On gamma factors and Bessel functions for representations of finite general linear groups

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Plan

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- Gamma factors
- Bessel functions
- Kloosterman sheaves

Generic representations

Let $\mathbb{F}$ be a finite field with q elements. Let $\psi : \mathbb{F} \to \mathbb{C}^\times$ be a non-trivial character.

Let $U_n \le \mathrm{GL}_n(\mathbb{F})$ be the subgroup of upper triangular unipotent matrices.

We define a character $\psi : U_n \to \mathbb{C}^\times$ by

$$\psi \begin{pmatrix} 1 & a_1 & * & * \\ & 1 & a_2 & * \\ & & \ddots & a_{n-1} \\ & & & 1 \end{pmatrix} = \psi\left(\sum_{i=1}^{n-1} a_i\right).$$

Let π be a (finite dimensional) representation of $\mathrm{GL}_n(\mathbb{F})$. A vector $0 \ne v \in \pi$ is called a *Whittaker vector* if $\pi(u)v = \psi(u)v$ for every $u \in U_n$.

- We say that π *is generic* if it admits a non-zero Whittaker vector.
- We say that π is of *Whittaker type* if π is generic and if it has a unique Whittaker vector, up to scalar multiplication.

Generic representations

By a well known result of Gelfand–Graev, irreducible generic representations are of Whittaker type. We also have the following well known results:

- Irreducible cuspidal representations are generic.
- If $\sigma_1, \dots, \sigma_r$ are representations of Whittaker type of $\mathrm{GL}_{n_1}(\mathbb{F}), \dots, \mathrm{GL}_{n_r}(\mathbb{F})$, respectively, then the parabolic induction $\sigma_1 \circ \dots \circ \sigma_r$ is also a representation of Whittaker type.

As a corollary, for every irreducible generic representation π, there exist irreducible cuspidal representations $\sigma_1, \dots, \sigma_r$ as above, such that π is the unique irreducible generic subrepresentation of $\sigma_1 \circ \dots \circ \sigma_r$. The multiset $\{\sigma_1, \dots, \sigma_r\}$ depends only on π, and we call it *the cuspidal support* of π.

Parabolic induction

Let π and σ be representations of Whittaker type of $\mathrm{GL}_n(\mathbb{F})$ and $\mathrm{GL}_m(\mathbb{F})$, respectively.

Suppose that π and σ are representations of Whittaker type, and let $v_{\pi,\psi}$ and $v_{\sigma,\psi}$ be ψ-Whittaker vectors. The representations $\pi \circ \sigma$ and $\sigma \circ \pi$ are also of Whittaker type. We may define a ψ-Whittaker vector $v_{\sigma,\pi,\psi}$ for $\sigma \circ \pi$ by the formula

$$v_{\sigma,\pi,\psi}(g) = \begin{cases} 0 & g \notin P_{m,n} \begin{pmatrix} & I_m \\ I_n & \end{pmatrix} U_{m+n}, \\ \psi(u) \, v_{\sigma,\psi} \otimes v_{\pi,\psi} & g = \begin{pmatrix} & I_m \\ I_n & \end{pmatrix} u, \text{ where } u \in U_{m+n}. \end{cases}$$

Similarly, we may define a ψ-Whittaker vector $v_{\pi,\sigma,\psi}$ for $\pi \circ \sigma$.

Intertwining operators and Shahidi's gamma factor

We have an intertwining operator $U_{\sigma,\pi} : \sigma \circ \pi \to \pi \circ \sigma$, defined by

$$\left(U_{\sigma,\pi} f\right)(g) = \sum_{X \in \mathrm{Mat}_{n \times m}(\mathbb{F})} \tilde{f}\left(\begin{pmatrix} & I_m \\ I_n & \end{pmatrix}\begin{pmatrix} I_n & X \\ & I_m \end{pmatrix} g\right),$$

where $\widetilde{v_\sigma \otimes v_\pi} = v_\pi \otimes v_\sigma$. Since the representations $\sigma \circ \pi$ and $\pi \circ \sigma$ are of Whittaker type, and since $U_{\sigma,\pi} v_{\sigma,\pi,\psi}$ is a ψ-Whittaker vector, we have that there exists a contant $\Gamma\left(\pi \times \sigma, \psi\right) \in \mathbb{C}$, such that

$$U_{\sigma,\pi} v_{\sigma,\pi,\psi} = \Gamma\left(\pi \times \sigma, \psi\right) v_{\pi,\sigma,\psi}.$$

