# Kazhdan-Lusztig Equivalence at the Iwahori Level

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February 22, 2022

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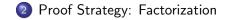
Iwahori Kazhdan-Lusztig

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#### Overview

Statement of Result



#### Quantum Side





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### Notations

G Reductive group over  $\mathbb C$  (for this talk, assumed simple)

G(O), G(K) Arc (resp. loop) group of G

- ${\mathfrak g}\,$  Lie algebra of G
- $h^{\vee}$  Dual Coxeter number
- $\check{\Lambda}, \Lambda$  Weight lattice / coweight lattice
  - W Weyl group for G
    - $\kappa\,$  Non-degenerate W-invariant symmetric bilinear form on  $\Lambda\,$
    - $\check{\kappa}$  Corresponding bilinear form on  $\check{\Lambda}$

$$c \ \check{\kappa} = rac{c-h^ee}{2h^ee}\check{\kappa}_{\mathsf{min}}$$
, where  $\check{\kappa}_{\mathsf{min}}(\check{lpha}_i,\check{lpha}_i) = 2$  for long roots  $\check{lpha}_i$ 

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# Kazhdan-Lusztig Equivalence

#### Theorem ([KL94])

If  $c \in \mathbb{C} \setminus \mathbb{Q}$ , or  $c \in \frac{m}{n} \in \mathbb{Q}^{<0}$  for (m, n) = 1 and m not too small, then there exists a braided monoidal equivalence  $KL_{\kappa}(G)^{\heartsuit} \simeq \operatorname{Rep}_{q}(G)^{\heartsuit}$ .

 $\hat{\mathfrak{g}}_{\kappa}$  Central extension of  $\mathfrak{g}((t))$  given by the 2-cocycle  $\kappa$ KL<sub> $\kappa$ </sub>(G)<sup> $\heartsuit$ </sup> Abelian category of finitely generated, smooth, G(O)-integrable  $\hat{\mathfrak{g}}_{\kappa}$ -modules at level  $\kappa$ 

- $U_q^{Lus}(\mathfrak{g})$  Lusztig's quantum group specialized at  $q:=e^{\frac{\pi i}{rc}}$ , where r is the lacing number of  $\mathfrak{g}$
- $\operatorname{Rep}_q(G)^{\heartsuit}$  Abelian category of finite dimensional  $\check{\Lambda}$ -graded  $U_q^{\operatorname{Lus}}(\mathfrak{g})$ -modules, where  $K_{\check{\alpha}_i} \in U_q^{\operatorname{Lus}}(\mathfrak{g})$  acts via grading

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At generic levels, both are equivalent to  $\mathfrak{g}$ -mod<sup>B</sup>. Rational levels are more interesting.

Theorem (Lin Chen and C.F.; Conjectured by D. Gaitsgory)

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- The RHS carries a braided monoidal structure (compatible with Rep<sub>q</sub>(G)<sup>♥</sup>); consequently it equips LHS with a (previously unknown) braided monoidal structure. We do not yet know how to describe it explicitly.



#### Proof Strategy: Factorization

#### 3 Quantum Side





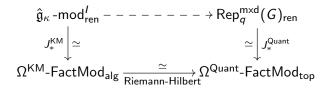
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$$\int_{\ast}^{\mathsf{KM}} \bigvee_{\simeq}^{\mathsf{MM}} \simeq \qquad \simeq \bigvee_{q}^{\mathsf{Quant}} \int_{\ast}^{\mathsf{Quant}} \Omega^{\mathsf{Quant}} - \operatorname{Fact}_{\mathsf{Mod}_{\mathsf{top}}}$$

In general, given a lax monoidal functor  $F : C \rightarrow D$  between monoidal categories, it automatically factors as

$$C \xrightarrow{F_{\mathsf{enh}}} F(\mathbf{1}_C)\operatorname{-mod}(D) \xrightarrow{\mathsf{oblv}} D;$$

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Assume A is locally constant. The behavior as two distinct points x and y collide into one then encode a certain (dg) algebra structure on  $A_{alg} := \bigoplus_{\tilde{\lambda} \in \tilde{\Lambda}^{<0}} \iota^!_{\tilde{\lambda} \cdot x}(A).$ 

\*-fibers encode a *coalgebra* structure on  $A_{\text{coalg}} := \bigoplus_{\check{\lambda} \in \check{\Lambda}^{<0}} \iota^*_{\check{\lambda} \cdot x}(A).$ 

