Dimers and embeddings

Marianna Russkikh

MIT

May 11, 2020

Based on:

[KLRR] “Dimers and circle patterns”

[CLR] “Dimer model and holomorphic functions on t-embeddings”
A dimer cover of a planar bipartite graph is a set of edges with the property: every vertex is contained in exactly one edge of the set.

(On the square lattice / honeycomb lattice it can be viewed as a tiling of a domain on the dual lattice by dominos / lozenges.)
Height function

Defined on $G^*$, fixed reference configuration, random configuration.

Note that $(h - \mathbb{E}h)$ doesn’t depend on the reference configuration.
Gaussian Free Field

GFF with zero boundary conditions on a domain $\Omega \subset \mathbb{C}$ is a conformally invariant random generalized function:

$$GFF(z) = \sum_k \xi_k \frac{\phi_k(z)}{\sqrt{\lambda_k}},$$

where $\phi_k$ are eigenfunctions of $-\Delta$ on $\Omega$ with zero boundary conditions, $\lambda_k$ is the corresp. eigenvalue, and $\xi_k$ are i.i.d. standard Gaussians.

[1d analog: Brownian Bridge]

The GFF is not a random function, but a random distribution.

GFF is a Gaussian process on $\Omega$ with Green’s function of the Laplacian as the covariance kernel.
Theorem As mesh goes to zero, fluctuations of height \( \Rightarrow \) Gaussian Free Field on \( \mathbb{D} \) with zero boundary conditions.

- [Kenyon '01+] conjectured for general lattices/domains, proved for lozenge tilings without facets in the limit shape.
- [Petrov '12], [Bufetov–Gorin '16-17]: certain polygons

[Kenyon'08], [Berestycki–Laslier–Ray' 16]: lozenge tilings
[Kenyon'00], [R.'16-18]: domino tilings

(open question: domains composed of 2 × 2 blocks on \( \mathbb{Z}^2 \))
\( \bar{h} = h - \mathbb{E}h \)

**Ambitious goal** [Chelkak, Laslier, R.]:
Given a big weighted bipartite planar graph to embed it so that

\[ \bar{h}^\delta \to \text{GFF} \]

**Q:** In which metric?

\[(G, K) \to (\mathcal{T}(G^*), K_\mathcal{T}), \quad K_{\text{gauge}} \sim K_\mathcal{T} \]

\( t \)-embedding

or

circle pattern embedding
Results

Theorem (Kenyon, Lam, Ramassamy, R.)

t-embeddings exist at least in the following cases:

- If $G^\delta$ is a bipartite finite graph with outer face of degree 4.
- If $G^\delta$ is a biperiodic bipartite graph.

Theorem (Chelkak, Laslier, R.)

Assume $G^\delta$ are perfectly t-embedded.

a) Technical assumptions on faces
b) The origami map is small in the bulk

$\Rightarrow$ convergence to $\pi^{-1/2}$ GFF$_\mathbb{D}$. 
Weighted dimers and gauge equivalence

Weight function $\nu : E(G) \to \mathbb{R}_{>0}$

Probability measure on dimer covers:

$$\mu(m) = \frac{1}{Z} \prod_{e \in m} \nu(e)$$

Definition

Two weight functions $\nu_1, \nu_2$ are said to be gauge equivalent if there are two functions $F : B \to \mathbb{R}$ and $G : W \to \mathbb{R}$ such that for any edge $bw$, $\nu_1(bw) = F(b)G(w)\nu_2(bw)$.

Gauge equivalent weights define the same probability measure $\mu$. 
Kasteleyn matrix

Complex Kasteleyn signs:
\[ \tau_i \in \mathbb{C}, \quad |\tau_i| = 1, \]
\[ \frac{\tau_1}{\tau_2} \cdot \frac{\tau_3}{\tau_4} \cdots \frac{\tau_{2k-1}}{\tau_{2k}} = (-1)^{(k+1)} \]

A (Percus–)Kasteleyn matrix \( K \) is a weighted, signed adjacency matrix whose rows index the white vertices and columns index the black vertices: \( K(w, b) = \tau_{wb} \cdot \nu(wb) \).