We call $\Gamma\left(\pi \times \sigma, \psi\right)$ the Shahidi gamma factor of π and σ with respect to ψ.

Shahidi's gamma factor

The Shahidi gamma factor enjoys the following properties:

- Symmetry:

$$\Gamma\left(\pi \times \sigma, \psi\right) = \Gamma\left(\sigma^{\vee} \times \pi^{\vee}, \psi^{-1}\right).$$

- Multiplicativity:

$$\Gamma\left(\pi \times \left(\sigma_1 \circ \sigma_2\right), \psi\right) = \Gamma\left(\pi \times \sigma_1, \psi\right)\Gamma\left(\pi \times \sigma_2, \psi\right).$$

We show the following property regarding the absolute value of $\Gamma\left(\pi \times \sigma, \psi\right)$. It is analogous to the property of the order of the pole of $L\left(s, \pi \times \sigma^{\vee}\right)$ in the local case.

Theorem (Soudry–Z. 2023): Suppose that π and σ are irreducible and that σ is cuspidal. Then

$$\left|q^{-\frac{nm}{2}}\Gamma\left(\pi \times \sigma, \psi\right)\right| = q^{-\frac{m}{2}d_{\pi}(\sigma)},$$

where $d_{\pi}\left(\sigma\right)$ is the number of times σ appears in the cuspidal support of π.

Jacquet–Piatetski-Shapiro–Shalika gamma factor

There exists another construction of gamma factors for irreducible generic representations of $\mathrm{GL}_n(\mathbb{F})$ and $\mathrm{GL}_m(\mathbb{F})$ representations due to Jacquet–Piatetski-Shapiro–Shalika. A finite field analog of this construction was studied in the master's thesis of Edva Roditty-Gershon, under the supervision of David Soudry. It is based on the notion of a Whittaker model: given an irreducible generic representation τ of $\mathrm{GL}_k(\mathbb{F})$, choose a $\mathrm{GL}_k(\mathbb{F})$-invariant inner product $\langle \cdot, \cdot \rangle$ on τ and a ψ-Whittaker vector $v_{\tau,\psi}$. The Whittaker model $\mathcal{W}(\tau, \psi)$ is the space of all functions on $\mathrm{GL}_k(\mathbb{F})$ of the form

$$\mathcal{W}(\tau, \psi) = \left\{ W_{v_\tau}(g) = \langle \tau(g) v_\tau, v_{\tau,\psi} \rangle \mid v_\tau \in \tau \right\}.$$

We have a linear isomorphism $\mathcal{W}(\tau, \psi) \to \mathcal{W}(\tau^\vee, \psi^{-1})$ by the formula $\tilde{W}_{v_\tau}(g) = W_{v_\tau}\left(w_k \left({}^t g\right)^{-1}\right)$, where $w_k = \begin{pmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{pmatrix}$.