Similarly, a  $\Lambda$ -graded factorization module M (supported at  $0 \in \mathbb{A}^1$ ) over A is (among other data) a sheaf on the moduli space

$$\mathfrak{Conf}_0 := \{\check{\lambda}_0 \cdot 0 + \sum_{i \in I, |I| < \infty} \check{\lambda}_i \cdot x_i \mid \check{\lambda}_0 \in \check{\Lambda}, \check{\lambda}_i \in \check{\Lambda}^{<0}, x_i \text{ disjoint}, x_i \neq 0\};$$

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Riemann-Hilbert allows the comparison between algebraic factorization modules (using D-modules) and topological ones (using constructible sheaves).

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$$\Delta(E_{\check{\alpha}_i}) = E_{\check{\alpha}_i} \otimes 1 + 1 \otimes E_{\check{\alpha}_i} \quad E_{\check{\alpha}_i} \in U_q^{\mathsf{Lus}}(\mathfrak{n}).$$

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The abelian category  $\operatorname{Rep}_q^{\mathsf{mxd}}(G)^{\heartsuit}$  consists of  $V \in \operatorname{Rep}_q(T)^{\heartsuit}$  with a *locally nilpotent*  $U_q^{\mathsf{Lus}}(\mathfrak{n})$  action and a compatible (arbitrary)  $U_q^{\mathsf{KD}}(\mathfrak{n}^-)$  action.

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The abelian category  $\operatorname{Rep}_q^{\mathsf{mxd}}(G)^{\heartsuit}$  consists of  $V \in \operatorname{Rep}_q(T)^{\heartsuit}$  with a *locally nilpotent*  $U_q^{\mathsf{Lus}}(\mathfrak{n})$  action and a compatible (arbitrary)  $U_q^{\mathsf{KD}}(\mathfrak{n}^-)$  action.

$$\operatorname{Rep}_{q}^{\mathsf{m} \times \mathsf{d}}(G)_{\mathsf{ren}}$$
 is the ind-completion of

$$\{V \in D^b(\operatorname{\mathsf{Rep}}_q^{\operatorname{\mathsf{mxd}}}(G)^\heartsuit) \text{ s.t.}$$

 $\operatorname{oblv}(V) \in U_q^{\mathsf{KD}}(\mathfrak{n}^-)\operatorname{-mod}(D(\operatorname{Rep}_q(\mathcal{T})^\heartsuit)) ext{ is compact}\}.$ 

There exists a topological factorization algebra  $\Omega^{\text{Quant}}$  and an equivalence of DG categories  $J^{\text{Quant}}_*$ :  $\text{Rep}_q^{\text{mxd}}(G)_{\text{ren}} \simeq \Omega^{\text{Quant}}$ -FactMod<sub>top</sub>.

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At abelian level, this is analogous to the main result of [BFS06].

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#### Remark

$$\iota_{\check{\lambda}.0}^{!}(J^{\text{Quant}}_{*}(M)) \text{ is the } \check{\lambda}\text{-component of } \mathsf{Ext}_{U_{q}^{\text{Lus}}(\mathfrak{n})}^{\bullet}(\mathbb{C}, M) \text{, and} \\ \iota_{\check{\lambda}.0}^{*}(J^{\text{Quant}}_{*}(M)) \text{ is the } \check{\lambda}\text{-component of } \mathsf{Tor}_{U_{q}^{\text{KD}}(\mathfrak{n}^{-})}^{\bullet}(\mathbb{C}, M).$$

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#### 2 Proof Strategy: Factorization

3 Quantum Side





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$$\mathfrak{g}((t))\operatorname{-mod}_{\mathsf{ren}}^{\mathcal{G}(O)}\simeq \mathsf{IndCoh}^!_{\mathsf{ren}}(\mathbb{B}\mathcal{G}(\mathcal{K})^\wedge_{\mathcal{G}(O)}),$$

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where renormalization on both sides mean taking the ind-completion of the category of objects induced from finite dimensional *smooth* representations of G(O).

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The infinite-dimensional theory bifurcates into the !-and the \*-versions; here !-version is considered.

