Z}$-form (1966) $\rightsquigarrow$ another construction of $G_{\mathbb{Z}}$.

- Lusztig (2009) gives another construction of $G_{\mathbb{Z}}$ using his theory of canonical bases.

Example: $SL_{2,\mathbb{Z}} \cong \operatorname{Spec}\left(\mathbb{Z}[x_{11}, x_{12}, x_{21}, x_{22}] \big/ \det = 1\right)$.

we will construct this

$SL_{2,\mathbb{Z}}$: Rings $\longrightarrow$ Groups.

$R \longmapsto SL_2(R) = \operatorname{Hom}\left(\mathbb{Z}[x_{11}, x_{12}, x_{21}, x_{22}] \big/ \det=1, R\right)$

we will construct this

$G_{\mathbb{C}} \rightsquigarrow$ root datum $(X, Y, I, \langle , \rangle, \cdots) \rightsquigarrow$ quantum groups.

$$U_q := \mathbb{Q}(q) \langle E_i, F_i, K_\mu \rangle / \sim \qquad i \in I, \ \mu \in Y.$$

- U_q is a non-cocommutative Hopf alg.

Examples:

- $G = SL_2(\mathbb{C})$.

$U = \langle E, F, K_n \rangle \quad n \in \mathbb{Z}$.

$K_n \cdot K_m = K_{n+m}, \quad K_n \cdot E = q^{2n} E K_n,$

$EF - FE = \dfrac{K_1 - K_{-1}}{q - q^{-1}}. \quad K_n F = q^{-2n} F K_n.$

- $G = PGL_2(\mathbb{C})$

$U = \langle E, F, K_n \rangle \quad n \in \mathbb{Z}$.

$K_n E = q^n E K_n. \quad \cdots$

$\dot{U}_q$: the modified form of U_q ($1 = \sum_{\lambda \in X} 1_\lambda \in \dot{U}_q$).

$\dot{B}$: the canonical basis on $\dot{U}_q$.

($\dot{U}_q \longrightarrow L(\lambda) \otimes^w L(\mu)$, $u \longmapsto u(\eta \otimes \xi)$, $\dot{B} \longmapsto$ canonical basis).

$_A\dot{U}_q$: A-subalg generated by $E_i^{(n)} 1_\lambda$, $F_i^{(n)} 1_\lambda$,

= the free A-subalg spanned by $\dot{B}$ ($A = \mathbb{Z}[q, q^{-1}]$).

$_R\dot{U}_q = R \otimes_A {_A\dot{U}_q}$ for any $A \to R$.

$q \mapsto ?$

$\dot{U}_q = \mathbb{Q}(q) \dot{U}_q$

universal enveloping algebra

kostant's $\mathbb{Z}$-form

......

Let $A \to \mathbb{Z}$, $q \mapsto 1$. Consider ${}_{\mathbb{Z}}\dot{U}_q \supset \dot{B}$

Def: $\mathcal{O}_{\mathbb{Z}} \subset \mathrm{Hom}_{\mathbb{Z}}({}_{\mathbb{Z}}\dot{U}_q, \mathbb{Z})$ is spanned by dual CB. that is,

$b^* \, (b \in \dot{B})$ such that $b^*(b') = \delta_{b,b'}$.

Thm (Lusztig 2009)

- $\mathcal{O}_{\mathbb{Z}}$ is a commutative Hopf alg. and an integral domain.
- $\mathrm{Spec}\, \mathcal{O}_{\mathbb{Z}} \cong G_{\mathbb{Z}}$, $\mathcal{O}_k$ is the coordinate ring of G_k $(k = \bar{k})$

group like element.