Jacquet–Piatetski-Shapiro–Shalika gamma factor

The Whittaker model $\mathcal{W}(\tau, \psi)$ is the space of all functions on $\mathrm{GL}_k(\mathbb{F})$ of the form $\mathcal{W}(\tau, \psi) = \{W_{v_\tau}(g) = \langle \tau(g) v_\tau, v_{\tau,\psi} \rangle \mid v_\tau \in \tau\}$.
If π and σ are irreducible generic representations of $\mathrm{GL}_n(\mathbb{F})$ and $\mathrm{GL}_m(\mathbb{F})$, respectively, with $n > m$, then one defines an "integral" $Z_j(W_{v_\pi}, W_{v_\sigma}; \psi)$ by the formula

$$\sum_{h \in U_m \backslash \mathrm{GL}_m(\mathbb{F})} \sum_{x \in \mathrm{Mat}_{(n-m-j-1)\times m}(\mathbb{F})} W_{v_\pi}\begin{pmatrix} h & & \\ x & I_{n-m-j-1} & \\ & & I_{j+1} \end{pmatrix} W_{v_\sigma}(h).$$

Under mild assumptions (for example, π is cuspidal), there exists a constant $\gamma(\pi \times \sigma, \psi) \in \mathbb{C}^\times$, such that for any j and v_π and v_σ,

$$Z_{n-j}\left(\pi^\vee \begin{pmatrix} I_m & \\ & w_{n-m} \end{pmatrix} \tilde{W}_{v_\pi}, \tilde{W}_{v_\sigma}; \psi^{-1}\right) = q^{mj} \gamma(\pi \times \sigma, \psi) Z_j(W_{v_\pi}, W_{v_\sigma}; \psi).$$

Comparsion between gamma factors

One can also define $\gamma\left(\pi \times \sigma, \psi\right)$ in the case that $n = m$. We show that if π and σ are irreducible generic representations such that $\gamma\left(\pi \times \sigma^\vee, \psi\right)$ is defined, then

$$\Gamma\left(\pi \times \sigma, \psi\right) = q^{\frac{m(2n-m-1)}{2}} \omega_\sigma\left(-1\right) \gamma\left(\pi \times \sigma^\vee, \psi\right).$$

The proof goes through the Bessel functions of π and σ.
Theorem (Soudry–Z 2023): if $n > m$, then for any irreducible generic π and σ

$$\Gamma\left(\pi \times \sigma, \psi\right) = q^{\frac{m(2n-m-1)}{2}} \omega_\sigma\left(-1\right) \sum_{g \in U_m \backslash \mathrm{GL}_m(\mathbb{F})} J_{\pi,\psi}\begin{pmatrix} & I_{n-m} \\ g & \end{pmatrix} J_{\sigma^\vee,\psi^{-1}}\left(g\right).$$

By the work of Roditty-Gershon, this establishes the result above.

Expression in terms of Gauss sums

Together with Rongqing Ye, we computed $\gamma(\pi \times \sigma, \psi)$ for π and σ irreducible cuspidal representations. It follows that if π and σ are irreducible generic representations of $\mathrm{GL}_n(\mathbb{F})$ and $\mathrm{GL}_m(\mathbb{F})$, associated with the (Galois orbits of the) regular characters $\alpha : \mathbb{F}_\lambda^\times \to \mathbb{C}^\times$ and $\beta : \mathbb{F}_\mu^\times \to \mathbb{C}^\times$, respectively. Then

$$\Gamma(\pi \times \sigma, \psi) = (-1)^{\ell(\mu)n+\ell(\lambda)m} \beta(-1)^n \times \sum_{\xi \in (\mathbb{F}_\lambda \otimes_{\mathbb{F}} \mathbb{F}_\mu)^\times} \alpha^{-1}(\mathrm{N}_{/\mathbb{F}_\lambda}(\xi)) \beta(\mathrm{N}_{/\mathbb{F}_\mu}(\xi)) \psi(\mathrm{tr}_{/\mathbb{F}} \xi).$$

Here, for a partition $\lambda = (n_1, \dots, n_r) \vdash n$ we denote $\ell(\lambda) = r$ and $\mathbb{F}_\lambda = \mathbb{F}_{n_1} \times \cdots \times \mathbb{F}_{n_r}$, where $\mathbb{F}_k / \mathbb{F}$ is a finite field extension of $\mathbb{F}$ of degree k.

