q-RATIONAL DAHA

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Introduction

The double affine Heche algebra (DAHA) has been introduced by Cherednik (see [C04]) for the bridging between several branches inside mathematics, from representation theory and algebraic combinatorics to mathematical physics.

Among different variants of DAHAs, one of the most important is the rational DAHA, which is a kind of deformation for the smash product algebra of the algebra of differential operators with the symmetric group.

We take the definition of rational DAHA from [S14]. We start with the smash product:

Definition 0.1. [S14] *Let A be an associative algebra with an action by a finite group G. The smash product $A\#G$ is isomorphic to $A \otimes \mathbb{C}G$ as vector space, while the multiplication is defined by $(f_1 \otimes f_2)(g_1 \otimes g_2) = f_1 g_1(f_2) \otimes g_1 g_2$.*

We would also introduce the space $\mathfrak{h}_{reg}$, which is the locus on the affine space $\mathbb{A}^n$ which S_n acts freely, or the space $\mathbb{A}^n \backslash (\bigcup P_{ij})$, $P_{i,j} = \{x_i = x_j\}$. With this, we consider the algebra of differential operators on $\mathfrak{h}_{reg}$, denoted $\mathcal{D}_{\mathfrak{h}_{reg}}$, and define the Dunkl operators:

Definition 0.2. [S14, Definition 1.3] *The Dunkl operator D_i is defined by the formula*

$$D_i = t\frac{\partial}{\partial x_i} - \sum_{j \neq i} \frac{1}{x_i - x_j}(1 - s_{ij}) \in \mathcal{D}_{\mathfrak{h}_{reg}}\#S_n.$$

Inside the formula, t, c are fixed constants taking value in the base ring or field.

In fact, we have a relatively the following relation between the Dunkl operator [S14, Definition 1.3, Lemma 1.4]:

$$\begin{aligned} [D_i, D_j] &= 0, \\ [D_i, x_j] &= cs_{ij}, \\ [D_i, x_i] &= t - \sum_{j \neq i} cs_{ij}. \end{aligned}$$

With this, we could define the rational DAHA $\mathcal{H}_{t,c}$, which is the subalgebra generated by x_i, D_i and S_n. Explicitly, it is defined by the following:

Lemma 0.3. *The algebra $\mathcal{H}_{t,c}$ is the quotient of $R\{x_i, y_i\}_{i=1}^n \# S_n$ by the following relations:*

$$[x_i, x_j] = 0, [y_i, y_j] = 0, [y_i, x_j] = cs_{ij}, [y_i, y_j] = t - \sum_{j \neq i} cs_{ij}.$$

There is an analogy for the differential operators in characteristic p, namely crystalline differental operators [BCEY21]. It is an algebra of differential operators with good functorial properties. Notice that there is a large center for this algebra due to the non-faithful action of it on the polynomial ring. For the characteristic p case of rational DAHA, the crystalline differential operators are usually used as a replacement for the algebra of differential operators. [BFG21]

There's a key property for the prismatic differential operators : the Azumaya property. With this property, the algebra structure could be simplified locally into matrices, and it would bring more connections, such as the ones connecting the Calogero-Moser spaces and Morita equivalence. For these two algebras, with a large center, they are in fact Azumaya over their centers. [BMR08, BFG21]

Recently, in study of crystalline cohomology, the algebra of q-differential operators was introduced, see e.g. a talk by Vologodsky [V24]. It is a variant of differential operators on $\mathbb{G}_m$ with relation $\partial x - x\partial = 1$ replaced with the q-commutator relation: $\partial x - qx\partial = x$. It is defined over the base ring $\mathbb{Z}_p[q^{\pm 1}]^{\wedge}_{(q-1)}$. Although it is in characteristic 0, the p-adic structure gives it the properties similar to the characteristic p crystalline differential operators, such as the large center and the Azumaya property.

In this article, we're interested in generalizing the rational DAHA into the q-differential case, or specifically, the case with $q^p = 1$ over $\mathbb{Z}_p$.

We start by giving the generalization of the q-differential operators into multivariable, which is also mentioned in [V24]. Then, we substituted $\epsilon = \sqrt[p]{1}$ as q, and developed the proof for its Azumaya condition. After this, we give the definition of ϵ-Dunkl operators, and then give a generator-relation definition for the q-DAHA structure.