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Iwahori Kazhdan-Lusztig

February 22, 2022

To each  $\kappa$  one can assign a *twisting* (an infinitesimal gerbe) on  $\mathbb{B}G(\kappa)^{\wedge}_{G(O)}$  and use it to twist the IndCoh category. A slight variant of above is

$$\mathsf{KL}_{\kappa}(G)_{\mathsf{ren}} := \mathsf{IndCoh}^!_{\mathsf{ren},\kappa}(\mathbb{B}G(K)^{\wedge}_{G(O)}).$$

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#### Proposition ([Ras20])

When restricted to bounded-below objects, the functor

$$\mathsf{KL}_{\kappa}(B)_{\mathsf{ren}} \simeq \mathsf{IndCoh}^!_{\mathsf{ren},\kappa}(\mathbb{B}B(\mathcal{K})^{\wedge}_{\mathcal{B}(\mathcal{O})}) \xrightarrow{\bigstar} \mathsf{IndCoh}^*_{\mathsf{ren},\kappa-\kappa_{\mathsf{crit}}}(\mathbb{B}B(\mathcal{K})^{\wedge}_{\mathcal{B}(\mathcal{O})})$$

$$\xrightarrow{*-\text{push}} \text{IndCoh}_{\text{ren},\kappa-\kappa_{\text{crit}}}^{*}(\mathbb{B}T(K)^{\wedge}_{T(O)}) \simeq \text{KL}_{\kappa-\kappa_{\text{crit}}}(T)_{\text{ren}}$$
  
coincides with Feigin's semi-infinite cohomology  $C_{*}^{\frac{\infty}{2}}(\mathfrak{n}((t)), N(O), -).$ 

Here  $\kappa_{crit}$  is the *critical* (a.k.a. *Tate*) shift, corresponding to c = 0. Existence of ( $\blacklozenge$ ) is a distinguished feature of the renormalized theory.

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In the present work, we extend this theory to the factorizable setting.

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There exists an unital factorizable crystal of categories  $\mathcal{KL}_{\kappa}(G)_{ren}$  whose 1-point fiber is  $KL_{\kappa}(G)_{ren}$ .

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 For every x ∈ A<sup>1</sup>(C) on the diagonal, the corresponding base change gives Vect ⊗<sub>DMod(A<sup>2</sup>)</sub> KL<sub>κ</sub>(G)<sub>[2]</sub> ≃ KL<sub>κ</sub>(G)<sub>ren</sub>;

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The behavior as we approach the diagonal encodes the *fusion* structure of  $KL_{\kappa}(G)$ .

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Unitality means, for instance, that  $\{x\} \hookrightarrow \{x, y\}$  yields a map

$$\mathsf{ins}_{x \leadsto (x,y)} : \mathsf{KL}_{\kappa}(G)_{\mathsf{ren}} \to \mathsf{KL}_{\kappa}(G)_{\mathsf{ren}} \otimes \mathsf{KL}_{\kappa}(G)_{\mathsf{ren}}$$

given by  $M \mapsto \mathbb{V}^0_{\kappa} \boxtimes M$ , where

$$\mathbb{V}^{\mathsf{0}}_{\kappa}:=\mathsf{Ind}_{\mathsf{Rep}(G(O))^{\heartsuit}}^{\mathsf{KL}_{\kappa}(G)^{\heartsuit}}(\mathbb{C})$$

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Similarly, to  $\hat{\mathfrak{g}}_{\kappa}$ -mod<sup> $\prime$ </sup><sub>ren</sub> we attach a factorizable module category  $\mathcal{IKL}_{\kappa}(G)$ . Over  $\mathbb{A}^2$ , its diagonal fiber is  $\hat{\mathfrak{g}}_{\kappa}$ -mod<sup> $\prime$ </sup><sub>ren</sub>, and off-diagonal fiber is KL<sub> $\kappa$ </sub>(G)<sub>ren</sub>  $\otimes \hat{\mathfrak{g}}_{\kappa}$ -mod<sup> $\prime$ </sup><sub>ren</sub>.

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It encodes the *fusion action* of  $KL_{\kappa}(G)$  on  $\hat{\mathfrak{g}}_{\kappa}$ -mod<sup>1</sup>.

