# Frobenius-twisted conjugacy classes of loop groups

# & Demazure product of Iwahori-Weyl groups

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## §1. $\sigma$-Conjugacy classes of loop groups.

$F$ non arch local field, eg. $F = \mathbb{F}_q((\varepsilon))$.

$\breve{F}$ completion of $F^{un}$, eg. $\breve{F} = \bar{\mathbb{F}}_q((\varepsilon))$

$G$ conn reductive gp $/F$. $\sigma$ Frob on $G(\breve{F})$

For this talk, for simplicity, we assume that $G$ is split over $F$.

$B(G)$ set of $\sigma$-conj classes of $G(\breve{F})$. $g \cdot_\sigma g' = g g' \sigma(g)^{-1}$.

discrete set, classified by Kottwitz

Let $\breve{I}$ be a $\sigma$-stable Iwahori sbgp of $G(\breve{F})$.

Then $G(\breve{F}) = \coprod_{w \in \breve{W}} \breve{I} w \breve{I}$, $\breve{W}$ Iwahori-Weyl gp.

Under our assumption, $\sigma$ acts trivially on $\hat{W}$.

## § 2. $B(\mathbb{G})$, $B(\breve{W})$, & $B(\breve{W})_{str}$.

Let $B(\breve{W})$ be the ($\sigma$-) conjugacy classes on $\breve{W}$.

Then $B(\breve{W}) \longrightarrow B(\mathbb{G})$ surjective, but not injective

Def. Let $w \in \breve{W}$. For $n \in \mathbb{N}$, $w^n \longleftarrow$ ordinary power.

$w$ is called straight if $\ell(w^n) = n\,\ell(w) \quad \forall n \in \mathbb{N}$.

A conjugacy class of $\breve{W}$ is called straight if it contains a straight elt.

$B(\breve{W})_{str} \subseteq B(\breve{W})$ the set of straight conjugacy classes of $\breve{W}$.

H.

$$\begin{tikzcd}
B(\breve{W}) \arrow[r, two heads] \arrow[d, dashed, bend right, "\pi"'] & B(\mathbb{G}) \\
B(\breve{W})_{str} \arrow[u, hook, bend right] \arrow[ur, leftrightarrow, "1\text{-}1"'] &
\end{tikzcd}$$

## §3. Affine Deligne-Lusztig varieties

Def. (Rapoport) Let $b \in G(\breve{F})$ and $w \in \breve{W}$. Then

$$X_w(b) = \{ g\breve{I} \in G(\breve{F})/\breve{I} ; \ g^{-1} b \sigma(g) \in \breve{I} w \breve{I} \}.$$

Fact: $\{ [b] \in B(G) ; X_w(b) \neq \emptyset \}$ contains a unique max elt, which we denote by $[b_w]$.

generic $\sigma$-conj class associated to $w$.

$[b_w]$ is the unique $\sigma$-conjugacy classes in $G(\breve{F})$ s.t. $[b_w] \cap \breve{I} w \breve{I}$ is dense in $\breve{I} w \breve{I}$.

Q: What is the explicit formula of $[b_w]$ and $\dim X_w(b_w) = ?$

## §4. Several descriptions of $[b_w]$.

$N(T)/T_0 \simeq \breve{W} \hookrightarrow G(\breve{F})$

- Via Bruhat order [Viehmann].

$\{[w'] ; w' \leq w\} \subseteq B(G)$ contains a unique max elt

and this max elt is $[b_w]$.

- Via $B(\breve{W})_{str}$ [H.] ← involves conjugation action

$\{\mathcal{O} \in B(\breve{W})_{str} ; \mathcal{O} \leq_\sigma w\}$ contains a unique max elt $\mathcal{O}_w$.

and $\mathcal{O}_w \longleftrightarrow [b_w]$

- Via 0-Hecke alg ← involves conjugation action

(skip here)

## §5. Demazure product. (monoid product)

Define $* : \check{W} \times \check{W} \longrightarrow \check{W}$ by

$$w * w' = w w' \quad \text{if } \ell(ww') = \ell(w) + \ell(w')$$

For simple reflections $s$, $\quad w * s = \begin{cases} ws & \text{if } ws > w \\ w & \text{if } ws < w. \end{cases}$ / in particular $s * s = s$

Then $(\check{W}, *)$ is a monoid.

$*$ occurs in

- $\overline{\check{I} w \check{I}} \cdot \overline{\check{I} w' \check{I}} = \overline{\check{I} (w * w') \check{I}}$
- Lusztig's theory of total positivity
- Lusztig-Vogan Hecke alg & involutions in Weyl grps.