1. q-DIFFERENTIAL OPERATORS ON $\mathbb{G}_m$

1.1. q-differential operators.

Here, we define the algebra of q-differential operators, which we learned from [V24].

The q-differential operator is an analogue of the algebra of differential operators on $\mathbb{G}_m$. Fix a prime p.

Definition 1.1. [V24] *The ring (algebra) of q-differential operators on $\mathbb{G}_m$, $\mathcal{D}_{q,\mathbb{G}_m}$, is defined by*

$$\mathcal{D}_{q,\mathbb{G}_m} = \left(\frac{\mathbb{Z}[q^{\pm 1}]\{x^{\pm 1}, \partial\}}{(\partial x - qx\partial - x)} \right)^{\wedge}_{(q-1,p)}.$$

Remark 1.2. *$\mathcal{D}_{q,\mathbb{G}_m}$ is an algebra over the base ring $\mathbb{Z}_p[[q-1]]$.*

Remark 1.3. [V24] *When taking $q = 1$, we have $\mathcal{D}_{q,\mathbb{G}_m}$ becomes the differential operator $\mathcal{D}_{\mathbb{G}_m}$, where the operator ∂ acts as $x\frac{\partial}{\partial x}$.*

There is an action of $\mathcal{D}_{q,\mathbb{G}_m}$ on the space $\mathbb{Z}_p[x^{\pm 1}][[q-1]]_p^{\wedge}$, where x acts by multiplication, and $\partial f(x) := \frac{f(qx)-f(x)}{q-1}$.

1.2. **Generalization to multivariable.** Similar with differential operators on $\mathbb{G}_m^n$, there is also a multivariable generalization for q-differential operators.

To write the relation inside the ring, we introduce the notation $[x, y]_q = xy - qyx$, and call it the "q-commutator".

Definition 1.4. [V24] *The ring (algebra) of q-differential operators on $\mathbb{G}_m^n$, $\mathcal{D}_{q,\mathbb{G}_m^n}$, is defined by*

$$\mathcal{D}_{q,\mathbb{G}_m^n} = \left(\frac{\mathbb{Z}[q^{\pm 1}]\{x_i^{\pm 1}, \partial_i\}}{(relation)} \right)^{\wedge}_{(q-1,p)},$$

The relation mentioned inside are the following ($i \neq j$):

$$[x_i, x_j] = 0,$$
$$[\partial_i, \partial_j] = 0,$$
$$[\partial_i, x_j] = 0,$$
$$[\partial_i, x_i]_q = x_i.$$

Again, this is an algebra over the ring $\mathbb{Z}_p[[q-1]]$, and it would degenerate into the algebra of differential operators on $\mathbb{G}_m^n$ when taking $q = 1$.

Similar with the one variable case, $\mathcal{D}_{q,\mathbb{G}_m^n}$ admits an action on $\mathbb{Z}_p[x_i^{\pm 1}][[q-1]]_p^{\wedge}$, where x_i acts by multiplication, and $\partial_i f(x) = \frac{f(\hat{x}_i)-f(x)}{q-1}$, with $x = (x_1, \cdots, x_n)$, and $\hat{x}_q$ is the vector with the ith coordinate scaled by q.

1.3. $q = \epsilon$ **substitution.**

Proposition 1.5. *When substituting q by a pth root of unity ϵ, the ring of q-differential operators in n variables becomes*

$$\left(\frac{\mathbb{Z}_p[\epsilon]\{x^{\pm 1}, \partial\}}{(relations)} \right)^{\wedge}_{p},$$

where the relations are the same with above, with q replaced by ϵ.