Gel'fand–Graev Bessel function

We move to discuss the Bessel function. Let π be an irreducible generic representation of $\mathrm{GL}_n(\mathbb{F})$. Let $0 \neq v \in \pi$ be a Whittaker vector. Choose a $\mathrm{GL}_n(\mathbb{F})$-invariant inner product $\langle \cdot, \cdot \rangle$ on π. The Bessel function $J_{\pi,\psi} : \mathrm{GL}_n(\mathbb{F}) \to \mathbb{C}$ is defined as the following matrix coefficient

$$J_{\pi,\psi}(g) = \frac{\langle \pi(g) v, v \rangle}{\langle v, v \rangle}.$$

It satisfies the following properties:

1. $J_{\pi,\psi}(I_n) = 1$.
2. $J_{\pi,\psi}(u_1 g u_2) = \psi(u_1) \psi(u_2) J_{\pi,\psi}(g)$ for $u_1, u_2 \in U_n$.
3. $J_{\pi,\psi}(g^{-1}) = \overline{J_{\pi,\psi}(g)} = J_{\pi^\vee,\psi^{-1}}(g)$.
4. $J_{\pi,\psi}(zg) = J_{\pi,\psi}(gz) = \omega_\pi(z) J_{\pi,\psi}(g)$, where $z \in \mathbb{F}^\times$ and $\omega_\pi : \mathbb{F}^\times \to \mathbb{C}^\times$ is the central character of π.

Gel'fand–Graev Bessel function

Recall that by the Bruhat decomposition, any $g \in \mathrm{GL}_n(\mathbb{F})$ can be written as $g = u_1 d w u_2$, where $u_1, u_2 \in U_n$, d is a diagonal matrix and w is a permutation matrix. Thanks to the property $J_{\pi,\psi}(u_1 g u_2) = \psi(u_1)\psi(u_2) J_{\pi,\psi}(g)$, it suffices to know the values of $J_{\pi,\psi}$ on matrices of the form dw. It turns out that $J_{\pi,\psi}(dw) = 0$ unless dw is of the form

$$dw = \begin{pmatrix} & & & c_1 I_{n_1} \\ & & c_2 I_{n_2} & \\ & \iddots & & \\ c_s I_{n_s} & & & \end{pmatrix},$$

where $n_1 + \cdots + n_s = n$ and $c_1, \ldots, c_s \in \mathbb{F}^\times$.

Gel'fand–Graev Bessel function

Why is this function called the Bessel function? If π is an irreducible cuspidal representation of $\mathrm{GL}_2(\mathbb{F})$, corresponding to the (Galois orbit of the) regular character $\theta : \mathbb{F}_2^\times \to \mathbb{C}^\times$, then we have for $a \in \mathbb{F}^\times$

$$\mathcal{J}_{\pi,\psi}\begin{pmatrix} 0 & 1 \\ a & 0 \end{pmatrix} = -q^{-1} \sum_{\substack{x \in \mathbb{F}_2^\times \\ \mathrm{N}_{2:1}(x) = -a^{-1}}} \theta^{-1}(x)\psi(\mathrm{tr}_{\mathbb{F}_2/\mathbb{F}}\, x).$$

Compare with the classical Bessel function

$$J_n(x) = \frac{1}{2\pi i} \int_{|z|=1} z^{-n} \exp\left(\frac{ix}{2}\left(\mathrm{tr}_{\mathbb{C}/\mathbb{R}}(-iz)\right)\right)\frac{dz}{z}.$$

Exotic Kloosterman sums

Let $\alpha : \mathbb{F}_{\lambda}^{\times} \to \mathbb{C}^{\times}$. We denote

$$J(\alpha, \psi, a) = \sum_{\substack{x \in \mathbb{F}_{\lambda}^{\times} \\ \mathrm{N}_{/\mathbb{F}}(x) = a}} \alpha^{-1}(x) \psi\left(\mathrm{tr}_{/\mathbb{F}} x\right).$$

We call $J(\alpha, \psi, a)$ an exotic Kloosterman sum.