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Factorization modules internal to  $\mathcal{IKL}_{\kappa}(G)$  are similarly defined (diagonal: M; off-diagonal:  $A \boxtimes M$ ), and unitality gives

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For simplicity we write  $C^{\frac{\infty}{2}} := C^{\frac{\infty}{2}}_*(\mathfrak{n}((t)), N(O), -)$ . The map  $\hat{\mathfrak{g}}_{\kappa}$ -mod<sup>*I*</sup><sub>ren</sub>  $\xrightarrow{\operatorname{Res}} \operatorname{KL}_{\kappa}(B)_{\operatorname{ren}} \xrightarrow{C^{\frac{\infty}{2}}} \operatorname{KL}_{\kappa-\kappa_{\operatorname{crit}}}(\mathcal{T})_{\operatorname{ren}}$  is a *lax-unital factorizable* functor, and thus factors through an "enhanced" map

$$C^{\frac{\infty}{2}}_{\mathsf{enh}}: \hat{\mathfrak{g}}_{\kappa}\operatorname{-mod}'_{\mathsf{ren}} \to C^{\frac{\infty}{2}}(\mathbb{V}^0_{\kappa})\operatorname{-FactMod}(\mathcal{KL}_{\kappa}(\mathcal{T})).$$

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#### Proposition ("Torus FLE")

There exists an equivalence of factorizable crystals of categories

 $\mathsf{FLE}_{\mathcal{T}}: \mathcal{KL}_{\kappa}(\mathcal{T})_{\mathsf{ren}} \simeq \mathsf{DMod}_{\check{\kappa}}(\mathsf{Gr}_{\check{\mathcal{T}}});$ 

where  $Gr_{\check{T}}$  is the affine Grassmannian for the dual torus  $\check{T}$ .

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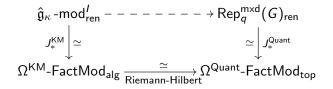
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factors through a map

$$J^{\mathsf{KM}}_* := \mathsf{FLE}_{\mathcal{T}} \circ C^{\frac{\infty}{2}}_{\mathsf{enh}} : \hat{\mathfrak{g}}_{\kappa}\operatorname{-mod}'_{\mathsf{ren}} \to \Omega^{\mathsf{KM}}\operatorname{-FactMod}_{\mathsf{alg}}.$$

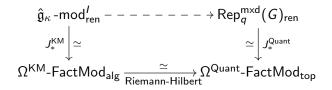
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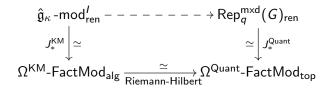


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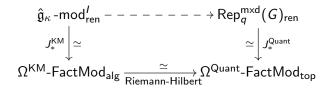
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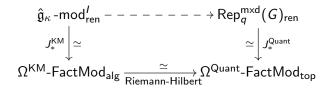
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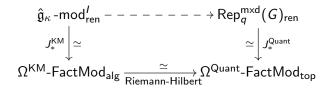


We have argued that  $J_*^{\text{Quant}}$  is an equivalence. The remaining tasks are:

Showing that Ω<sup>KM</sup> and Ω<sup>Quant</sup> match up under Riemann-Hilbert; and
Showing that J<sup>KM</sup><sub>\*</sub> is an equivalence for c > 0.

Let us do the first part.

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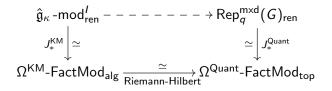


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Let us do the first part. Recall that, !-fiber of  $\Omega^{Quant}$  are components of  $\operatorname{Ext}_{U^{\operatorname{Lus}}(\mathfrak{n})}^{\bullet}(\mathbb{C},\mathbb{C})$ , and that of  $\Omega^{\operatorname{KM}}$  are components of  $C^{\frac{\infty}{2}}(\mathbb{V}_{\kappa}^{0})$ .

Charles Fu (Harvard University)



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Let us do the first part. Recall that, !-fiber of  $\Omega^{Quant}$  are components of  $\operatorname{Ext}_{U^{\operatorname{Lus}}(\mathfrak{n})}^{\bullet}(\mathbb{C},\mathbb{C})$ , and that of  $\Omega^{\operatorname{KM}}$  are components of  $C^{\frac{\infty}{2}}(\mathbb{V}_{\kappa}^{0})$ .