Proof. We notice that $(\epsilon - 1)^p = \sum_{i=0}^{p} \epsilon^i \binom{p}{i} = \epsilon^p - 1 = 0$ when considering in characteristic p, thus $(\epsilon - 1)^{kp}$ would be zero modulo p^k,

Thus, since $(p)^p \subset (p, \epsilon - 1)^p \subset (p)$, we know the completion of the two ideals would be the same. The ring of q-differentials would then become a completion by ideal (p), which gives the base ring $\mathbb{Z}_p$. ∎

Lemma 1.6. *The center of $\mathcal{D}_{\epsilon,\mathbb{G}_m}$ is isomorphic to the subalgebra*

$$\mathbb{Z}_p[\epsilon]\left[x^{\pm p}, \frac{((\epsilon-1)\partial+1)^p-1}{(\epsilon-1)^p}\right]_p^\wedge$$

Proof. We consider the ring $\frac{K[\epsilon]\{x^{\pm1},y\}}{(yx-\epsilon xy)}$, where K is a field with characteristic 0, and x, y are generators. Thus, given a polynomial $P = \sum_{i\in I} c_i x^{m_i} y^{n_i}$, $c \in K[\epsilon]$, we have $xP - Px = \sum_{i\in I} c_i(1-\epsilon^{n_i})x^{m_i+1}y^{n_i}$, $yP - Py = \sum_{i\in I} c_i(\epsilon^{-m_i}-1)x^{m_i}y^{n_i+1}$. Thus, all elements commute with both x and y must have all terms vanishing, thus lies in the ring $K[\epsilon][x^{\pm p}, y^p]$. It is obvious that this ring indeed lies in the center, thus is itself the center.

Now, we consider the ring $\frac{\mathbb{Q}_p[\epsilon]\{x^{\pm1},\partial\}}{\partial x-\epsilon x\partial - x}$. Notice that the base ring is a field, thus using $(\epsilon-1)\partial+1$ to replace ∂ would give the same ring as a generator. As $((\epsilon-1)\partial+1)x = (\epsilon-1)\partial x + x = (\epsilon-1)(\epsilon x\partial + x) + x = \epsilon((\epsilon-1)x\partial + x)$, we have $\mathbb{Q}_p[\epsilon]\{x^{\pm1}, (\epsilon-1)\partial+1\}$ indeed have the relation above, and has center $\mathbb{Q}_p[\epsilon][x^{\pm p}, ((\epsilon-1)\partial+1)^p]$.

Since we're working on $\mathbb{Z}_p[\epsilon]$ instead of its quotient field, we have the center being the intersection of $\mathbb{Q}_p[\epsilon][x^{\pm p}, ((\epsilon-1)\partial+1)^p]$ and $\frac{\mathbb{Z}_p[\epsilon]\{x^{\pm1},\partial\}}{\partial x - qx\partial - x}$. As ∂ has integral coefficients, we could at most put $(\epsilon-1)^p$ on the denominator. With noticing $\frac{((\epsilon-1)\partial+1)^p-1}{(\epsilon-1)^p}$ is indeed in the ring, we have it as the second generator.

Since taking center commutes with completion, we can conclude the result. □

Moreover, we could extend this result into multivariable case:

Proposition 1.7. *The center of $\mathcal{D}_{\epsilon,\mathbb{G}_m^n}$ is isomorphic to the subalgebra*

$$\mathbb{Z}_p[\epsilon]\left[x_i^{\pm p}, \frac{((\epsilon-1)\partial_i+1)^p-1}{(\epsilon-1)^p}\right]_p^\wedge$$

Proof. We first consider the ring $\frac{\mathbb{Q}_p[\epsilon]\{x_i^{\pm1},y_i\}_{i=1}^n}{(y_i x_i - \epsilon x_i y_i)}$.

Notice that this ring is isomorphic to $\bigotimes_{\mathbb{Q}_p[\epsilon]}^n \frac{\mathbb{Q}_p[\epsilon]\{x^{\pm1},y\}}{yx-\epsilon xy}$. We have show that the center of $\frac{\mathbb{Q}_p[\epsilon]\{x^{\pm1},y\}}{yx-\epsilon xy}$ is isomorphic to $\mathbb{Q}_p[\epsilon][x^{\pm p}, y^p]$ in Lemma 1.6. Hence, we have the center also being the tensor product of n copies for $\mathbb{Q}_p[\epsilon][x^{\pm p}, y^p]$.

Now we would get the center of $\mathcal{D}_{\epsilon,\mathbb{G}_m^n}$ being the intersection of it with $\mathbb{Q}_p[\epsilon][x_i^{\pm p}, ((\epsilon - 1)\partial_i+1)^p]$, which gives the subalgebra in the proposition after taking the completion. □

Notice that although our algebra is of characteristic 0, it behaves similarly to the crystalline differential operators in characteristic p. It would indeed drop into the crystalline operators when taking modulo p, as we show below.

