

Frobenius-twisted conjugacy classes of loop groups

& Demazure product of Iwahori-Weyl groups

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§1. σ -Conjugacy classes of loop groups.

F non arch local field, eg. $F = \mathbb{F}_q((\varepsilon))$.

\check{F} completion of F^{un} , eg. $\check{F} = \widehat{\mathbb{F}_q((\varepsilon))}$

G conn reductive gp / F . σ Frob on $G(\check{F})$

For this talk, for simplicity, we assume that G is split over F .

$B(G)$ set of σ -conj classes of $G(\check{F})$. $g \cdot \sigma g' = g g' \sigma(g)^{-1}$.

discrete set, classified by Kottwitz

Let \check{I} be a σ -stable Iwahori subgroup of $G(\check{F})$.

Then $G(\check{F}) = \bigsqcup_{w \in \check{W}} \check{I} w \check{I}$, \check{W} Iwahori-Weyl gp.

Under our assumption, σ acts trivially on \check{W} .

§2. $B(G)$, $B(\check{W})$, & $B(\check{W})_{str}$.

Let $B(\check{W})$ be the $(G-)$ conjugacy classes on \check{W} .

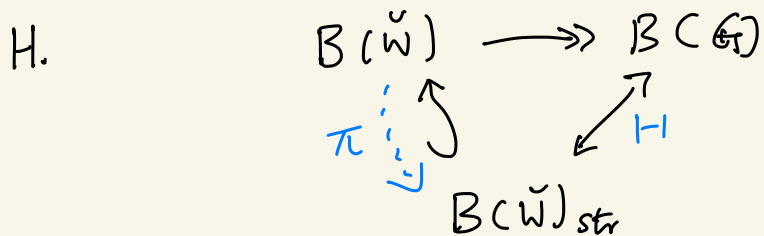
Then $B(\check{W}) \rightarrow B(G)$ *surjective, but not injective*

Def. Let $w \in \check{W}$. For $n \in \mathbb{N}$, $w^n \leftarrow$ ordinary power.

w is called *straight* if $l(w^n) = n l(w) \quad \forall n \in \mathbb{N}$.

A conjugacy class of \check{W} is called *straight* if it contains a straight elt.

$B(\check{W})_{str} \subseteq B(\check{W})$ the set of straight conjugacy classes of \check{W} .



§3. Affine Deligne - Lusztig varieties.

Def. (Rapoport) Let $b \in G(\check{F})$ and $w \in \check{W}$. Then

$$X_w(b) = \{ g \check{I} \in G(\check{F}) / \check{I}; g^{-1} b \sigma(g) \in \check{I} w \check{I} \}$$

Fact: $\{ [b] \in B(G); X_w(b) \neq \emptyset \}$ contains a unique max elt, which we denote by $[bw]$.

$[bw]$ is the unique σ -conjugacy classes in $G(\check{F})$ s.t.

$$[bw] \cap \check{I} w \check{I} \text{ is dense in } \check{I} w \check{I}.$$

← generic σ -conj
class associated
to w .

Q: What is the explicit formula of $[bw]$ and $\dim X_w(bw) = ?$

§4. Several descriptions of $[bw]$. $MT/T_0 \cong \check{W} \hookrightarrow G(\check{F})$

- Via Bruhat order [Viehmann].

$\{ [w'] ; w' \leq w \} \subseteq B(G)$ contains a unique max elt

and this max elt is $[bw]$.

- Via $B(\check{W})_{str}$ [H.] \leftarrow involves conjugation action

$\{ \sigma \in B(\check{W})_{str} ; \sigma \leq_{\sigma} w \}$ contains a unique max elt σ_w .

and $\sigma_w \leftrightarrow [bw]$

- Via \mathcal{O} -Hecke alg \leftarrow involves conjugation action
(skip here)

§5. Demazure product. (monoid product)

Define $*$: $\check{W} \times \check{W} \rightarrow \check{W}$ by

$$w * w' = ww' \quad \text{if } l(ww') = l(w) + l(w')$$

For simple
reflections s ,

$$w * s = \begin{cases} ws & \text{if } ws > w \\ w & \text{if } ws < w. \end{cases}$$

/ in particular
 $s * s = s$

Then $(\check{W}, *)$ is a monoid.

$*$ occurs in

$$- \overline{\check{I} w \check{I}} \cdot \overline{\check{I} w' \check{I}} = \overline{\check{I} (w * w') \check{I}}$$

- Lusztig's theory of total positivity

- Lusztig-Vogan Hecke alg & involutions in Weyl grps.

§6. Main result.

gen σ -conj class in $G(F)$ w.r.t. w .

$$\text{Th (H.)} \quad \dim X_w(bw) = \ell(w) - \lim_{n \rightarrow \infty} \frac{\ell(w^{*,n})}{n}$$

\uparrow
in \check{w}

Here $w^{*,n} = \underbrace{w * w * \dots * w}_{n \text{ times}}$ Demazure power.