- $G_{\mathbb{Z}}(R) = \mathrm{Hom}_{\mathbb{Z}\text{-alg}}(\mathcal{O}_{\mathbb{Z}}, R) \longrightarrow {}_R\hat{\dot{U}}_q$, $\phi \longmapsto \sum_{b \in \dot{B}} \phi(b^*)\, b$

Remarks:

- $$\begin{tikzcd} \mathrm{Hom}_{\mathbb{Z}}({}_{\mathbb{Z}}\dot{\mathcal{U}}_{\mathfrak{z}}, \mathbb{Z}) \times \mathrm{Hom}_{\mathbb{Z}}({}_{\mathbb{Z}}\dot{\mathcal{U}}_{\mathfrak{z}}, \mathbb{Z}) \arrow[r] & \mathrm{Hom}_{\mathbb{Z}}({}_{\mathbb{Z}}\dot{\mathcal{U}}_{\mathfrak{z}}, \mathbb{Z}) \\ \mathcal{O}_{\mathbb{Z}} \times \mathcal{O}_{\mathbb{Z}} \arrow[u, hook] \arrow[r, "\star"] & \mathcal{O}_{\mathbb{Z}} \arrow[u, hook] \end{tikzcd}$$

dual to the stability condition of CB of $\dot{\mathcal{U}}_{\mathfrak{z}}$

§1.2. Quantum groups at $\sqrt[p]{1}$.

Let p be a prime. ($\neq 2,3$), $\xi = \sqrt[p]{1}$, $\Bbbk = \bar{\Bbbk}$ of char p.

$$\dot{U}_\xi = \mathbb{Z}[\xi] \otimes_{\mathcal{A}} {}_{\mathcal{A}}\dot{U}_q, \quad q \longmapsto \xi.$$

$$\dot{U}_1 = \mathbb{Z}[\xi] \otimes_{\mathcal{A}} {}_{\mathcal{A}}\dot{U}_q, \quad q \longmapsto 1.$$

☆ $\dot{U}_{\Bbbk} = \Bbbk \otimes_{\mathbb{Z}[\xi]} \dot{U}_\xi = \Bbbk \otimes_{\mathbb{Z}[\xi]} \dot{U}_1$ for any field $\Bbbk$ of char p.

($\sqrt[p]{1} = 1$, $x^p - 1 = (x-1)^p$).

Thm (Lusztig 1990). There is a quantum Frobenius morphism

$$\mathrm{Fr}: \dot{U}_{\xi} \longrightarrow \dot{U}_1, \qquad E_i^{(n)} 1_\lambda \longmapsto E^{(n/p)} 1_{\lambda/p} \quad \text{or } 0.$$

$$F_i^{(n)} 1_\lambda \longmapsto F^{(n/p)} 1_{\lambda/p}.$$

Moreover

$$\begin{tikzcd}
\dot{U}_{\xi} \arrow[r, "\mathrm{Fr}"] \arrow[d, "\Delta"] \arrow[dr, phantom, "\circlearrowright"] & \dot{U}_1 \arrow[d, "\Delta"] \\
\dot{U}_{\xi} \hat{\otimes} \dot{U}_{\xi} \arrow[r, "\mathrm{Fr} \otimes \mathrm{Fr}"] & \dot{U}_1 \hat{\otimes} \dot{U}_1
\end{tikzcd}$$

$\boxed{\text{Thm (Lusztig 1994)}}$ There is a quantum Frobenius <u>splitting</u> (alg hom)

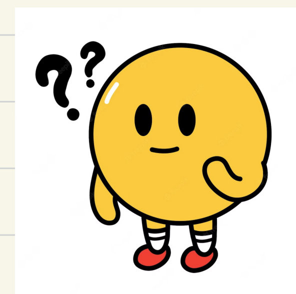
$$Fr' : \dot{U}_1 \longrightarrow \dot{U}_{\varepsilon}, \qquad E^{(n)} 1_\lambda \longmapsto E^{(pn)} 1_{p\lambda}.$$

$$F^{(n)} 1_\lambda \longmapsto F^{(pn)} 1_{p\lambda}.$$

such that. $Fr \circ Fr' = 1$.

Moreover ?? .

$$\begin{tikzcd}
\dot{U}_1 \arrow[r, "Fr'"] \arrow[d, "\Delta"] \arrow[dr, phantom, "\times"] & \dot{U}_{\varepsilon} \arrow[d, "\Delta"] \\
\dot{U}_1 \hat{\otimes} \dot{U}_1 \arrow[r, "Fr' \otimes Fr'"] & \dot{U}_{\varepsilon} \hat{\otimes} \dot{U}_{\varepsilon}
\end{tikzcd}$$



§1.3. Frobenius splitting.

