February 19: Bertram Kostant (MIT), "Recent developments on powers of the Dedekind functio and representation theory"

Let K be a simply-connected simple compact Lie group, T a maximal torus of K, and C a Weyl chamber in $\mathfrak{h} = i\, Lie\, T$. Let $D \subset C$ be the set of dominant integral linear forms on $\mathfrak{h}$ and, for each $\lambda \in D$ let $\pi_\lambda : K \to Aut\, V_\lambda$ be an irreducible representation with highest weight λ. Some time ago, as a generalization of the Jacobi triple product formula (case where K= SU(2)) and using a Dedekind eta function result of MacDonald we proved

$$(\prod_{n=1}^{\infty} 1 - x^n)^{dim\, K} = \sum_{\lambda \in D} \chi_\lambda(a_P)\ \ dim\, V_\lambda\ \ x^{Cas(\lambda)} \tag{1}$$

where χ_λ is the character of π_λ, a_P is a remarkable element in K with the property that $\chi_\lambda(a_P) \in \{-1, 0, 1\}$ for any $\lambda \in D$ and $Cas(\lambda)$ is the (unique) eigenvalue of the Casimir operator for π_λ. For $K = SU(5)$ the left side of (1) is (upon multiplication by x) the Ramanujuan tau function.

Obviously the only summands contributing to the right side of (1) are those when $\lambda \in D_{alcove}$ where, by definition, $D_{alcove} = \{\lambda \in D \mid \chi_\lambda(a_P) \in \{1, -1\}\}$. We can now determine the set D_{alcove}. In fact the elements of D_{alcove} parameterize all the alcoves in the Weyl chamber C in the following way: As one knows the set $\mathcal{A}$ of all alcoves A in C is parameterized by a well-known subset W^C of the affine Weyl group so that we can write $\mathcal{A} = \{A_\sigma\}$, $\sigma \in W^C$. One readily defines the centrum of A_σ.

Theorem 1. *We can write $D_{alcove} = \{\lambda^\sigma \mid \sigma \in W^C\}$ where for any $\sigma \in W^C$,*

$$2(\rho + \lambda^\sigma) \text{ is the centrum of } A_\sigma$$

Also

$$\chi_\lambda(a_P) = (-1)^{\ell(\sigma)}$$

If K is of type A-D-E then $2(\rho + \lambda^\sigma)$ is the inCenter of A_σ so that one has a sphere packing of the Weyl chamber by spheres of radius $(Coxeter\ number)^{1/2}$.

If $K = SU(k)$ then D_{alcove} corresponds to the null k-core Young diagrams of length at most $k-1$. There are exactly $\binom{k-m+2}{k-2}$ such diagrams of size mk.

In general $Cas(\lambda)$ is not an integer. However $Cas(\lambda^\sigma)$ is an integer for any $\sigma \in W^C$. In fact for any positive root φ let $n_\varphi(\sigma)$ be the number of $\varphi = integer$ walls crossed in joining the centrum of $A(\sigma)$ to the centrum of the fundmental simplex so that $\ell(\sigma) = \sum_{\varphi>0} n_\varphi(\sigma)$. One has

Theorem 2. *Let $\sigma \in W^C$. Then $Cas(\lambda^\sigma)$ is the integer given by*

$$Cas(\lambda^\sigma) = \sum_{\varphi>0} n_\varphi(\sigma)(n_\varphi(\sigma) + 1)/2$$

In particular

$$Cas(\lambda^\sigma) \geq \ell(\sigma)$$

where equality occurs exactly for the 2^ℓ alcoves in 2 times the fundamental simplex.

Remark 3. The significance of the last statement in Theorem 2 is that these 2^ℓ alcoves correspond (Peterson, Cellini, Papi) to the 2^ℓ abelian ideals in a Borel subalgebra. I show that the corresponding λ^σ are the "weights" of these ideals. I am then able to explicitly compute the coefficient b_n of x^n for the left side of (1) for $n \leq$ dual Coxeter number in terms of the dimensions of certain subspaces of the exterior algebra $\wedge(Lie\, K)_{\mathbb{C}}$. For example up to the alternation of signs the first 6 coefficients of Ramanujian tau function are

$$1, \quad 24, \quad 252, \quad 1472, \quad 4830, \quad 6048$$

These are exactly the dimensions of the eigen subspace S_j of $\wedge^j sl(5, \mathbb{C})$ belonging to the eigenvalue j (maximal eigenvalue) of the Casimir operator where $j = 0, \ldots, 5$.