$w^n = w \cdot w \dots w$
ordinary power.

Rmk. If w is straight, then $w^n = w^{*,n}$ and RHS = 0.

Also $\dim X_w(bw) = 0$

In general, $\ell(w) - \lim_{n \rightarrow \infty} \frac{\ell(w^{*,n})}{n}$ estimates the "non-straightness" of w .

||
 $\dim X_w(bw)$

§7. Idea of proof.

(char O result \Rightarrow char F result)

total positivity

Frobenius-conjugacy.

Let $g \in \check{I} w \check{I}$ generic elt. Then $\exists h \in G(\check{F})$ s.t.

$h^{-1} g \sigma(h) \in \check{I} w' \check{I}$, where w' is a straight elt in

$$\mathcal{O}_w \in B(G) \leftrightarrow [bw]$$

Then $(h^{-1} g \sigma(h))^{\sigma, n} := (h^{-1} g \sigma(h)) \sigma(h^{-1} g \sigma(h)) \dots \sigma^{n-1}(h^{-1} g \sigma(h))$

$$\parallel \in \check{I} w'^n \check{I} \quad \text{since } w' \text{ is straight.}$$

$$\underline{h^{-1} g^{\sigma, n} \sigma^n(h)}.$$

$$g^{\sigma, n} \in (\check{I} w \check{I}) \cdot (\check{I} w \check{I}) \dots (\check{I} w \check{I}) \subseteq \overline{\check{I} w^{*, n} \check{I}}$$

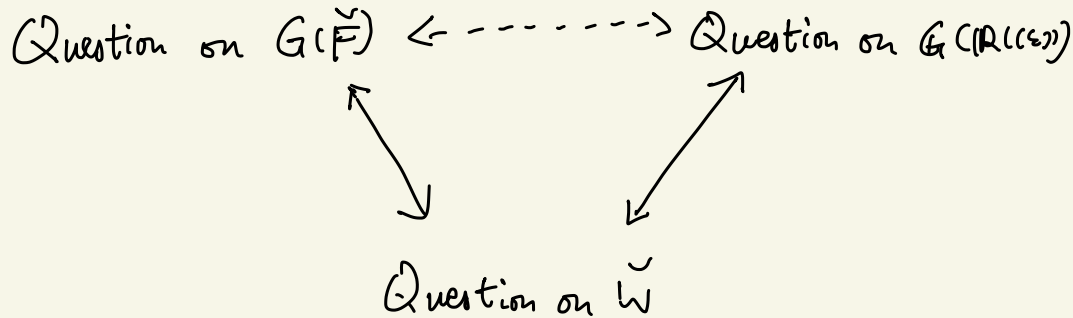
Q: For generic $g \in \check{I} w \check{I}$, does $g^{\mathbb{F}, n} \in \check{I} w^{*, n} \check{I}$?

Trick: Lusztig's total positivity

Consider $G(\mathbb{R}(C_2))$ instead of $G(\check{\mathbb{F}})$.

We may take "generic" element $g \in \check{I} w \check{I}$ to be in U_w , so

Then $g^n \in U_{w^{*, n}}^- \subset \check{I} w^{*, n} \check{I}$.



§ 8. Explicit formula on $*$. (for almost all elements)

Fomin-Gelfand-Postnikov : Quantum Brauer graph. on $W_0 \leftarrow$ finite Weyl gr

For $x, y \in W_0$, def $\text{wt}(x, y)$, an element in the coroot lattice.

$\check{W} = W_0 \times \Lambda \leftarrow$ coweight lattice.

H. If $\lambda_1, \lambda_2 \in \Lambda_+$ s.t. $\langle \lambda_i, \alpha \rangle \geq 2 \quad \forall$ simple root α .

Then $\forall x_i, y_i \in W_0, (x_1 t^{\lambda_1} y_1) * (x_2 t^{\lambda_2} y_2) = x_1 t^{\lambda_1 + \lambda_2 - \text{wt}((y_1, x_2)^{-1}, 1)} y_2$

§9. Application. (assume that $w_0 = -1$)

Lusztig-Vogan: $\check{W} \rightarrow \check{W}$, $w \mapsto w * w^{-1}$

$$\rightsquigarrow \Lambda \rightarrow \Lambda_+$$

coweight dom coweight.

$\lambda \mapsto \mu$, where $t^\lambda * t^{-\lambda} \in W_0 t^\mu w_0$.

Q: What is the explicit formula for $\Lambda \rightarrow \Lambda_+$?

H. If $\lambda = x(\lambda_0)$, where $\lambda_0 \in \Lambda_+$, $x \in W_0 I(\lambda_0)$

then $\mu = 2\lambda_0 - wt((x^{-1} \Delta(xw_0))^{-1}, 1)$