Def: Let X/k be a scheme. we define the absolute Frobenius

$F: X \to X$.
- identity on the underlying space X.
- $F^\#: \mathcal{O}_X \longrightarrow F_*\mathcal{O}_X$, $f \mapsto f^p$.

we say X is Frobenius split if there exits

$\varphi \in \mathrm{Hom}_{\mathcal{O}_X}(F_*\mathcal{O}_X, \mathcal{O}_X)$ such that. $\varphi \circ F^\# = \mathrm{id}$.

$f \cdot x = f^p x$,

- $\varphi(f^p g) = f\varphi(g)$.
- $\varphi(1) = 1$.

There is also a notion of compatily split subscheme.

Example: $\mathbb{A}^1_k$ for alg closed k of char p.

twisted k-linear structure

$F^\#: k[x] \longrightarrow k[x]$, $\quad f \longmapsto f^p$.

$\varphi: k[x] \longrightarrow k[x]$. $\quad a x^t \longmapsto a^{1/p} x^{t/p}$ if $p \mid t$.

$= a \cdot x^{t/p}$

$\left.\begin{array}{l} \varphi(f^p g) = f\,\varphi(g). \ (k[x]\text{-linear}) \\ \varphi(1) = 1. \end{array}\right\} \Rightarrow \varphi$ is a splitting.

- $\varphi: (x) \to (x) \Rightarrow$ compatibly split the closed reduced point 0.

  Ideal sheaf of 0.

- Frobenius splitting was introduced by Mehta-Ramanathan 1985
Ramanan-Ramanathan 1985.

- Original motivation & application (even for char 0)

  Study the geometry of Schubert varieties.

  (I will recall some results later).

Back to quantum groups.

$$\dot{U}_1 \xrightarrow{Fr'} \dot{U}_q \xrightarrow{Fr} \dot{U}_1 \Rightarrow \dot{U}_k \xrightarrow{Fr'} \dot{U}_k \xrightarrow{Fr} \dot{U}_{\Bbbk}$$ ($\Bbbk = \bar{\Bbbk}$ of char p.)

$$\Rightarrow O_k \xrightarrow{Fr^*} O_{\Bbbk} \xrightarrow{Fr'^*} O_k.$$ in $\mathrm{Hom}_{\Bbbk}(\dot{U}_k, \Bbbk)$.

"Thm (B-Song 2023)"

- $O_k \subset \mathrm{Hom}_{\Bbbk}(\dot{U}_k, \Bbbk)$ is stable under Fr^* / Fr'^*.

- $Fr^* = p$-th power map.

- Fr'^* is a Frobenius split of $G_{\Bbbk}$. (or O_k).

$\boxed{\text{Thm (Lusztig 1994)}}$ There is a quantum Frobenius splitting

(alg hom) $Fr' : \dot{U}_1 \to \dot{U}_{\varepsilon}$, $\quad E^{(n)} 1_\lambda \mapsto E^{(pn)} 1_{p\lambda}$.

$\qquad\qquad\qquad\qquad\qquad F^{(n)} 1_\lambda \mapsto F^{(pn)} 1_{p\lambda}$.

such that. $Fr \circ Fr' = 1$.

The quantum Frobenius splitting

Moreover.

$$\begin{tikzcd}
\dot{U}_1 \arrow[r, "Fr'"] \arrow[d, "\Delta"] & \dot{U}_{\varepsilon} \arrow[r, "\Delta"] & \dot{U}_{\varepsilon} \hat{\otimes} \dot{U}_{\varepsilon} \arrow[d, "id \otimes Fr"] \\
\dot{U}_1 \hat{\otimes} \dot{U}_1 \arrow[rr, "Fr' \otimes id"] & & \dot{U}_{\varepsilon} \hat{\otimes} \dot{U}_1
\end{tikzcd}$$

Dual: $\quad Fr'^{,*}(f^p g) = Fr'^{,*}(Fr^*(f)\, g) = f \cdot Fr'^{,*}(g)$. in $\mathcal{O}_K$.

§1.4. Frobenius splitting of flag varieties. (over $k = \bar{k}$).

- $B \curvearrowright G/B$ has finite many orbits. $\xleftrightarrow{\text{posets}} W$.
- $X_w = \overline{BwB/B}$ is normal with rational sigularities.
- The scheme theoretical intersection of X_w and X_v

is reduced.