First, we define the ring of crystalline differential operators.

Definition 1.8. [BCEY21, Corollary 9.7]*Let k be a field with characteristic $p > 0$, and X be a smooth variety. Then the sheaf of algebras of crystalline differential operators $\mathcal{D}_X$ is locally generated by the function algebra $\mathcal{O}_X$ and the tangent Lie algebroid $\mathcal{T}_X$, with the relation $[\partial, f] = \partial(f)$.*

Remark 1.9. *For $X = \mathbb{G}_m^n$, we have the crystalline algebra being defined as*

$$\mathcal{D}_{\mathbb{G}_m^n} = \frac{k[x_i^{\pm 1}, \partial_i]}{(\partial_i x_i - x_i \partial_i - x_i)},$$

with the center $k[x_i^{\pm p}, \partial_i^p]$.

To develop more structure for this algebra, we start with recalling equivalent definitions for Azumaya algebra. In what follows, the subscript of a ring or a field means the base change.

Lemma 1.10. [B23, Lemma 18.1]*Let R be a commutative ring and A be a finitely generated and projective R-module. Then the followings are equivalent definitions for Azumaya property.*

(a): The map $A \otimes A^{op} \to \mathrm{End}_R(A)$ is an isomorphism.

(b): For every algebraically closed field k and a homomorphism $R \to k$, A_k is isomorphic to $\mathrm{Mat}_n(k)$.

(c): For every maximal ideal $\mathfrak{m} \subset R$ with $k = R/\mathfrak{m}$, the ring A_k is a central simple algebra over k.

If A satisfies any of these, we say A is Azumaya over R.

We'll also introduce an Azumaya property for differential operators on $\mathbb{G}_m$ inside characteristic p fields.

Lemma 1.11. *Let k be an algebraically closed field with characteristic p, and X being a smooth k-variety. The ring of crystalline differential operators on X is Azumaya over its center.*

Proof. This is [BMR08, Theorem 2.2.3]. □

Remark 1.12. *For finite field F and a smooth F-variety X with dimension n, the differential operator $\mathcal{D}_X$ is finitely generated and projective over its center.*

Proof. The finite generated situation is clear. For the projective case, we consider $\mathcal{D}_X \otimes_R R/\mathfrak{m}$ for R being the center. For a fixed ideal $\mathfrak{m}$, we have a pair (x, ζ) inside the cotangent bundle, and throught this we could take out the central maximal ideal inside R, thus we have the quotient $\mathcal{D}_X \otimes_R R/\mathfrak{m}$ being a p^n dimension $R/\mathfrak{m}$ vector space.

We consider the action of $\mathcal{D}_X \otimes_R R/\mathfrak{m}$ on this vector space, and would get an injective from this algebra to the endomorphism algebra. Through some dimension count, we notice that the map is indeed isomorphism, thus proving the dimension of the vector space $\mathcal{D}_X \otimes_R R/\mathfrak{m}$ is always p^{2n}. □

Now, we would prove the Azumaya property in Lemma 1.11 also holds for all finite fields:

Lemma 1.13. *Let X be a $\mathbb{F}_{p^n}$-variety. The algebra of crystalline differential operators on X is Azumaya over its center.*

Proof. This is a local statement, so we check it in the affine case.

Let A be the algebra of crystalline differential operators on X. We consider the second condition in Lemma 1.10. Let $R = Z(A)$. Given a morphism $R \to k$, with k algebtaically closed, it extends to a morphism $R_{\overline{\mathbb{F}_p}} \to k$. Given that $A_{\overline{\mathbb{F}_p}} = \mathcal{D}_{X_{\overline{\mathbb{F}_p}}}$ is Azumaya over its center, we have A_k satisfies the second condition for all algebraically closed k with morphism $Z(A_{\overline{\mathbb{F}_p}}) \to k$, we have all A_k with $Z(A) \to k$ satisfying this condition. Thus, we have the algebra of crystalline differential operators for all the varieties over $\mathbb{F}_{p^n}$ being Azumaya over their center. □

We return to our algebra of q-differential operators.

