Langlands classification

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Topic: representation theory of a reductive group G over a local or finite field F.

Fundamental example: G=GL(n) n x n matrices.

A parabolic subgroup P of G is one with G/P projective. Example: Grassmann variety X = m-diml subspaces of n-diml vec space = GL(n)/P_{m,n-m}.

Parabolic subgroups are important because ANY G/H can be deformed to G/Hd with Hd normal in parabolic P

Original motivation: big idea from 1950s was that many representations are parabolically induced: sections of vector bundles over G(F)/P(F).

Langlands idea from 1960s: all reps of G(F) indexed by f: $W'_F \dashrightarrow {}^LG$; here W'_F is the Weil-Deligne group of F, and ${}^LG = {}^\vee G \rtimes \Gamma$; ${}^\vee G$ is cplx dual group, Γ is Gal(F).

MORE PRECISELY: G//F = all F-forms inner to G(F) <---> action of Γ on based root datum $(\Pi, X^*, \Pi^\vee, X_*)$

<---> action of Γ on dual based root datum $(\Pi^\vee, X_*, \Pi, X^*)$

<---> action of Γ on complex dual group ${}^\vee G$

<---> Langlands L-group ${}^LG = {}^\vee G \rtimes \Gamma$

${}^\vee G$ and a covering group ${}^\vee G^{can}$ act on Langlands parameters f ---> stabilizer $S^{can}(f)$, component group $A^{can}(f)$.

COMPLETE LANGLANDS PARAMETER is pair (f,x) with x irreducible of $A^{can}(f)$.

LOCAL LANGLANDS CONJECTURE: pairs ("rigid inner twist" G(F), irr p of G(F))/G conjugacy <----> pairs (f,x) / ${}^\vee G^{can}$ conjugacy

This is proven by Adams-Barbasch-V in 1992 for F=R, and formulated by Tasho Kaletha for F characteristic zero p-adic.

Details A: p-adic case

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F p-adic [char 0], $\Gamma = Gal(\bar{F}/F)$

$\cup$

$\mathcal{O}_F$ integers

$\cup$

$\mathfrak{p}_F$ maximal ideal

$\boxed{\mathcal{O}_F/\mathfrak{p}_F = \mathbb{F}_q}$

$$\begin{array}{ccccccccc} 1 & \to & \overbrace{I(F)}^{\text{inertia group}} & \to & \Gamma & \to & \overbrace{\hat{\mathbb{Z}}}^{Gal(\bar{\mathbb{F}}_q/\mathbb{F}_q)} & \to & 1 \\ & & \| & & \cup & & \cup & & \\ 1 & \to & I(F) & \to & \underbrace{W_F}_{\text{Weil group}} & \to & \mathbb{Z} & \to & 1 \end{array}$$

$\mathbb{Z} = \langle Fr \rangle$

$\|\cdot\| : W_F \to \{q^{\mathbb{Z}}\}$ homomorphism

$\|\cdot\|\big|_{I_F} = 1, \quad \|Fr\| = q$

Langlands parameter starts with $\varphi_I : I(F) \to {}^LG = {}^\vee G \rtimes \Gamma$

$$\begin{array}{c} {}^LG \\ \downarrow \\ \Gamma \end{array}$$

$\varphi_0 : W_F \to {}^LG$

generated by $\varphi_I, \varphi(Fr)$

${}^\vee G \supset$ reductive subgroup ${}^\vee G^{\varphi_I}$

$\varphi(Fr)$ is semisimple automorphism of ${}^\vee G^{\varphi_I}$

${}^\vee G^{\varphi_0}$ = fixed points of $\varphi(Fr)$ on ${}^\vee G$

Details B: p-adic case

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Langlands parameter starts with $\varphi_I : I(F) \to {}^L G = {}^\vee G \rtimes \Gamma$

$\varphi(Fr)$ is semisimple automorphism of ${}^\vee G^{\varphi_I}$

${}^\vee G \supset$ reductive subgroup ${}^\vee G^{\varphi_I}$

$G \downarrow \Gamma$

$\varphi_0 : W_F \to {}^L G$ generated by φ_I, $\varphi(Fr)$

${}^\vee G^{\varphi_0}$ = fixed points of $\varphi(Fr)$ on ${}^\vee G^{\varphi_I}$ REDUCTIVE

