

Elliptic stable envelopes
and 3d mirror symmetry.

based on joint works with
Andrey Smirnov.

K -theoretic stable env.

X -symp. res of. sing $\xi \in \text{Lie } A$ \mathcal{C}

(1)

$$A \subset \text{Aut}(X, \omega) \quad \text{Attr}(F) = \{x \in X \mid \lim_{t \rightarrow \infty} e^{-t\xi} x = F\}$$

$$\text{Attr}^{\text{full}} \subset X^A \times X$$

$$\text{Stab}_\chi \in K(X^\Delta \times X) \quad \mathcal{L} \in \text{Pic}_A(X) \otimes R$$

$$T^{1/2} \oplus \nexists T^{1/2} = T$$

1) $\text{Supp Stab} \subset \text{Attr}^{\text{full}}$

2) Near diagonal $\text{Stab} = \theta_{\text{Attr}} \otimes \text{weight}$

3) $F_2 < F_1 \quad \deg_A \text{Stab}_\chi|_{F_1 \times F_2} \otimes \mathcal{L} \subset \deg_A \text{Stab}_\chi|_{F_2 \times F_2}$

$$\boxtimes \text{ Hilb}(\mathbb{C}^2, n) \rightarrow \bullet \circlearrowleft$$

$$U_h(\mathfrak{gl}(2)) \quad e_{mn} \quad \frac{m}{h} \quad \text{Pic}(X) \otimes R = R.$$

Elliptic

Aganagic, Okounkov : $\text{Stab}_{X, e}^{E\parallel}(\lambda)$ Fixed point

$$E_T(x) = E\parallel_T(x) \times \mathcal{E}_{\text{Pic}(x)} \quad \mathcal{E}_{\text{Pic}(x)} = E \otimes \mathbb{Z}$$

$$T_{\lambda\mu}(a, z) = \text{Stab}_{\lambda}^{E\parallel} / \mu$$

$$\theta(x) = (x^{\frac{1}{2}} - x^{-\frac{1}{2}}) \prod_{j \geq 1} (1 - x q^j)(1 - q^j/x)$$

$$E = \mathbb{C}^*/q\mathbb{Z}$$

$S(a, z)$ is balanced in a

$$S(a, z) = \sum \frac{\theta(a^l \dots)}{\theta(a^e \dots)} \quad a^l = a_1^{e_1} \dots a_k^{e_k}$$

Ex

$$S(a, z) = \frac{\theta(a z)}{\theta(a) \theta(z)} + \frac{\theta(a^2 z) \theta(a)}{\theta(a^4) \theta(a z)}$$

Define

$$S(a, z) = \frac{T_\lambda \mu(a, z)}{\theta(N_\lambda^-)}$$

Proposition

- 1) S is balanced in a
- 2) ... in z

- 3) it has poles separately in a, z

$$\frac{1}{\theta(a^2) \theta(z)}, \text{ but not } \frac{1}{\theta(az)}$$

} for Nakajima varieties

Lemma

s-balanced

$$\sqrt{a} \lim_{q \rightarrow 0} s(aq^w, z) \in \mathbb{C}(a, z)$$

3d mirror symmetry

X - smooth resolution of singularities $\theta \in H^2(X, \mathbb{R})$

$$\begin{matrix} \uparrow \\ T \supset A \end{matrix}$$

$$T = A \times \mathbb{C}_\hbar^+$$

$$K = \text{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{C}^* \quad \text{K\"ahler torus} \quad (z)$$

$$\theta \in \text{Lie}_{\mathbb{R}}(K)$$

X' - dual variety ; $T', A', K', \hbar', \theta'$

Def. X, X' - symplectic dual if

1) $K: A \xrightarrow{\sim} K', K \rightarrow A', \mathbb{C}_A^+ \xrightarrow{*} \mathbb{C}_{K'}^+$

$$\begin{cases} G = dK^{-1}(\theta') \in \text{Lie}_R A \\ G' = dF'(\theta) \in \text{Lie}_R A' \end{cases} \quad \text{- define attr, repelling directions.}$$

2) bijection $X^A \xrightarrow{\sim} (X')^{A'}$ inverting the partial ordering
on fixed points.

