

Langlands classification

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Topic: **representation theory of a reductive group G** over a local or finite field F .

Fundamental example: $G = \mathrm{GL}(n)$ $n \times n$ matrices.

A **parabolic subgroup** P of G is one with G/P projective. Example: Grassmann variety $X = m$ -diml subspaces of n -diml vec space = $\mathrm{GL}(n)/P_{\{m, n-m\}}$.

Parabolic subgroups are important because ANY G/H can be **deformed** to G/H_d with H_d normal in parabolic P

Original motivation: big idea from 1950s was that many representations are **parabolically induced**: sections of vector bundles over $G(F)/P(F)$.

Langlands idea from 1960s: all reps of $G(F)$ indexed by $f: W'_F \rightarrow {}^L G$; here W'_F is the Weil-Deligne group of F , and ${}^L G = {}^V G \rtimes G$; ${}^V G$ is cplx dual group, G is $\mathrm{Gal}(F)$.

MORE PRECISELY: $G//F =$ all F -forms inner to $G(F)$ \leftrightarrow action of G on based root datum (P, X^*, P^V, X^*)

\leftrightarrow action of G on dual based root datum (P^V, X^*, P, X^*)

\leftrightarrow action of G on complex dual group ${}^V G$

\leftrightarrow Langlands L-group ${}^L G = {}^V G \rtimes G$

${}^V G$ and a covering group ${}^V G^{\mathrm{can}}$ act on Langlands parameters $f \rightarrow$ stabilizer $s^{\mathrm{can}}(f)$, component group $A^{\mathrm{can}}(f)$.

COMPLETE LANGLANDS PARAMETER is pair (f, x) with x irreducible of $A^{\mathrm{can}}(f)$.

LOCAL LANGLANDS CONJECTURE: pairs ("rigid inner twist" $G(F)$, irr ρ of $G(F)) / G$ conjugacy \leftrightarrow pairs $(f, x) / {}^V G^{\mathrm{can}}$ conjugacy

This is proven by **Adams-Barbasch-V** in 1992 for $F = \mathbb{R}$, and formulated by **Tasho Kaletha** for F characteristic zero p -adic.

Details A: p-adic case

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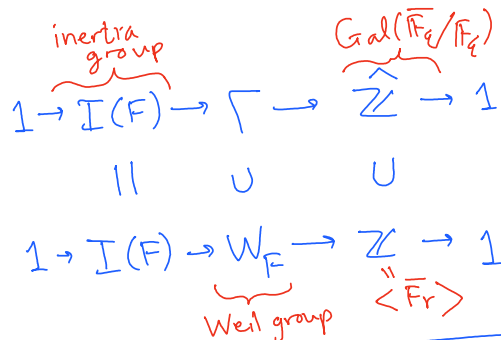
F p-adic [char 0], $\Gamma = \text{Gal}(\bar{F}/F)$

\mathcal{O}_F integers
 \mathfrak{p}_F maximal ideal

$$\mathcal{O}_F / \mathfrak{p}_F = \mathbb{F}_q$$

$\|\cdot\|: W_F \rightarrow \{q^{\mathbb{Z}}\}$ homomorphism

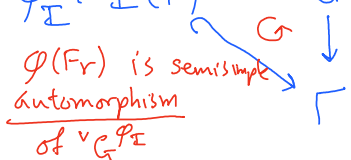
$\|\cdot\|_{\mathbb{I}_F} = 1, \|\text{Fr}\| = q$



Langlands parameter starts with $\varphi_{\mathbb{I}}: \mathbb{I}(F) \rightarrow {}^L G = {}^v G \rtimes \Gamma$

${}^v G \supset$ reductive subgroup $({}^v G, \varphi_{\mathbb{I}})$

${}^v G^{\varphi_0}$ = fixed points of $\varphi(\text{Fr})$
 on v



$\varphi_0: W_F \rightarrow {}^L G$
 generated by $\varphi_{\mathbb{I}}, \varphi(\text{Fr})$