**Problem:** neither is easy to compute / explicitly known.

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# Matching Factorization Algebras

### Proposition ([Gai21])

There exists an unique  $\check{\Lambda}^{<0}$ -graded factorization algebra  $\Omega$  such that:

- if  $\check{\lambda} \notin \check{\Lambda}^{<0}$ , then the !-fiber at  $\check{\lambda}x$  is zero;
- the !-fiber at every  $\check{\lambda}x$  has no negative cohomology;
- if λ̃ is a simple negative root, then either the \*-fiber at λ̃x is C[1], or the !-fiber at λ̃x is C[−1];
- if λ̃ equals w(ρ̃) − ρ̃ for some ℓ(w) = 2, then the !-fiber at λ̃x vanishes at H<sup>0</sup> and H<sup>1</sup>, and \*-fiber at λ̃x vanishes at H<sup>0</sup> and H<sup>-1</sup>;
- otherwise, the !-fiber at  $\check{\lambda}x$  vanishes at  $H^0$ , and \*-fiber at  $\check{\lambda}x$  vanishes at  $H^0$ ,  $H^{-1}$  and  $H^{-2}$ .

One can use direct computation (using e.g. Kashiwara-Tanisaki localization) to verify this for both  $\Omega^{\text{KM}}$  and  $\Omega^{\text{Quant}}$ .



3 Quantum Side





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# Proving $J_*^{\text{KM}}$ is an Equivalence

The category  $\Omega^{\text{KM}}$ -FactMod<sub>alg</sub> has a *highest weight* structure: it contains *standard objects* which are compact generators, and *costandard objects* which are their right orthogonals.

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The category  $\Omega^{\text{KM}}$ -FactMod<sub>alg</sub> has a *highest weight* structure: it contains *standard objects* which are compact generators, and *costandard objects* which are their right orthogonals. It suffices to show that (co)standards map to (co)standards under  $J_*^{\text{KM}}$ .

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Standards

Costandards

 $\hat{\mathfrak{g}}_{\kappa}$ -mod<sup>*I*</sup><sub>ren</sub>  $\Omega$ -FactMod<sub>alg</sub>

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Wakimoto modules are the  $\frac{\infty}{2}$ -analogues of Verma modules.

At generic *c*,  $M_{\mathrm{KM}}^{!,\check{\lambda}}$  becomes the *affine Verma* module  $\mathrm{Ind}_{\mathrm{Lie}(I)}^{\hat{\mathfrak{g}}_{\kappa}}(\mathbb{C})$ , and  $M_{\mathrm{KM}}^{*,\check{\lambda}}$  becomes the dual affine Verma module.

Our choice is made such that  $\operatorname{Hom}_{\hat{\mathfrak{g}}_{\kappa}\operatorname{-mod}_{\operatorname{ren}}'}(M_{\operatorname{KM}}^{!,\tilde{\lambda}},N)$  gives the  $\check{\lambda}$ -component of  $C^{\frac{\infty}{2}}(N)$ . It follows from definition that  $M_{\operatorname{KM}}^{*,\check{\lambda}}$  are right orthogonals to  $M_{\operatorname{KM}}^{!,\check{\lambda}}$  and  $J_{*}^{\operatorname{KM}}(M_{\operatorname{KM}}^{*,\check{\lambda}}) \simeq M_{\operatorname{fact}}^{*,\check{\lambda}}$ .

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To show  $M_{\mathrm{KM}}^{!,\check{\lambda}} \mapsto M_{\mathrm{fact}}^{!,\check{\lambda}}$  it suffices to compute the \*-fiber of  $M_{\mathrm{KM}}^{!,\check{\lambda}}$  at every  $\check{\mu}x$ .

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To show  $M_{\mathrm{KM}}^{!,\check{\lambda}} \mapsto M_{\mathrm{fact}}^{!,\check{\lambda}}$  it suffices to compute the \*-fiber of  $M_{\mathrm{KM}}^{!,\check{\lambda}}$  at every  $\check{\mu}x$ . Using contraction principle, this can be done by:

- Placing another *costandard* object  $M^{*,2\check
  ho-\check\mu}_{\mathsf{K}\mathsf{M}}$  at  $\infty\in\mathbb{P}^1;$
- !-pushing  $J_*^{\text{KM}}(\mathcal{M}_{\text{KM}}^{!,\check{\lambda}}, \mathcal{M}_{\text{KM}}^{*,2\check{\rho}-\check{\mu}})_{0,\infty}|_{\text{tot.deg.}=2\check{\rho}}$  along the Abel-Jacobi map AJ :  $\mathfrak{Conf}_{0,\infty} \to \text{Bun}_{\check{\mathcal{T}}}(\mathbb{P}^1)$ ;
- Pairing with the (pushforward of) dualizing sheaf of the  $(\omega_{\mathbb{P}^1}^{1/2})^{2\check{
  ho}}$ -component of  $\operatorname{Bun}_{\check{T}}(\mathbb{P}^1)$ .

