In a natural way, the faces of ranks 1 and 2 in a 4-polytope $\mathcal{P}$ provide the vertices of a bipartite graph $\mathcal{G}$. Recently, Asia Weiss and I have examined this construction when $\mathcal{P}$ is a finite, abstract regular (or chiral) polytope of Schläfli type $\{3, q, 3\}$. If, in this case, $\mathcal{P}$ is also self-dual, then $\mathcal{G}$ must be a 3-transitive (or 2-transitive) trivalent graph. With Egon Schulte and Tomáš Pisanski, we have also proved that if $\mathcal{P}$ is not self-dual, then $\mathcal{G}$ is no more symmetric then it has right to be. Indeed, $\mathcal{G}$ is then a trivalent semisymmetric graph, so that $\text{Aut}(\mathcal{G})$ is transitive on edges but not on vertices. (Such graphs are a little elusive.)

After covering some background ideas, I’ll illustrate the theorems through some beautiful examples: for example, when $\mathcal{P}$ is the 4-simplex (which of course can realized as a regular convex polytope), the graph $\mathcal{G}$ is the Levi graph for the Desargues configuration. And when $\mathcal{P}$ is the universal locally toroidal abstract regular polytope

\[ \{ \{3,6\}_{(3,0)}, \{6,3\}_{(1,1)} \}, \]

we find that $\mathcal{G}$ is the Gray graph, smallest among all semisymmetric trivalent graphs.

To get a feel for these things try a related problem: from a $3 \times 3 \times 3$ cube construct a bipartite graph $\mathcal{G}$ whose red vertices are the $27 = 3^3$ cubelets and whose blue nodes are the $27 = 9 + 9 + 9$ columns of 3 cubelets parallel to an edge, with red adjacent to blue when incident. Just how symmetrical is this trivalent graph? What is its order? Structure? Is $\text{Aut}(\mathcal{G})$ transitive on vertices? What does this have to do with the Pappus configuration in plane geometry?