- $H^0(G/B, \mathcal{L}) \twoheadrightarrow H^0(X_w, \mathcal{L})$ for any semi-ample $\mathcal{L}$.

- There can all be proved by Frobenius splittings ([MR], [RR].)

- Kumar-Littelmann (2002) constructed an alg. Frobenius splitting of G/B using Lusztig's quantum Frobenius splitting.

(concrete & flexible)

I will recall an example

Example : $G_k = SL_{2,k}$ $\quad G_k/B_k \cong \mathbb{P}^1_k$. $\quad$ Affine cone $= \operatorname{Spec}(k[x,y])$.

we have $Fr^{\#} : k[x,y] \longrightarrow k[x,y]$, : $f \longmapsto f^p$.

$$\begin{array}{ccc} \| & & \| \\ \bigoplus_{n=0}^{\infty} V^*(n), & & \bigoplus_{n=0}^{\infty} V^*(n) \end{array} . \quad \mathcal{J}_n^* \longrightarrow \mathcal{J}_{pn}^* \text{ of } \mathcal{U}_k\text{-mod}.$$

(Dual : $V(pn) \longrightarrow V(n)^{Fr,}$)

Fr' : $k[x,y] \longrightarrow k[x,y]$, : $x^a y^b \longmapsto x^{a/p} y^{b/p}$.

$V^*(pn) \longrightarrow V^*(n)$. $\quad \mathcal{J}_{pn}^* \longmapsto \mathcal{J}_n^*$ of $\mathcal{U}_k$-mod.

Goal : $G \rightsquigarrow G^\theta = K \subset G$. (over $k = \bar{k}$ of char $\neq 2$).

§2. Quantum symmetric pairs.

§2.1 Symmetric subgroup scheme.

§2.2 Quantum Frobenius splitting.

§2.3. Frobenius splitting of K.

§2.4 Frobenius splitting of flag varieties.

§2.1 Symmetric subgroup scheme.

- $\theta: G \to G$ be an involution. (The classification is independent of k by Springer).

θ-anisotropic. contains a max θ-aniso torus.

- Assume θ is quasi-split, that is $B \cap \theta(B) = T$.
- Let $K = G^{\theta}$. ($K \neq K^{\circ}$ in general), $K = K^{\circ} \cdot T^{\theta}$.
- Then $K \cdot B$ is the unique open orbit on G/B.

- The symmetric subgroup scheme can be constructed in full generality.
- The Frobenius splitting is only for quasi-split cases.

- θ induces $\tau : I \to (-)I$.

- $\theta : X \to X$, $\theta : Y \to Y$, $X_\imath = X/X^\theta$, $Y^\imath = Y^\theta$.

$(G, \theta) \longleftrightarrow$ $\imath$-root datum $\rightsquigarrow$ quantum symmetric pairs.

$(\mathcal{U}_q, \mathcal{U}^\imath_q)$

$\imath$-quantum group

Def: $U_q^\imath \subset U_q$ is generated by

$$B_i = F_i + \varsigma_i E_{\tau i} K_i^{-1} \quad (i \in I) \qquad K_\mu \quad (\mu \in Y^\imath).$$

coideal subalg: $\Delta : U_q^\imath \longrightarrow U_q^\imath \otimes U_q$.

- QSP was originally defined by Letzter
- Joint with Wang (B-Wang), we developed the theory of

canonical bases. $\rightsquigarrow$ (geometric construction, categorifications, Hall algebras, ...)

$\dot{U}^\imath_q$: the modified form of $U^\imath_q$ ($1 = \sum_{\lambda \in X_\imath} 1_\lambda \in \dot{U}_q$).

$\dot{B}^\imath$: the $\imath$-canonical basis on $\dot{U}^\imath_q$.

(No satisfactory crystal theory, some progress by [Watanabe]).

${}_{\mathcal{A}}\dot{U}^\imath_q$: $\mathcal{A}$-subalg generated by $B^{(m)}_{i,\lambda}$. ($\imath$divided powers).

$=$ the free $\mathcal{A}$-subalg spanned by $\dot{B}^\imath$ ($\mathcal{A} = \mathbb{Z}[q, q^{-1}]$).