- [Percus’69, Kasteleyn’61]: \( Z = |\det K| = \sum_{m \in M} \nu(m) \)
- The local statistics for the measure \( \mu \) on dimer configurations can be computed using the inverse Kasteleyn matrix.
Kasteleyn matrix as a discrete Cauchy–Riemann operator

Kasteleyn \(\mathbb{C}\) signs proposed by Kenyon for the uniform dimer model on \(\mathbb{Z}^2\) [flat case]:

\[
\begin{pmatrix}
1 \\ -i \\ i \\ -1
\end{pmatrix} = \begin{pmatrix}
1 & 1 & 0 \\
1 & 1 & 1 \\
0 & 1 & 1
\end{pmatrix}
\]

Relation for 4 values of \(K^{-1}_\Omega\):

\[
1 \cdot K^{-1}_\Omega(v + 1, v') - 1 \cdot K^{-1}_\Omega(v - 1, v') +
\]

\[
i \cdot K^{-1}_\Omega(v + i, v') - i \cdot K^{-1}_\Omega(v - i, v') = \delta_{\{v=v'\}}
\]

Discrete Cauchy-Riemann:

\[
F(c) - F(a) = -i \cdot (F(d) - F(b))
\]
Kasteleyn matrix as a discrete Cauchy–Riemann operator

What about non-flat case / general weights / other grids?

A function $F^\bullet : B \to \mathbb{C}$ is discrete holomorphic at $w \in W$ if

$$[\bar{\partial}F^\bullet](w) := \sum_{b \sim w} F^\bullet(b) \cdot K(w, b) = [F^\bullet K](w) = 0.$$ 

For a fixed $w_0 \in W$ the function $K^{-1}(\cdot, w_0)$ is a discrete holomorphic function with a simple pole at $w_0$.

Q: How do discrete holomorphic functions correspond to their continuous counterparts? [gauge + Kasteleyn signs + embedding]

(+) [flat] uniform dimer model on $\mathbb{Z}^2$, isoradial graphs

(?) General weighted planar bipartite graphs [Chelkak, Laslier, R.]
Circle pattern

[Kenyon, Lam, Ramassamy, R.]

An embedding of a bipartite graph with cyclic faces.

Assume that each bounded face contains its circumcenter.

The circumcenters form an embedding of the dual graph.
Circle pattern

[Kenyon, Lam, Ramassamy, R.]

Circle pattern realisations with an embedded dual, where the dual graph is the graph of circle centres.

(!) Circle patterns themselves are not necessarily embedded.
Circle pattern

A circle pattern realization with an embedded dual.
Circle pattern = t-embedding

[Chelkak, Laslier, R.]

A t-embedding $\mathcal{T}$:

- **Proper:** All edges are straight segments and they don't overlap.
- **Bipartite dual:** The dual graph of $\mathcal{T}$ is bipartite.
- **Angle condition:** For every vertex $v$ one has

$$\sum_{f \text{ white}} \theta(f, v) = \sum_{f \text{ black}} \theta(f, v) = \pi,$$

where $\theta(f, v)$ denotes the angle of a face $f$ at the neighbouring vertex $v$. 
Proposition (Kenyon, Lam, Ramassamy, R.)

Suppose $G$ is a bipartite graph and $u : V(G^*) \to \mathbb{C}$ is a convex embedding of the dual graph (with the outer vertex at $\infty$). Then there exists a circle pattern $\mathcal{C} : V(G) \to \mathbb{C}$ with $u$ as centers if and only if the alternating sum of angles around every dual vertex is 0.
Proposition (Kenyon, Lam, Ramassamy, R.)

Suppose $\mathcal{G}$ is a bipartite graph and $u : \mathcal{V}(\mathcal{G}^*) \to \mathbb{C}$ is a convex embedding of the dual graph (with the outer vertex at $\infty$). Then there exists a circle pattern $\mathcal{C} : \mathcal{V}(\mathcal{G}) \to \mathbb{C}$ with $u$ as centers if and only if the alternating sum of angles around every dual vertex is 0.
Kasteleyn weights

\[ T \rightarrow (G, K_T), \quad \text{where} \quad \sum_b K_T(w, b) = \sum_w K_T(w, b) = 0 \]

Then \( K_T \) is a Kasteleyn matrix.

\[
\prod \frac{K_T(w_i, b_i)}{K_T(w_{i+1}, b_i)} \in (-1)^{k+1} \mathbb{R}_+ \\

\sum \text{white} = \pi \mod 2\pi
\]
Existence of $t$-embeddings

$$(\mathcal{G}, K) \rightarrow (\mathcal{G}^*, K) \rightarrow (\mathcal{T}(\mathcal{G}^*), K_T), \quad \text{where} \quad K_{\text{gauge}} \sim K_T.$$ 

Theorem (Kenyon, Lam, Ramassamy, R.)

$t$-embeddings of the dual graph $\mathcal{G}^*$ exist at least in the following cases:

- If $\mathcal{G}$ is a bipartite finite graph with outer face of degree 4, with an equivalence class of real Kasteleyn edge weights under gauge equivalence.