$$p \mapsto p'$$

3) $X = X \times \{p'\} \rightarrow X \times X' \leftarrow \{p\} \times X' = X'$

$$E_T(x) \xrightarrow{i_{p!}} E_{T \times T'}(x \times x') \xleftarrow{i_p} E_{T'}(x')$$

$$\underline{\text{Stab}}_G^{X, E^I}(p) = \theta(N_{p^I}^-) \cdot \text{Stab}_G^{X, E^I}(p)$$

$$\text{Stab}_G^{X, E^I}(p)|_p = \text{Stab}_{G^!}^{X^!, E^I}(p^!)|_{p^!} = \theta(N_p^-) \cdot \theta(N_{p^!}^-)$$

$$\exists \text{ a line bundle } \mathcal{L}, \quad , \quad m \in \Gamma(\mathcal{L})$$

\downarrow
 $E^I_{T_x T^!}(x, x^!)$

$$i_{p^!}^*(m) = \underline{\text{Stab}}_G^{X, E^I}(p), \quad i_p^*(m) = \underline{\text{Stab}}_G^{X^!, E^I}(p^!)$$

$$\tilde{T}_{p,r}^X(z, a) = k^*(\tilde{T}_{r^!, p^!}^{X^!}(z, a))$$

Theorem (Aganagic - Okounkov) For $s \in H^2(X, \mathbb{R})$

$$\lim_{q \rightarrow 0} \tilde{T}_{p,r}(zq^s, a)$$

- is a piecewise function of s .
- changes when s crosses certain hyperplane arrangement $\text{Walls}(X) \subset H^2(X, \mathbb{R})$
- For generic $s \in H^2 \setminus \text{Walls}(X)$

$$\lim_{q \rightarrow 0} \tilde{T}_{p,r}(zq^s, a) = \tilde{A}_{p,r}^{[s], X}$$

$$\tilde{A}_{p,r}^{[s], X} = \frac{\text{Stab}^{k\text{-th}}(p)|_r}{\text{Stab}^{k\text{-th}}(r)|_r} \quad - k\text{-th. stable envelopes}$$

$$\omega = (w_1, \dots, w_n) \in \text{Lie}_{\mathbb{R}}(A) \cong \mathbb{R}^n$$

$$\omega = e^{2\pi i w} = (e^{2\pi i w_1}, \dots, e^{2\pi i w_n}) \in A$$

$$\gamma_\omega = \langle \omega \rangle$$

$$\text{Res}(x) = \{ w \mid x^{v_w} \neq x^A \} \subset \text{Lie}_{\mathbb{R}}(A)$$

$$\text{Res}(x) = \text{Walls}(x^!) , \quad \text{Res}(x^!) = \text{Walls}(x)$$

Theorem (A.Smirnov - Y.K) $s \in H^2(X, \mathbb{R})$, s' - regular
slope in neighborhood of s .

$$\lim_{q \rightarrow 0} \tilde{\tau}(zq^s, a) = \tilde{z}'' \cdot \tilde{x}^{[s]}, x \\ \equiv \text{the limit for } s'$$

$$\tilde{z}'' = L_s \cdot z' \cdot L_s^{-1}$$

classical
multibl.

by $L_s \in \text{Pic}(X) \otimes \mathbb{Q}$

$$z' \in \mathbb{Q}(z, t)$$

$$0 \subset U_0 \subset H^2(X, \mathbb{R}) \quad \text{Walls}_0$$

$$(U_0 \setminus \text{Walls}_0(X)) = \bigsqcup \text{chambers}.$$

$\mathcal{D}_+(X)$ - ample

$\mathcal{D}_-(X)$ - anti-ample

$S \in H^2(X, \mathbb{R})$ not regular, then

$$S \in \text{Walls}(X) = \text{Res}(X!) \rightsquigarrow \boxed{Y_S \subset X!}$$

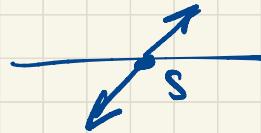
Theorem (A. Sunnar - Y.K.) $S \in \text{Walls}(X), \epsilon \in \mathcal{D}_+(X)$

$$S' = S + \epsilon$$

$$\tilde{\mathcal{Z}}'_{r,p} = \underset{(t)}{H} \tilde{\mathcal{Z}} H^{-1}$$

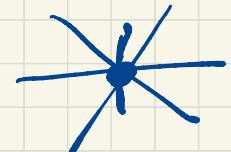
$$\tilde{\mathcal{Z}}'_{p!, t!} = \text{Stab}_{G!}^{\mathcal{D}_+(Y_S), k+t!} (P!).$$

