February 19: David Vogan (MIT), *Bruhat order on representations of K II: nonunitarity certificates.*

Last week I sketched a construction of a partial (pre)order on the set $\widehat{K}$ of irreducible representations of the complexified maximal compact subgroup of a real reductive algebraic group. A central property of this order is that $\mu \leq \mu'$ if μ' appears in a standard $(\mathfrak{g}, K)$-module with lowest K-type μ. For this week I will just *define* the Bruhat order to be the transitive closure of this relation; the geometric relation I considered last week at least includes all these relations, and probably they are exactly the same.

In order to classify unitary $(\mathfrak{g}, K)$-modules, it is important for each μ to find a small finite set of *nonunitary certificates* $\{\mu_1, \ldots, \mu_m\}$. These are K-representations $\mu_i \geq \mu$ with the property that if J is an irreducible nonunitary Hermitian representation of lowest K-type μ, then the form must be indefinite on some pair of K-types (μ, μ_i). (Another way to say this is that if the form is positive on μ, then it must fail to be positive on some μ_i.)

For $G = GL(n, \mathbb{C})$ (regarded as a real group), $K = GL(n, \mathbb{C})$, and μ the trivial representation of K, the set of $[n/2]$ K-types

$$\mu_i = \text{irr of highest weight } (1, \ldots, 1, 0, \ldots, 0, -1, \ldots, -1)$$

is a set of nonunitarity certificates for μ. (Here there are i 1s and -1s, and $n - 2i$ 0s.) This statement is the main step in the classification of the unitary dual of $GL(n, \mathbb{C})$.

Just as teaser about Why This Is (related to) Interesting Mathematics, I will mention that if G is a split group over a p-adic field and $K = G(\mathcal{O})$ the usual maximal compact, then a set of nonunitary certificates for the trivial representation of K is indexed by the nontrivial representations of the Weyl group. This result of Barbasch and Moy was used by Barbasch and Ciubotaru to classify the spherical unitary representations of G.

I will outline an idea for finding (well, for *guessing*) a set of nonunitarity certificates for some μ and general G. The idea has some connection to the geometry from last week.