Remark 1.14. *$\mathcal{D}_{\epsilon,\mathbb{G}_m^n}$ is finitely generated and projective over its center.*

Proof. Again, finitely generatedness is clear. The projective condition comes from the same proof: we first take a maximal ideal and consider it as a vector field in $\mathbb{F}_{p^k}$, and through a dimension comparison with injectivity toward endomorphism ring and the count of generators, we would get a fixed dimension for the vector field, thus proving the projectivity. □

Theorem 1.15. [V24] *$\mathcal{D}_{\epsilon,\mathbb{G}_m^n}$ is Azumaya over its center.*

Proof. Let $A = \mathbb{F}_p[\epsilon][x^{\pm 1}, \partial]/(\partial x - \epsilon x \partial - x)$, then $\mathcal{D}_{\epsilon,\mathbb{G}_m^n} = A_p^{\wedge}$. We consider the base ring $R = \mathbb{Z}_p[\epsilon][x^{\pm p}, \frac{((\epsilon-1)\partial+1)^p-1}{(\epsilon-1)^p}]$, and work on the third condition. Notice that for all maximal ideals for this ring, the residue field would be in shape of $\mathbb{F}_{p^n}$. Now we focus on the base change $A_{\mathbb{F}_{p^n}} = \mathbb{F}_{p^n}[x_i^{\pm 1}, \partial_i]$.

As $A_{\mathbb{F}_{p^n}} = \mathcal{D}_{\mathbb{G}_m^n}$ for the smooth $\mathbb{F}_{p^n}$-variety $\mathbb{G}_m^n$, we have $A_{\mathbb{F}_{p^n}}$ being Azumaya over its center from Lemma 1.13, and since it has a base ring being a field, it is indeed a central simple algebra over $\mathbb{F}_{p^n}$. Thus we proved the result that the algebra A being Azumaya over its center.

Since Azumaya property and center are preserved under completion, we have $\mathcal{D}_{\epsilon,\mathbb{G}_m^n}$ over $\mathbb{Z}_p[\epsilon]$ being Azumaya over its center. □

2. Definition and first properties of q-rational DAHA

Similar with the Dunkl operators, we would take the space that S_n acts on freely, which is the space $\mathfrak{h}_r = \mathbb{G}_m^n \backslash P_{ij}$, with $P_{ij} := \{x_i = x_j\}$. To ensure x_i is invertible, we begin with $\mathbb{G}_m$ instead of $\mathbb{G}_a$.

We consider the ϵ-differential operators on $\mathfrak{h}_r$, $\mathcal{D}_{\epsilon,\mathfrak{h}_r}$, which is isomorphic to localization of $\mathcal{D}_{\epsilon,\mathbb{G}_m^n}$ by $(x_i - x_j)$.

Now, we define the new ϵ-Dunkl operators as generalizations for the Dunkl operator, on the space $\mathcal{D}_{\epsilon,\mathfrak{h}_r} \# S_n$:

Definition 2.1. *The ϵ-Dunkl operator, D_i, is defined by the formula*

$$D_i = tx_i^{-p}\partial_i - c\sum_{j\neq i}\frac{1}{x_i^p - x_j^p}(1-s_{ij}).$$

Inside the formula, t, c are fixed constants taking value in $\mathbb{Z}_p[\epsilon]$.

Similar with the standard case, we also want commutativity for the q-Dunkl operators, which indeed holds true:

Lemma 2.2. *The q-Dunkl operators commute.*

Proof. We consider the Dunkl operator action on an arbitrary monomial $P = \prod x_i^{a_i}$, which results to

$$tx_i^{-p}[a_i]_\epsilon P - \sum_{k\neq i} cp\frac{(1-x_i^{a_k-a_i}x_k^{a_i-a_k})P}{x_i^p - x_j^p}.$$

We again apply D_j to D_iP, and compare it with D_iD_jP. Since two operators would commute if they operate on disjoint coordinates, we only check if the two have the same coordinate. As the t^2 term would be automatically symmetric, we only need to consider the other two.