${}_R\dot{U}^\imath_q = R \otimes_{\mathcal{A}} {}_{\mathcal{A}}\dot{U}^\imath_q$ for any $\mathcal{A} \to R$.

$q \mapsto ?$

Let $\,_{\mathbb{Z}}\dot{U}^{\imath}_{q} = \mathbb{Z} \otimes_{A} \dot{U}^{\imath}_{q}$ $\quad q \mapsto 1$.

Define $O^{\imath}_{\mathbb{Z}} \subset \mathrm{Hom}_{\mathbb{Z}}(\,_{\mathbb{Z}}\dot{U}^{\imath}_{q}, \mathbb{Z})$ spanned by dual $\imath$CB.

Thm (B-Song) • $O^{\imath}_{\mathbb{Z}[\frac{1}{2}]} = O^{\imath}_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Z}[\frac{1}{2}]$ is reduced, commutative Hopf alg.

• $O_{\mathbb{Z}[\frac{1}{2}]} \twoheadrightarrow O^{\imath}_{\mathbb{Z}[\frac{1}{2}]}$ defines a smooth affine $\mathbb{Z}[\frac{1}{2}]$-gp scheme $G^{\imath}_{\mathbb{Z}[\frac{1}{2}]} \subset G_{\mathbb{Z}}$.

• The geometric fibers of $G^{\imath}_{\mathbb{Z}[\frac{1}{2}]}$ is $K \subset G$ (functorially).

☆ • Closure of K-orbits on G/B can be defined over $\mathcal{U} \subset \mathrm{Sp}\,\mathbb{Z}[\frac{1}{2}]$.

• we call $G^{\imath}_{\mathbb{Z}}$ the symmetric subgroup scheme.

Remark:

- $\mathcal{O}_Z^2 \times \mathcal{O}_Z^2 \longrightarrow \mathcal{O}_Z^2$ depends the stability of iCB.

(Conj by B-Wang, proved by Watanabe).

- $\mathcal{O}_Z \twoheadrightarrow \mathcal{O}_Z^2$ follows from compatibility of iCB and CB.

- we expect G_Z^2 is the scheme theoretical fixed point of G_Z.

Example :

$$G_{\mathbb{Z}} = SL_{2,\mathbb{Z}} . \qquad \theta : SL_2(R) \to SL_2(R) \; : \; \begin{bmatrix} a & b \\ c & d \end{bmatrix} \longmapsto \begin{bmatrix} d & c \\ b & a \end{bmatrix}$$

$$G_{\mathbb{Z}}^{1}(R) = \left\{ \begin{bmatrix} a & b \\ b & a \end{bmatrix} \middle| \; a^2 - b^2 = 1 , a, b \in R \right\} .$$

$$G_{\mathbb{Z}}^{2} = \operatorname{Spec} \left(\mathbb{Z}[x_{11}, x_{12}, x_{21}, x_{22}] \big/ \det = 1 ,\; x_{11} = x_{22} ,\; x_{12} = x_{21} \right) .$$

$$= \operatorname{Spec} \left(\mathbb{Z}[a, b] \big/ a^2 - b^2 = 1 \right) .$$

§2.2 Quantum Frobenius Splitting

Let p be a prime. $(\neq 2,3)$, $\xi = \sqrt[p]{1}$, $\Bbbk = \overline{\Bbbk}$ of char p.

$$\dot{\mathcal{U}}^{i}_{\xi} = \mathbb{Z}[\xi] \otimes_{\mathcal{A}} {}_{\mathcal{A}}\dot{\mathcal{U}}^{i}_{q} , \quad q \longmapsto \xi .$$

$$\dot{\mathcal{U}}^{i}_{1} = \mathbb{Z}[\xi] \otimes_{\mathcal{A}} {}_{\mathcal{A}}\dot{\mathcal{U}}^{i}_{q} , \quad q \longmapsto 1 .$$

$\bigstar$. $\dot{\mathcal{U}}^{i}_{\Bbbk} = \Bbbk \otimes_{\mathbb{Z}[\xi]} \dot{\mathcal{U}}^{i}_{\xi} = \Bbbk \otimes_{\mathbb{Z}[\xi]} \dot{\mathcal{U}}^{i}_{1}$ for any field $\Bbbk$ of char p.