- If $\mathcal{G}$ is a biperiodic bipartite graph, with an equivalence class of biperiodic real Kasteleyn edge weights under gauge equivalence.

$$K_{\text{gauge}} \sim K_T \quad \iff \quad K_T(wb) = G(w)K(wb)F(b)$$
Coulomb gauge for finite planar graphs

**Def:** Functions $G : W \to \mathbb{C}$ and $F : B \to \mathbb{C}$ are said to give Coulomb gauge for $G$ if for all internal white vertices $w$

\[
\sum_{b} G(w)K_{wb}F(b) = 0,
\]

and for all internal black vertices $b$

\[
\sum_{w} G(w)K_{wb}F(b) = 0.
\]

\[
\sum_{w} G(w)K_{wb_i}F(b_i) = B_i
\]

\[
\sum_{b} G(w_i)K_{w_ib}F(b) = -W_i.
\]
Coulomb gauge for finite planar graphs

Closed 1-form: \( \omega(wb) = G(w)K_{wb}F(b) \).

Define \( \phi: G^* \to \mathbb{C} \) by the formula \( \phi(f_1) - \phi(f_2) = \omega(wb) \).

Theorem (Kenyon, Lam, Ramassamy, R.)
Suppose \( G \) has an outer face of degree 4. The mapping \( \phi \) defines a convex \( t \)-embedding into \( P \) of \( G^* \) sending the outer vertices to the corresponding vertices of \( P \).
Circle patterns and elementary transformations

[Kenyon, Lam, Ramassamy, R.]:
T-embeddings of $G^*$ are preserved under elementary transformations of $G$. 
t-embedding of a finite planar graph with an outer face of degree 4

[A. Postnikov]:
Any nondegenerate planar bipartite graph with 4 marked boundary vertices $w_1, b_1, w_2, b_2$ can be built up from the 4-cycle graph with vertices $w_1, b_1, w_2, b_2$ using a sequence of elementary transformations; moreover the marked vertices remain in all intermediate graphs.
Origami map

To get an origami map $\mathcal{O}(G^*)$ from $\mathcal{T}(G^*)$ one can choose a root face $\mathcal{T}(w_0)$ and fold the plane along every edge of the embedding.
Uniqueness of biperiodic t-embeddings

To get an origami map $O(G^*)$ from $T(G^*)$ one can choose a root face $T(w_0)$ and fold the plane along every edge of the embedding.

\[
T(G^*) \xrightarrow{O} O(G^*)
\]

\[
\xi = O(w_0) = O(b) = O(w_1) = O(G)
\]

Theorem (Chelkak; Kenyon, Lam, Ramassamy, R.)

1. The boundedness of the origami map $O$ is equivalent to the boundedness of the radii in any circle pattern.
2. If $G$ is biperiodic with biperiodic real Kasteleyn edge weights. There exists unique periodic t-embedding with a bounded $O$. 
Assumption ($\text{Lip}(\kappa, \delta)$)

Given two positive constants $\kappa < 1$ and $\delta > 0$ we say that a t-embedding $\mathcal{T}$ satisfies assumption $\text{Lip}(\kappa, \delta)$ in a region $U \subset \mathbb{C}$ if

$$|O(z') - O(z)| \leq \kappa \cdot |z' - z|$$

for all $z, z' \in U$ such that $|z - z'| \geq \delta$.

Remark:

- We think of $\delta$ as the ‘mesh size’;
- All faces have diameter less than $\delta$;
- The actual size of faces could be in fact much smaller than $\delta$. 

[Chelkak, Laslier, R.]
T-holomorphicity, assumptions

[Chelkak, Laslier, R.]

Assumption \( (\text{Lip}(\kappa, \delta)) \)

Given two positive constant \( \kappa < 1 \) and \( \delta > 0 \) we say that a t-embedding \( T \) satisfies assumption \( \text{Lip}(\kappa, \delta) \) in a region \( U \subset \mathbb{C} \) if

\[
|\mathcal{O}(z') - \mathcal{O}(z)| \leq \kappa \cdot |z' - z| \quad \text{for all } z, z' \in U \text{ such that } |z - z'| \geq \delta.
\]

Assumption \( (\text{Exp-Fat}(\delta), \text{triangulations}) \)

A sequence \( T^\delta \) of t-embeddings with triangular faces satisfies assumption \( \text{Exp-Fat}(\delta) \) on a region \( U^\delta \subset \mathbb{C} \) as \( \delta \to 0 \) if the following is fulfilled for each \( \beta > 0 \):

If one removes all ‘\( \exp(-\beta \delta^{-1}) \)-fat’ triangles from \( T^\delta \), then the size of remaining vertex-connected components tends to zero as \( \delta \to 0 \).
T-holomorphicity

[Chelkak, Laslier, R.]