For the cross term, we only consider the case when $k_i = j$ in D_i or $k_j = i$ in D_j. With ignoring the common factor $-tcp\prod_{k\neq i,j} x_k^{a_k}$, the term becomes

$$x_j^{-p}\frac{[a_j]_\epsilon x_i^{a_i}x_j^{a_j} - [a_i]_\epsilon x_i^{a_j}x_j^{a_i}}{x_i^p - x_j^p} + [a_i]_\epsilon\frac{x_i^{a_i-p}x_j^{a_j} - x_i^{a_j}x_j^{a_i-p}}{x_j^p - x_i^p} = \frac{[a_j]_\epsilon x_j^{-p} - [a_i]_\epsilon x_i^{-p}}{x_i^p - x_j^p}x_i^{a_i}x_j^{a_j},$$

which is invariant under s_{ij}.

For the c^2 term, notice that when $k_i = j$ in D_i and $k_j = i$ in D_j, the two operators only differ by a sign, thus they would be symmetric. Thus, we just need to consider $k_i = j$, $k_j = i$, or $k_i = k_j$. Hence, we could label the only term differ with i and j inside being k.

We fix some k. With relabeling $x_i^{a_i}x_j^{a_j}x_k^{a_k} = x^a y^b z^c$. The sum becomes:

$$\begin{aligned}
&\frac{1}{y^p - z^p}\left(\frac{x^ay^bz^c - x^cy^bz^a}{x^p - z^p} - \frac{x^ay^cz^b - x^cy^az^b}{x^p - y^p}\right) + \frac{1}{y^p - z^p}\left(\frac{x^ay^bz^c - x^by^az^c}{x^p - y^p} - \frac{x^ay^cz^b - x^by^cz^a}{x^p - z^p}\right)\\
&\quad + \frac{1}{y^p - x^p}\left(\frac{x^ay^bz^c - x^cy^bz^a}{x^p - z^p} - \frac{x^by^az^c - x^by^cz^a}{y^p - z^p}\right)\\
=&\frac{1}{(x^p - y^p)(y^p - z^p)(z^p - x^p)}\Big((y^p - x^p)(x^ay^bz^c - x^cy^bz^a - x^ay^cz^b + x^by^cz^a)\\
&\quad + (z^p - x^p)(x^ay^bz^c - x^by^az^c + x^by^az^c - x^by^cz^a + x^cy^az^b - x^ay^cz^b) + (y^p - z^p)(x^ay^bz^c - x^cy^bz^a)\Big)\\
=&\frac{1}{(x^p - y^p)(y^p - z^p)(z^p - x^p)}\Big(x^ay^bz^c(2y^p - 2x^p) + x^ay^cz^b(2x^p - y^p - z^p) + x^cy^bz^a(x^p + z^p - 2y^p)\\
&\quad + x^by^cz^a(y^p - z^p) + x^cy^az^b(z^p - x^p)\Big)
\end{aligned}$$

Which is invariant under s_{ij}. By adding through all $k \neq i, j$, the c^2 term itself would be invariant, and so D_iD_j would be invariant. Thus all D_i are commutative over the polynomials. □

Let us investigate the presentation of this algebra by generators and relations, which we call q-DAHA:

Definition 2.3. *The q-DAHA $\mathcal{H}_{t,c}$ is a subalgebra of $\mathcal{D}_{\epsilon,\mathfrak{h}_r}\#S_n$ generated by $x_i^{\pm 1}, D_i, S_n$ over $\mathbb{Z}_p[\epsilon]$.*

In fact, the q-DAHA is connected with the crystalline differential operator even if they are in different characteristics.

Theorem 2.4. *When doing the mod $\epsilon - 1$ reduction, q-DAHA would reduce to different forms based on if t and c vanish or not. Specially, when $\epsilon - 1$ does not divide t and c, it becomes $\mathcal{D}_{\mathbb{G}_m^n}[\frac{1}{x_i - x_j}]$ over $\mathbb{F}_p$.*

Proof. We denote the new space being A.

We have $\mathbb{Z}_p[\epsilon]$ becoming its residue field $\mathbb{F}_p$, since $\epsilon - 1$ with the valuation $\frac{1}{p-1}$ is a uniformizer. Moreover, since ϵ becomes 1, we have the differential operator behave as $[\partial, x] = 1$, which is the same with the operator in the algebra of crystalline differential operators.

