# COMBINATORICS SEMINAR

## SYMMETRIC GRAPHS AND POLYTOPES

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**ABSTRACT:**

In a natural way, the faces of ranks 1 and 2 in a 4-polytope $\mathcal{P}$ provide the vertices of a bipartite graph $\mathcal{G}$. Recently, Asia Weiss and I have examined this construction when $\mathcal{P}$ is a finite, abstract regular (or chiral) polytope of Schläfli type $\{3, q, 3\}$. If, in this case, $\mathcal{P}$ is also *self-dual*, then $\mathcal{G}$ must be a 3-transitive (or 2-transitive) trivalent graph. With Egon Schulte and Tomaž Pisanski, we have also proved that if $\mathcal{P}$ is *not self-dual*, then $\mathcal{G}$ is no more symmetric then it has right to be. Indeed, $\mathcal{G}$ is then a trivalent *semisymmetric* graph, so that $\mathrm{Aut}(\mathcal{G})$ is transitive on edges but not on vertices. (Such graphs are a little elusive.)

After covering some backgound ideas, I'll illustrate the theorems through some beautiful examples: for example, when $\mathcal{P}$ is the 4-simplex (which of course can realized as a regular convex polytope), the graph $\mathcal{G}$ is the *Levi graph* for the Desargues configuration. And when $\mathcal{P}$ is the universal locally toroidal abstract regular polytope

$$\{\,\{3,6\}_{(3,0)}\,,\,\{6,3\}_{(1,1)}\,\}\,,$$

we find that $\mathcal{G}$ is the *Gray graph*, smallest among all semisymmetric trivalent graphs.

To get a feel for these things try a related problem: from a $3 \times 3 \times 3$ cube construct a bipartite graph $\mathcal{G}$ whose *red* vertices are the $27 = 3^3$ cubelets and whose *blue* nodes are the $27 = 9 + 9 + 9$ columns of 3 cubelets parallel to an edge, with red adjacent to blue when incident. Just how symmetrical is this trivalent graph? What is its order? Structure? Is $\mathrm{Aut}(\mathcal{G})$ transitive on vertices? What does this have to do with the Pappus configuration in plane geometry?

Wednesday, November 15, 2006
4:15 p.m.

M.I.T. Room 2-136

Refreshments will be served at 3:30 PM in Room 2-349.

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