Curtis–Shinoda showed the following relation to the Bessel function.

Theorem (Curtis–Shinoda 2004): If π is an irreducible generic representation of $\mathrm{GL}_n(\mathbb{F})$ associated with the character $\alpha : \mathbb{F}_{\lambda}^{\times} \to \mathbb{C}^{\times}$ then for any $c \in \mathbb{F}^{\times}$,

$$J_{\pi,\psi}\begin{pmatrix} & I_{n-1} \\ c & \end{pmatrix} = (-1)^{n+\ell(\lambda)} q^{-n+1} J\left((-1)^{n-1} c^{-1}, \psi, a\right).$$

I generalized this result of Curtis–Shinoda.

Exotic Kloosterman sums

When $\lambda = (1, \ldots, 1) \vdash n$ and $\alpha = 1$, we have that π is the Steinberg representation St_n of $\mathrm{GL}_n(\mathbb{F})$ and that $J(1, \psi, a)$ is the usual Kloosterman sum

$$J(1, \psi, a) = \mathrm{Kl}(\psi, a) = \sum_{\substack{x_1, \ldots, x_n \in \mathbb{F}^\times \\ x_1 \cdot \cdots \cdot x_n = a}} \psi(x_1 + x_2 + \cdots + x_n).$$

Curtis–Shinoda's result reads

$$J_{\mathrm{St}_n, \psi} \begin{pmatrix} & I_{n-1} \\ c & \end{pmatrix} = q^{-n+1} \mathrm{Kl}\left(\psi, (-1)^{n-1} c^{-1}\right).$$

Exotic Kloosterman sheaves

Consider the exotic Kloosterman sheaf
$\mathcal{K}^* = \mathbf{R}\,\mathrm{Norm}_!(\mathrm{Trace}^*\,\mathrm{AS}_\psi \otimes \mathcal{L}_{\alpha^{-1}})\,[n-1]\,(\frac{n-1}{2})$, where

$$\begin{array}{ccc} & \mathrm{Res}_{\mathbb{F}_\lambda/\mathbb{F}}\,\mathbb{G}_m & \\ {\scriptstyle\mathrm{Norm}}\swarrow & & \searrow{\scriptstyle\mathrm{Trace}} \\ \mathbb{G}_m & & \mathbb{A}^1 \end{array} .$$

For every m, we have

$$\mathrm{tr}\left(\mathrm{Fr}^m, \mathcal{K}^* \mid_a\right) = (-1)^{n-1}\, q^{-\frac{m(n-1)}{2}} \sum_{\substack{x\in(\mathbb{F}_\lambda\otimes_{\mathbb{F}}\mathbb{F}_m)^\times \\ \mathrm{N}_{/\mathbb{F}_m}(x)=a}} \alpha^{-1}\left(\mathrm{N}_{/\mathbb{F}_\lambda}(x)\right)\psi\left(\mathrm{tr}_{/\mathbb{F}}\, x\right).$$

Exotic Kloosterman sheaves

Theorem (Z. 2022): for any $c \in \mathbb{F}^\times$ we have

$$\operatorname{tr}\left(\mathrm{Fr}, \wedge^m \mathcal{K}^* \mid_{(-1)^{n-1}c^{-1}}\right) = (-1)^{m(\ell(\lambda)-1)} q^{\frac{m(n-m)}{2}} J_{\pi,\psi} \begin{pmatrix} & I_{n-m} \\ cI_m & \end{pmatrix}.$$

As a corollary, we get that for the usual Kloosterman sheaf $\mathrm{Kl}(\psi)$, the characteristic polynomial of $\mathrm{Fr}, \mathrm{Kl}(\psi) \mid_{(-1)^{n-1}c^{-1}}$ is

$$\sum_{m=0}^{n} (-1)^{n(m-n)} q^{\frac{m(n-m)}{2}} J_{\mathrm{St}_n,\psi} \begin{pmatrix} & I_m \\ cI_{n-m} & \end{pmatrix} T^m.$$

This is a deep relation between the Steinberg representation and Kloosterman sheaves.