Since $\frac{cs_{ij}}{(x_i^p - x_j^p)} \in A$ and $s_{ij} \in A$, we have $\frac{c}{(x_i - x_j)^{p-1}} \in A$, thus we have $\frac{1}{x_i - x_j} \in A$ given $(\epsilon - 1) \nmid c$. Thus, we have $\frac{1}{(x_i - x_j)^k} \in A$ for all k, and we could eliminate all the c term inside the q-Dunkl operator to get $tx_i^{-p}\partial_i \in A$, thus $tx_i^{-p+1} \in A$ and $x_i^{-1} \in A$ given $(\epsilon - 1) \nmid t$.

Notice that given $(\epsilon - 1)|t$ or $(\epsilon - 1)|c$, we have the term of ∂_i or $\sum \frac{1}{x_i^p - x_j^p}$ completely vanish. Thus, we have the q-DAHA modding $(\epsilon - 1)$ as the following:

$$\begin{cases} \mathcal{D}_{\mathbb{G}_m^n}\{(x_i - x_j)^{-1}\}, & (\epsilon - 1) \nmid c, t; \\ \mathcal{D}_{\mathbb{G}_m^n}, & (\epsilon - 1)|c, (\epsilon - 1) \nmid t; \\ \mathbb{F}_p[x_i, (x_i - x_j)^{-1}]_{1 \le i < j \le n}, & (\epsilon - 1)|t, (\epsilon - 1) \nmid c; \\ \mathbb{F}_p[x_i]_{i=1}^n, & (\epsilon - 1)|t, c. \end{cases}$$

□

Instead of the simple relations inside rational case, there's some more complex relations.

Lemma 2.5. *Given $(\epsilon - 1) \nmid t, c$, the relations in the q-DAHA are the following:*

$$[D_i, x_j] = c\frac{x_i - x_j}{x_i^p - x_j^p}s_{ij},$$

$$[D_i, x_i]_\epsilon = tx_i^{1-p} + c\sum_{j \neq i} \frac{(\epsilon - 1)x_i - (\epsilon x_i - x_j)s_{ij}}{x_i^p - x_j^p},$$

$$D_is_{ij} = s_{ij}D_j, \qquad x_is_{ij} = s_{ij}x_j.$$

Proof. We first calculate the first two relations.

Consider $[D_i, x_j]$, clearly x_j would commute with all the terms not containing x_j, thus the result would be the same with the commutator $[-c\frac{1}{x_i^p - x_j^p}(1 - s_{ij}), x_j]$, which gives value

$$-c\frac{1}{x_i^p - x_j^p}(1 - s_{ij})x_j + cx_j\frac{1}{x_i^p - x_j^p}(1 - s_{ij}) = c\frac{(x_j - x_j) + (x_i - x_j)s_{ij}}{x_i^p - x_j^p} = \frac{c(x_i - x_j)s_{ij}}{x_i^p - x_j^p}.$$

For $[D_i, x_i]$, we notice that we need to eliminate the coefficient for ∂_i, thus we need to take the ϵ-commutator. notice for all $j \neq i$, the latter term is symmetric, we could just do it once:

For the first part, we have

$$[tx_i^{-p}\partial_i, x_i]_\epsilon = tx_i^{-p}[\partial_i, x_i] = tx_i^{-p+1}.$$

For the second part, we have

$$[-c\frac{1}{x_i^p - x_j^p}(1 - s_{ij}), x_i]_\epsilon = c\frac{(\epsilon x_i - x_i) - (\epsilon x_i - x_j)s_{ij}}{x_i^p - x_j^p}.$$

Now we prove that there's no other relations.

Suppose there's one more relation: $\sum P_k = 0$, where P_i are all monomials in shape of $P_{k,x}P_{k,\partial}s_k$, since the previous two relations would allow us to change all polynomials into sum of monomial in this way. Moreover, we could divide everything by $\epsilon - 1$, until at least one term is not divisible by $\epsilon - 1$.

After this, we consider the reduction of this mod $\epsilon - 1$. Notice this becomes a nonzero expression inside the crystalline differential operators smash product with S_n, which would be nonzero. □

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