#### Localization

Set  $\operatorname{Bun}_{G}(\mathbb{P}^{1})_{0,\infty} := \operatorname{Bun}_{G}(\mathbb{P}^{1}) \times_{(\operatorname{pt}/G \times \operatorname{pt}/G)} (\operatorname{pt}/B \times \operatorname{pt}/B)$ . There exists a *localization* functor

$$\mathsf{Loc}_{\mathcal{G}}^{0,\infty}:\hat{\mathfrak{g}}_{\kappa}\operatorname{\mathsf{-mod}}^{\prime}\otimes\hat{\mathfrak{g}}_{\kappa}\operatorname{\mathsf{-mod}}^{\prime}\to\mathsf{DMod}_{\kappa}(\mathsf{Bun}_{\mathcal{G}}(\mathbb{P}^{1})_{0,\infty}),$$

where the !-fiber at the trivial bundle is given by conformal block of the two modules (placed at 0 and  $\infty$ ) over  $\mathbb{P}^1$ .

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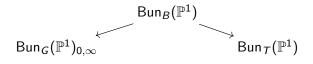
where the !-fiber at the trivial bundle is given by conformal block of the two modules (placed at 0 and  $\infty$ ) over  $\mathbb{P}^1$ .

Work of N. Rozenblyum [Roz11] tells us that there is also a *chiral localization* functor

$$\mathsf{Loc}_{\mathcal{T},\Omega}: C^{\frac{\infty}{2}}(\mathbb{V}^0_\kappa)\operatorname{\!-Fact}\mathsf{Mod}(\mathsf{KL}_{\kappa-\kappa_{\mathsf{crit}}}(\mathcal{T})_{\mathsf{ren}}) \to \mathsf{DMod}_{\kappa-\kappa_{\mathsf{crit}}}(\mathsf{Bun}_{\mathcal{T}}(\mathbb{P}^1));$$

the !-fiber is more interesting here (intuitively, it computes conformal block with  $C^{\frac{\infty}{2}}(\mathbb{V}^0_{\kappa})$  occupying everywhere away from  $0,\infty$ ).

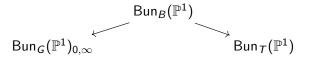
Let  $CT_* : DMod_{\kappa}(Bun_{G}(\mathbb{P}^1)_{0,\infty}) \to DMod_{\kappa-\kappa_{crit}}(Bun_{T}(\mathbb{P}^1))$  denote the !-pull-\*-push along



(followed by a  $\kappa_{crit}$  shift).

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(followed by a  $\kappa_{crit}$  shift).

The final piece of folklore that we prove is the commutativity of the following diagram:

$$\begin{split} \hat{\mathfrak{g}}_{\kappa}\operatorname{-mod}' & \xrightarrow{C^{\frac{\infty}{2}}} C^{\frac{\infty}{2}}(\mathbb{V}^{0}_{\kappa})\operatorname{-FactMod}(\mathsf{KL}_{\kappa-\kappa_{crit}}(\mathcal{T})_{ren}) \xrightarrow{\mathsf{FLE}_{\mathcal{T}}} \Omega^{\mathsf{KM}}\operatorname{-FactMod}_{\mathsf{alg}} \\ & \downarrow_{\mathsf{Loc}_{G}^{0,\infty}} & \downarrow_{\mathsf{Loc}_{\mathcal{T},\Omega}} & \downarrow_{\mathsf{AJ}_{!}} \\ \mathsf{DMod}_{\kappa}(\mathsf{Bun}_{G}(\mathbb{P}^{1})_{0,\infty}) \xrightarrow{\mathsf{CT}_{*}} \mathsf{DMod}_{\kappa-\kappa_{crit}}(\mathsf{Bun}_{\mathcal{T}}(\mathbb{P}^{1})) \xrightarrow{\mathsf{Fourier-Mukai}} \mathsf{DMod}_{(\kappa-\kappa_{crit})^{-1}}(\mathsf{Bun}_{\check{\mathcal{T}}}(\mathbb{P}^{1})) \end{split}$$

from which the \*-fibers can be computed.

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