Relation to exotic Kloosterman sheaves

Using the formula

$$\operatorname{tr}\left(\mathrm{Fr}, \wedge^m \mathcal{K}^* \mid_{(-1)^{n-1}c^{-1}}\right) = (-1)^{m(r-1)} q^{\frac{m(n-m)}{2}} J_{\pi,\psi}\begin{pmatrix} & I_{n-m} \\ cI_m & \end{pmatrix},$$

the relation

$$J_{\pi,\psi}\begin{pmatrix} & I_{n-m} \\ cI_m & \end{pmatrix} = \overline{J_{\pi,\psi}\begin{pmatrix} & c^{-1}I_m \\ I_{n-m} & \end{pmatrix}} = \omega_\pi(c)\overline{J_{\pi,\psi}\begin{pmatrix} & I_m \\ cI_{n-m} & \end{pmatrix}}$$

can be seen as a corollary of the isomorphism

$$\wedge^m(V) \cong \det V \otimes \left(\wedge^{n-m}(V)\right)^\vee,$$

where V is an n-dimensional complex representation of a group.

About the proof – Recursive formula for the Bessel function

As we explained before, $\gamma(\pi \times \sigma, \psi)$ can be expressed in terms of the Bessel functions of π and σ:

$$\gamma(\pi \times \sigma, \psi) = \sum_{g \in U_m \backslash \mathrm{GL}_m(\mathbb{F})} J_{\pi,\psi} \begin{pmatrix} & I_{n-m} \\ g & \end{pmatrix} J_{\sigma,\psi^{-1}}(g).$$

On the other hand, we have that matrix coefficients of non-isomorphic representations are orthogonal. This allows us (by inverting a matrix) to find a formula for $J_{\pi,\psi} \begin{pmatrix} & I_{n-m} \\ g & \end{pmatrix}$ in terms of $J_{\sigma,\psi}(g)$:

$$J_{\pi,\psi} \begin{pmatrix} & I_{n-m} \\ g & \end{pmatrix} = \frac{1}{[\mathrm{GL}_m(\mathbb{F}) : U_m]} \sum_{\sigma} \dim \sigma \cdot \gamma\left(\pi \times \sigma^{\vee}, \psi\right) \cdot J_{\sigma,\psi}(g),$$

where σ runs over the irreducible generic representations of $\mathrm{GL}_m(\mathbb{F})$.

About the proof – Recursive formula for the Bessel function

We are able to parameterize irreducible generic representations of $\mathrm{GL}_m(\mathbb{F})$ by characters of multiplicative groups of étale algebras over $\mathbb{F}$ of degree m. Using this parameterization, we have:

Theorem (Z. 2022)

$$J_{\pi,\psi}\begin{pmatrix} & I_{n-m} \\ g & \end{pmatrix} = \sum_{\mu \vdash m} \frac{1}{Z_\mu} \frac{1}{\left|\widehat{\mathbb{F}}_\mu^\times\right|} \sum_{\beta \in \widehat{\mathbb{F}}_\mu^\times} \gamma\left(\pi \times \Pi_\mu(\beta)^\vee, \psi\right) J_{\Pi_\mu(\beta),\psi}(g)$$

Where, for a partition $\mu = (m_1, \dots, m_t) \vdash m$, $\widehat{\mathbb{F}}_\mu^\times$ is the group of characters $\beta : \mathbb{F}_\mu^\times \to \mathbb{C}^\times$, and $Z_\mu = \prod_{k=1}^\infty k^{\mu(k)} \mu(k)!$, where $\mu(k)$ is the number of $1 \le i \le t$ for which m_i equals k.
This formula allows us to recursively compute $J_{\pi,\psi}$. Substituting $g = cI_m$ and the expression for the γ-factor yields the result. When g is not a scalar matrix, we get more complicated expressions.

Thank you for your attention!