- t-holomorphicity:
  Fix $\tilde{w} \in W$. Given a function $F^\bullet_{\tilde{w}}$ on $B$, s.t. $F^\bullet_{\tilde{w}}(b) \in \eta_b \mathbb{R}$ and $K_T F^\bullet_{\tilde{w}} = 0$ at $w$, there exists $F^\circ_{\tilde{w}}$ such that $F^\bullet_{\tilde{w}}(b_i)$ are projections of $F^\circ_{\tilde{w}}(w)$

- bounded t-holomorphic functions are uniformly (in $\delta$) Hölder and their contour integrals vanish as $\delta \to 0$.

$K_T^{-1}(\cdot, w_0)$ is a t-holomorphic function for a fixed white vertex $w_0$.
**T-graph = t-embedding + Origami map**

**[Kenyon-Sheffield]:** A pairwise disjoint collection \( L_1, L_2, \ldots, L_n \) of open line segments in \( \mathbb{R}^2 \) forms a T-graph in \( \mathbb{R}^2 \) if \( \bigcup_{i=1}^n L_i \) is connected and contains all of its limit points except for some set of boundary points.

**[Chelkak, Laslier, R.]:**

- For any \( \alpha \) with \( |\alpha| = 1 \), the set \( \mathcal{T} + \alpha \mathcal{O} \) is a T-graph, possibly non-proper and with degenerate faces.

- A t-white-holomorphic function \( F_w \), can be integrated into a real harmonic function on a T-graph (\( \text{Re}(\mathcal{I}_\mathbb{C}[F_w]) \) is harmonic on \( \mathcal{T} + \mathcal{O} \)).

- Lipschitz regularity of harmonic functions on \( \mathcal{T} + \alpha \mathcal{O} \).
Height function $\rightarrow$ GFF

**Theorem (Chelkak, Laslier, R.)**

Assume that $T^\delta$ satisfy assumptions $\text{LIP}(\kappa, \delta)$ and $\text{EXP-FAT}(\delta)$ on compact subsets of $\Omega$ and

(I) **The origami map is small:** $O^\delta(z) \xrightarrow{\delta \to 0} 0$

(II) $K_{T^\delta}^{-1}(b^\delta, w^\delta)$ is uniformly bounded as $\delta \to 0$ \(\Rightarrow\) convergence to $\pi^{-1/2} \text{GFF}_\mathbb{D}$.

(III) **the correlations** $\mathbb{E}[h^\delta(v_1^\delta) \ldots h^\delta(v_n^\delta)]$ are uniformly small near the boundary of $\Omega$.

A similar (though more involved) analysis can be performed assuming that the origami maps $O^\delta \xrightarrow{\delta \to 0} \vartheta$, which is a graph of a Lorenz-minimal surface in $\mathbb{R}^{2+2}$.

[Chelkak, Laslier, R.]: "Bipartite dimer model: perfect t-embeddings and Lorentz-minimal surfaces" (In preparation)
T-embeddings

Boundary of degree $2k$:

[Kenyon, Lam, Ramassamy, R.]:

- For each (generic) polygon $P$, there exists a t-embedding “realisation onto $P$”.
- Usually not unique (finitely many)
- Maybe self-intersections.

Open question: Is it always a proper embedding?
Perfect t-embeddings

[Chelkak, Laslier, R.]

**Definition.** Perfect t-embeddings:

- $P$ tangential to $\mathbb{D}$
- $\mathcal{T}(f_i)\mathcal{T}(f_i')$ bisector of the $\mathcal{T}(f_{i-1})\mathcal{T}(f_i)\mathcal{T}(f_{i+1})$

**Remark:**

- proper embeddings (no self-intersections) [at least if $P$ is convex]
- Not unique: $(F, G) \sim$ perfect t-embedding, then for all $|\tau| < 1$ \((F + \tau \bar{F}, G + \tau \bar{G}) \sim\) perfect t-embedding.

**Open question:** existence of perfect t-embeddings.

**Conjecture:** perfect t-embedding always exists.
Generalization

Theorem (Chelkak, Laslier, R.)

Let $G\delta$ be finite weighted bipartite planar graphs. Assume that

- $T\delta$ are perfect $t$-embeddings of $(G\delta)^*$ satisfying assumption $\text{Exp-Fat}(\delta)$
- $(T\delta, O\delta)$ converge to a Lorentz-minimal surface $S$.

Then the height fluctuations converge to the standard Gaussian Free Field in the intrinsic metric of $S$.

Chelkak, Laslier, R. “Bipartite dimer model: perfect $t$-embeddings and Lorentz-minimal surfaces” (In preparation)

Chelkak, Ramassamy “Fluctuations in the Aztec diamonds via a Lorentz-minimal surface” (arXiv:2002.07540)
Thank you!