$(1 = \sqrt[p]{1} , \quad x^p - 1 = (x-1)^p)$.

complicated.

↓

- [B-Sales 2021]. $\mathrm{Fr}: \dot{U}_\zeta \to \dot{U}_1$ restricts to. $\mathrm{Fr}: \dot{U}^\imath_\zeta \to \dot{U}^\imath_1$.

- Fr' does not restricts to $U^\imath_1$. (e.g. Fr' is NOT compatible with Δ)

$\boxed{\text{Thm (B-Song)}}$ There is a quantum Frobenius splitting.

$\mathrm{Fr}'_\imath : \dot{U}^\imath_1 \to \dot{U}^\imath_\zeta$ $\quad B^{(n)}_{i,\lambda} \longmapsto B^{(pn)}_{i,p\lambda}$ if $i \neq \tau i \in \mathbb{I}$.

such that.

$$B^{(n)}_{i,\lambda} \longmapsto \begin{cases} B^{(pn)}_{i,p\lambda} & \langle \alpha_i^\vee, \lambda \rangle \not\equiv n \mod 2 \\ \sum_{t=0}^{(p-1)/2} \begin{bmatrix} (p-1)/2 \\ t \end{bmatrix}_{\zeta^{2d_i}} B^{(np-2t)}_{i,p\lambda}, & \text{o.w.} \end{cases}$$

$\mathrm{Fr} \circ \mathrm{Fr}'_\imath = \mathrm{id}$.

Main difficulties:

- ıSerre relations [Chen-Lu-Wang, I, II, III],

- ıDivided powers. [Berman-Wang, 2019].

  - $\tau i \neq i$: $B^{(n)}_{i,\lambda} = \dfrac{B_i^n}{[n]_i^!} 1_\lambda$.

  - $\tau i = i$: $\langle \alpha_i^\vee, \lambda \rangle = 1 \mod 2$. $\left(\langle \alpha_i^\vee, \lambda \rangle = 0 \mod 2 \ \cdots\cdots \right)$

$$B^{(n)}_{i,\lambda} := \frac{1}{[n]_i^!} \cdot \begin{cases} B_i \prod_{j=1}^{k} \left(B_i^2 - [2j-1]_i^2 \right) 1_\lambda, & \text{if } m = 2k+1 \\ \prod_{j=1}^{k} \left(B_i^2 - [2j-1]_i^2 \right) 1_\lambda, & \text{if } m = 2k. \end{cases}$$

§1.3 Frobenius splitting of K $\quad$ ($k = \bar{k}$ of char $p \neq 2$).

$$\dot{\mathcal{U}}^2_1 \xrightarrow{Fr'_1} \dot{\mathcal{U}}^2_q \xrightarrow{Fr} \dot{\mathcal{U}}^2_1 \Rightarrow \dot{\mathcal{U}}^2_k \xrightarrow{Fr'_2} \dot{\mathcal{U}}^2_k \xrightarrow{Fr} \dot{\mathcal{U}}^2_k$$

$$\Rightarrow O^2_k \xrightarrow{Fr^*} O^2_k \xrightarrow{Fr_2'^{,*}} O^2_k . \quad \text{in} \quad \mathrm{Hom}_k(\dot{\mathcal{U}}^2_k, k).$$

Thm (B-Song 2023) $\quad$ • O^2_k is stable under $Fr^* / Fr_2'^{,*}$

• Fr^* = p-th power map.

• $Fr_2'^{,*}$ is a Frobenius split of K_k (or O^2_k).
$$K_k \subset G_k.$$

§2.4 Frobenius splitting of flag varieties.

Goal: Use Fr_2' to study the geometry of K-orbits on G/B.

Complication: K-orbits are much more complicated than B-orbits.