## §6. Main result.

gen σ-conj class in $G(F)$ w.r.t. $w$.

Th (H.) $$\dim X_w(b_w) = \ell(w) - \lim_{n\to\infty} \frac{\ell(w^{*,n})}{n}$$

in $\breve{W}$

Here $w^{*,n} = \underbrace{w * w * \cdots * w}_{n \text{ times}}$ Demazure power.

$w^n = w \cdot w \cdots w$ ordinary power.

Rmk. If $w$ is straight, then $w^n = w^{*,n}$ and $RHS = 0$.

Also $\dim X_w(b_w) = 0$

In general, $\ell(w) - \lim_{n\to\infty} \frac{\ell(w^{*,n})}{n}$ estimates the "non-straightness" of $w$.

$$\ell(w) - \lim_{n\to\infty} \frac{\ell(w^{*,n})}{n} = \dim X_w(b_w)$$

§7. Idea of proof. ( char 0 result $\Longrightarrow$ char $p$ result)

total positivity Frobenius-conjugacy.

Let $g \in \check{I} w \check{I}$ generic elt. Then $\exists\, h \in G(\check{F})$ s.t.

$h^{-1} g \sigma(h) \in \check{I} w' \check{I}$, where $w'$ is a straight elt in

$\mathcal{O}_w \in B(G) \longleftrightarrow [\dot{w}]$

Then $(h^{-1} g \sigma(h))^{\sigma, n} := (h^{-1} g \sigma(h))\, \sigma(h^{-1} g \sigma(h)) \cdots \sigma^{n-1}(h^{-1} g \sigma(h))$

$\in \check{I} w'^{n} \check{I}$ since $w'$ is straight.

$\| $

$\underline{h^{-1}}\, g^{\sigma, n}\, \sigma^{n}\underline{(h)}$.

$g^{\sigma, n} \in (\check{I} w \check{I}) \cdot (\check{I} w \check{I}) \cdots (\check{I} w \check{I}) \subseteq \overline{\check{I} w^{*, n} \check{I}}$

Q: For generic $g \in \breve{I} w \breve{I}$, does $g^{\sigma, n} \in \breve{I} w^{*,n} \breve{I}$ ?

Trick: Lusztig's total positivity

Consider $G(\mathbb{R}((\epsilon)))$ instead of $G(\breve{F})$.

We may take "generic" element $g \in \breve{I} w \breve{I}$. to be in $U^-_{w,>0}$.

Then $g^n \in U^-_{w^{*,n},>0} \subset \breve{I} w^{*,n} \breve{I}$.

$$
\begin{tikzcd}
\text{Question on } G(\breve{F}) \arrow[rr, dashed, leftrightarrow] \arrow[dr, leftrightarrow] & & \text{Question on } G(\mathbb{R}((\epsilon))) \arrow[dl, leftrightarrow] \\
& \text{Question on } \breve{W} &
\end{tikzcd}
$$

## § 8. Explicit formula on $*$. (for almost all elements)

Fomin-Gelfand-Postnikov : Quantum Bruhat graph on $W_0$ $\leftarrow$ finite Weyl gp

For $x, y \in W_0$, def $wt(x,y)$, an element in the coroot lattice.

$\check{W} = W_0 \ltimes \Lambda$ $\leftarrow$ coweight lattice

H. If $\lambda_1, \lambda_2 \in \Lambda_+$ s.t. $\langle \lambda_i, \alpha \rangle \geq 2$ $\forall$ simple root $\alpha$.

Then $\forall x_i, y_i \in W_0$, $(x_1 t^{\lambda_1} y_1) * (x_2 t^{\lambda_2} y_2) = x_1 t^{\lambda_1 + \lambda_2 - wt((y_1 \triangleleft x_2)^{-1}, 1)} y_2$

§9. Application. (assume that $w_0 = -1$)

Lusztig-Vogan: $\tilde{W} \longrightarrow \tilde{W}$, $w \mapsto w * w^{-1}$

$\rightsquigarrow \Lambda \longrightarrow \Lambda_+$

coweight  dom coweight.

$\lambda \mapsto \mu$, where $t^{\lambda} * t^{-\lambda} \in W_0 t^{\mu} W_0$.

Q: What is the explicit formula for $\Lambda \to \Lambda_+$?

H. If $\lambda = x(\lambda_0)$, where $\lambda_0 \in \Lambda_+$, $x \in W_0^{I(\lambda_0)}$

then $\mu = 2\lambda_0 - wt\left( (x^{-1} \triangleright (x w_0))^{-1}, 1 \right)$