February 7: David Vogan (MIT), "Associated varieties for discrete series representations."

This talk concerns three closely related subjects: the classification of nilpotent conjugacy classes in a real reductive Lie algebra; the associated varieties of irreducible Harish-Chandra modules; and (behind the scenes) the geometry of conormal bundles of orbits on flag varieties.

Suppose $G(\mathbb{R})$ is the group of real points of a complex connected reductive algebraic group. Fix a Cartan involution θ, with fixed point group $K(\mathbb{R})$ a maximal compact subgroup of $G(\mathbb{R})$. The complexified group K acts on the flag variety X of Borel subalgebras of $\mathfrak{g}$ with finitely many orbits; so in particular there are finitely many closed orbits $Z_1, \dots, Z_r$. These orbits index the "irreducible fundamental series representations" of $G(\mathbb{R})$. We can introduce a partial preorder $\leq_c$ on these orbits, by saying that $Z \leq Z'$ if for any irreducible representations π and π' attached to Z and Z', there is a finite-dimensional representation F of G such that π is a composition factor of $\pi' \otimes F$. Write $\sim_c$ for the corresponding equivalence relation (called *cell equivalence*, because of its connection to Kazhdan-Lusztig cells). Equivalent representations have the same associated variety, giving a map

$$\{Z_i\}/\sim_c \quad \xrightarrow{AV} \quad \text{nilpotent orbits of } K \text{ on } (\mathfrak{g}/\mathfrak{k})^*.$$

Ben Harris proves in his thesis that the image of the map AV is contained in what Alfred Noël called "noticed nilpotents:" those for which the centralizer contains no split torus outside the center of $G(\mathbb{R})$. For the complex group G, the noticed nilpotents are precisely the Bala-Carter "distinguished" nilpotents. There is exactly one closed K orbit Z_1, which maps to the principal nilpotent element.

I will calculate the map AV in some other examples. The examples support the conjecture that the map AV is always one-to-one. In the case of the split group of type E_8, there are 32 noticed nilpotent classes (calculated by Djoković in 1988). There are 29 cell equivalence classes of discrete series, which map to 29 distinct noticed nilpotent classes. Of course this raises the question of how to characterize the image of AV precisely, or to modify its definition slightly to get the three missing nilpotent classes.