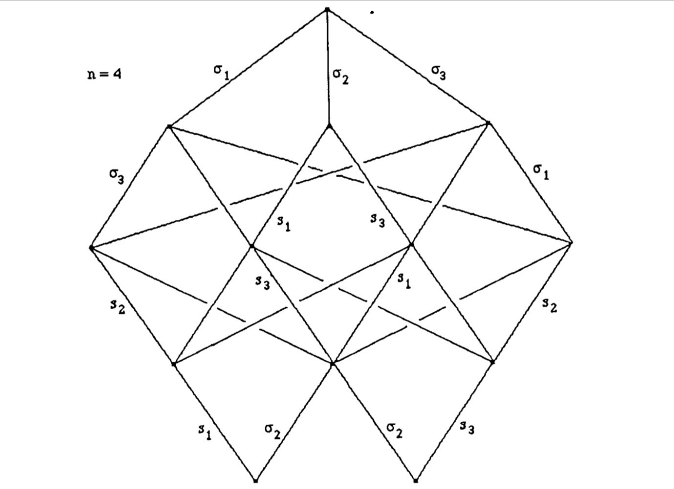
$k=\bar{k}$ char $p \neq 2$ $\quad K = G^\theta$ for quasi-split (G, θ).

- $K \curvearrowright G/B$ has finitely many orbits. [Springer, Richardson].

$\longrightarrow$ twisted involution on W.

Example: (SL_4, SO_4).

o—o—o.



- There are non-normal $\overline{\mathcal{O}}$. [Barbasch-Evens].

- There are non-reduced scheme-theoretical intersection $\overline{\mathcal{O}}_1$ and $\overline{\mathcal{O}}_2$ [Brion].

- There are non surjective $H^0(G/B, \mathcal{L}) \to H^0(\overline{\mathcal{O}}, \mathcal{L})$ [Brion-Helminck].

- We construct new splittings using Fr_2' to study K-orbits.

  (New tool).

- The precise thm is complicated, as K-orbits are.

  We explain some examples here.

Example : $G_{\Bbbk} = SL_{2,\Bbbk}$ $\quad G_{\Bbbk}/B_{\Bbbk} \cong \mathbb{P}^1_{\Bbbk}$. Affine cone $= \operatorname{Spec}(\Bbbk[x,y])$

we have $Fr^{\#} : \Bbbk[x,y] \longrightarrow \Bbbk[x,y]$; $f \longmapsto f^p$.

$$\bigoplus_{n=0}^{\infty} V^*(n) \quad\quad \bigoplus_{n=0}^{\infty} V^*(n) . \quad \mathcal{J}^*_n \longrightarrow \mathcal{J}^*_{pn} \text{ of } \underline{\mathcal{U}^2_{\Bbbk}\text{-mod}} .$$

(Dual: $V(pn) \longrightarrow V(n)^{Fr}$)

$Fr' : \Bbbk[x,y] \longrightarrow \Bbbk[x,y]$, : $x^a y^b \longmapsto x^{a/p} y^{b/p}$.

$$V^*(pn) \longrightarrow V^*(n) . \quad \mathcal{J}^*_{pn} \longmapsto \mathcal{J}^*_n + (\quad) \text{ of } \underline{\mathcal{U}^2_{\Bbbk}\text{-mod}} .$$

concrete & flexible.

$\boxed{\text{Thm (B-Song)}}$. Assume G is simply-laced, θ is quasi-split.

Then for any co-dim one K-orbit $\mathcal{O}$ of G/B, there is a

Frobenius splitting of G/B compatibly splits $\overline{\mathcal{O}}$.

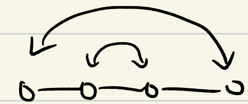
Cor: For any ample line bundle $\mathcal{L}$ on G/B, we have,

- $H^i(\overline{\mathcal{O}}, \mathcal{L}) = 0$ for $i > 0$. (Kempf vanishing).

- $H^0(G/B, \mathcal{L}) \twoheadrightarrow H^0(\overline{\mathcal{O}}, \mathcal{L})$.

$$\begin{array}{ccc} H^0(G/B, \mathcal{L}) & \twoheadrightarrow & H^0(\overline{\mathcal{O}}, \mathcal{L}) \\ \wr\| & & \wr\| \\ V^*(\lambda) & & \text{"} V_w^*(\lambda) \text{"} \end{array}$$

Thm (B-Song) For $(GL_{2n+1}, GL_n \times GL_{n+1})$, all codim one $\overline{O}$ (and many more) are normal.



Remark: • Frobenius splitting/normality for multiplicity-free divisors are known by [Knutson], [Brion]

(Our results apply to some non-multiplicity-free case **New!**)

Thank You!