Wall-crossing operators.



$$R^X(s, G) = \text{Stab}_G^{[s+\varepsilon], X, k} \circ (\text{Stab}_G^{[s-\varepsilon], X, k})^{-1}$$

$\text{End } (K_T(X^A)_{\text{localized}})$.



$$R^X(0, G) = \prod R^X(s, G)$$

Theorem (A. Smirnov, Y.E.) $s \in H^2(X, \mathbb{R})$, $Y_s \subset X^!$

$$R^X(s, G) = \frac{\partial_s H R^{Y_s}(0, -G^!)}{H^+ \partial_s^+}$$

Proof

$$\lim_{q \rightarrow 0} \text{Stab}^{E^!} = \sum_{Y_s} \text{Stab}_X^{s+\varepsilon, k} = \sum_X \text{Stab}_X^{s-\varepsilon, k}$$

Coroll. $p^!, r^!$ belong to different components of \mathcal{V}_S , then

$$R_{p,r}^X = 0.$$

Main application: $X = \text{Hilb}(\mathbb{C}^2, n)$.

$$X = X^!$$

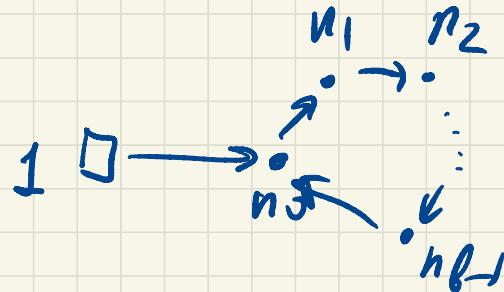
$$\text{Pic}(X) = \mathbb{Z} O(1)$$

Theorem 1) $\text{Walls}(X) = \left\{ \frac{a}{b} \in \mathbb{Q} \mid |b| \leq n \right\}$.

2) For $S = \frac{a}{b}$ $Y_S \subset X^!$

$$Y_S = \bigsqcup X(n_0, \dots, n_{B-1})$$

$$n_0 + \dots + n_{B-1} = n$$



For $\widehat{\mathfrak{gl}}(B)$ Fock module

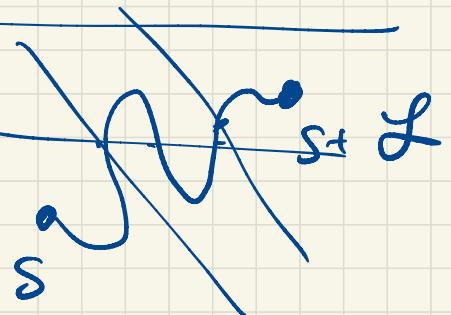
standard basis $\rightarrow \text{Stab}^{D_+, Y_S}$

co-standard $\rightarrow \text{Stab}^{D_-, Y_S}$

Theorem The well R-matrix $R^x(\frac{a}{\beta})$ coincides,
 up to conjugation, with the transition matrix between
 standard and costandard basis for $U_q(\widehat{\mathfrak{gl}}(\beta))$

Quantum difference equation:

$$\Psi(qz)L = M(z)\Psi(z)$$



$$\left. \begin{aligned} M(z) &= \prod_{w \in [0;1]} B_w \cdot L \\ w &\in [0;1] \end{aligned} \right\}$$

Theorem (A.Smirnov, Y.K.) $B_{\alpha\beta}$ in stable basis coincides,
 up to conjugation, with the R -matrix for
 $U_h(\hat{\mathfrak{gl}}(6))$ in the basis of fixed points.

$$\text{Stab}_G^{\varepsilon, \gamma_s, k-th} \cdot \left(\text{Stab}_{-e}^{\varepsilon, \gamma_s, k-th} \right)^{-1}$$

Follows from $\text{Mon}(z) = \text{Stab}_e^{\text{El}} \cdot \left(\text{Stab}_{-e}^{\text{El}} \right)^{-1}$

$$\text{Hilb}(A_{\Gamma_I}) \hookrightarrow \mathcal{M}(\varepsilon)$$

$$\mathcal{M}(\varepsilon, A_{S-1}) \hookrightarrow \mathcal{M}(s, A_{r-1})$$