Details B: p-adic case

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Langlands parameter starts with $\varphi_I: I(F) \rightarrow L_G = {}^vG \rtimes \Gamma$

${}^vG \supset$ reductive subgroup $({}^vG, \varphi_I)$ $\varphi(Fr)$ is semisimple automorphism of vG

$\varphi_0: W_F \rightarrow L_G$ generated by $\varphi_I, \varphi(Fr)$

${}^vG^{\varphi_0} = \text{fixed points of } \varphi(Fr) \text{ on } {}^vG^{\varphi_I}$ REDUCTIVE

\prod

${}^vG^{\varphi_0, q} = \langle {}^vG^{\varphi_0}, \text{eigenspaces of } \varphi(Fr) \text{ on } {}^vG^{\varphi_I} \text{ with eigenvalues POWERS OF } q \rangle$

${}^vG^{\varphi_0, q}$ is \mathbb{Z} -graded; 0 subspace is ${}^vG^{\varphi_0, 1}$

${}^vG^{\varphi_0, q} = q$ -eigenspace of $\varphi(Fr)$ on ${}^vG^{\varphi_I}$

(Weil-Deligne) Langlands parameter $\varphi: W'_F \rightarrow L$

is $\varphi_0: W_F \rightarrow L_G$ AND

$N \in {}^vG^{\varphi_0, 1}$

is prehomog vec space of nilp elts for ${}^vG^{\varphi_0}$

, nilpotent element for ${}^vG^{\varphi_0, q}$

Details: real case

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$$\Gamma = G \ltimes \mathfrak{a}(\mathbb{C}/\mathbb{R}) = \{1, \sigma\} \quad L_G = \vee G \rtimes \Gamma = \vee G \rtimes (\vee G \rtimes \sigma)$$

What matters about semisimple $\lambda \in \vee \mathfrak{g}$ are integer eigenspaces of $\text{ad}(\lambda)$

$\vee_G^{e(\lambda)}$

pseudo Levi in $\vee G$,

analogous to $\vee_G^{\rho, \mathbb{Z}} = \langle \mathbb{Z} \text{-eigenspaces of } \rho(\Gamma) \text{ on } \vee_G^{\rho, \mathbb{Z}} \rangle$

$\vee_{\mathfrak{g}}^{e(\lambda)}$

\mathbb{Z} -graded by eigenspaces of $\text{ad}(\lambda)$

\vee_G^{λ}

Levi subgp of $\vee_G^{e(\lambda)}$ and $\vee G$

Harish-Chandra

$$e: \vee \mathfrak{g} \rightarrow \vee G, \quad e(\lambda) = \exp(2\pi i \lambda)$$

semisimple conjugacy class $G \cdot \lambda \subset \vee \mathfrak{g}$

infinitesimal character for \mathfrak{g}

$e(\lambda) \in Z(\vee G) \iff$ infinitesimal character is INTEGRAL

CANONICAL FLAT of λ is

$$\Lambda = \lambda + \left[\sum_{j \in \{1, 2, 3, \dots\}} j \cdot \text{eigenspace of } \text{ad}(\lambda) \right] \subset G \cdot \lambda$$

e constant

on Δ $e(\Lambda) = e(\lambda)$

$\vee \mathfrak{n}(\lambda)$ nilpotent

LANGLANDS PARAMETER is (j, Λ) $j \in L_G \setminus \vee G$

$\text{Ad}(j) = \text{involutive}$

$\Lambda \subset \vee \mathfrak{g}$ canonical flat $j^2 = e(\Lambda)$

What happened to parabolic induction?

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Γ -fixed character ξ of

$H(F)$ torus
in $G(F)$

\rightsquigarrow

rational parabolic
 $P(F) = L(F) U(F)$
 \cup
 $H(F)$

\updownarrow

Γ -fixed one-parameter subgroup $\mathbb{C}^\times \xrightarrow{\xi} {}^\vee G$

${}^\vee L = \text{Cent}_G(\xi(\mathbb{C}^\times))$

rational