$\cap$

${}^\vee G^{\varphi_0, q} = \langle {}^\vee G^{\varphi_0}$, eigenspaces of $\varphi(Fr)$ on ${}^\vee \mathfrak{g}^{\varphi_I}$ with eigenvalues POWERS OF $q \rangle$

${}^\vee \mathfrak{g}^{\varphi_0, q}$ is $\mathbb{Z}$-graded; 0 subspace is ${}^\vee \mathfrak{g}^{\varphi_0}$.

$\boxed{{}^\vee \mathfrak{g}^{\varphi_0, q}_1 = q\text{-eigenspace of } \varphi(Fr) \text{ on } {}^\vee \mathfrak{g}^{\varphi_I}}$

is prehomog vec space of nilp elts for ${}^\vee G^{\varphi_0}$

(Weil-Deligne) Langlands parameter $\varphi : W_F' \to {}^L$

is $\varphi_0 : W_F \to {}^L G$ AND $N \in {}^\vee \mathfrak{g}^{\varphi_0, q}_1$, nilpotent element for ${}^\vee \mathfrak{g}^{\varphi_0, q}$

Details: real case

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$\Gamma = Gal(\mathbb{C}/\mathbb{R}) = \{1, \sigma\}$ $\quad {}^L G = {}^\vee G \rtimes \Gamma = {}^\vee G \amalg ({}^\vee G \sigma)$

$e : {}^\vee\mathfrak{g} \to {}^\vee G$, $e(\lambda) = \exp(2\pi i \lambda)$

What matters about semisimple $\lambda \in {}^\vee\mathfrak{g}$ are integer eigenspaces of $ad(\lambda)$

Harish-Chandra

$\boxed{\text{semisimple conjugacy class } {}^\vee G \cdot \lambda \subset {}^\vee\mathfrak{g}}$

$\boxed{\text{infinitesimal character for } \mathfrak{g}}$

$\boxed{e(\lambda) \in Z({}^\vee G) \Longleftrightarrow \text{infinitesimal character is INTEGRAL}}$

${}^\vee G^{e(\lambda)}$ pseudolevi in ${}^\vee G$,

analogous to ${}^\vee G^{\varphi_0, q} = \langle q^{\mathbb{Z}}\text{-eigenspaces of } \varphi(Fr) \text{ on } {}^\vee G^{\varphi_I} \rangle$

${}^\vee\mathfrak{g}^{e(\lambda)}$ $\mathbb{Z}$-graded by eigenspaces of $ad(\lambda)$

${}^\vee G^\lambda$ Levi subgp of ${}^\vee G^{e(\lambda)}$ and ${}^\vee G$

CANONICAL FLAT of λ is

$\Lambda = \lambda + \left[\sum_{j \in \{1,2,3\ldots\}} j\text{-eigenspace of } ad(\lambda)\right] \subset G \cdot \lambda$

e constant on Λ: $e(\Lambda) = e(\lambda)$

$\underbrace{}_{{}^\vee\mathfrak{n}(\lambda) \text{ nilpotent}}$

LANGLANDS PARAMETER is (j, Λ) $\quad j \in {}^L G \setminus {}^\vee G$ $\quad \Lambda \subset {}^\vee\mathfrak{g}$ canonical flat $\boxed{j^2 = e(\Lambda)}$

$Ad(j)$ = involuti

What happened to parabolic induction?

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Γ-fixed character ξ of $\boxed{H(F) \text{ torus in } G(F)}$ $\rightsquigarrow$ $\boxed{\text{rational parabolic } P(F) = L(F)U(F) \supset H(F)}$

$\updownarrow$

Γ-fixed one-parameter subgp $\boxed{\mathbb{C}^\times \xrightarrow{\xi} {}^\vee G}$

$$\boxed{{}^\vee L = \mathrm{Cent}_{{}^\vee G}(\xi(\mathbb{C}^\times))}$$