Properties of the Affine Grassmannian and its Equivariant Cohomology

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August 31, 2025

1 Introduction

Let G be a reductive group over the complex numbers. The affine Grassmannian is an infinite-dimensional complex variety defined as $\mathrm{Gr}_G = G(K)/G(\mathcal{O})$, where $K = \mathbb{C}((z))$ and $\mathcal{O} = \mathbb{C}[[z]]$.

One of the fundamental roles of Gr_G arise in the study of the Langlands dual group $\check{G}$ and its relation with G. A core question is how to construct $\check{G}$ naturally from G. This is answered by the *geometric Satake equivalence*, which states that $\mathrm{Rep}(\check{G})$, the category of representations on the Langlands dual group, is tensor equivalent to $\mathrm{Perv}_{G(\mathcal{O})}(\mathrm{Gr}_G)$, the category of perverse sheaves on Gr_G with an equivariant $G(\mathcal{O})$ action (see [6]). Therefore, the category $\mathrm{Perv}_{G(\mathcal{O})}(\mathrm{Gr}_G)$ is crucial to understanding $\check{G}$.

The perverse sheaves are defined as a special class of complexes of sheaves with constructible cohomology. Then, one may ask if the equivariant derived category $D_{G(\mathcal{O})}(\mathrm{Gr}_G)$ is also related to $\check{G}$. Bezrukavnikov and Finkelberg expressed this category as well as the loop-rotation equivariant version $D_{G(\mathcal{O})\rtimes\mathbb{C}^\times}(\mathrm{Gr}_G)$ in terms of representations on $\check{G}$ [2, Thm. 5]. One important functor in their approach is taking equivariant cohomology, which takes a complex of sheaves in $D_{G(\mathcal{O})\rtimes\mathbb{C}^\times}(\mathrm{Gr}_G)$ into a complex of $H^*_{G(\mathcal{O})\rtimes\mathbb{C}^\times}(\mathrm{Gr}_G)$-modules. Thus, it is important to understand the algebra $H^*_{G(\mathcal{O})\rtimes\mathbb{C}^\times}(\mathrm{Gr}_G)$.

The purpose of this paper is to discuss some geometric properties of Gr_G, and to compute the equivariant cohomology $H^*_{G(\mathcal{O})\rtimes\mathbb{C}^\times}(\mathrm{Gr}_G)$. Section 2 focuses on the geometry of Gr_G. Most importantly, we will discuss two stratifications of Gr_G. In section 3, we will briefly discuss some properties of equivariant cohomology, as well as the Atiyah-Bott localization theorem, which will be useful in section 4. Finally, in section 4, we compute the equivariant cohomology ring $H^*_{G(\mathcal{O})\rtimes\mathbb{C}^\times}(\mathrm{Gr})$. The proof mostly follows [2, §3.1]. I will provide more details than in [2], and give my original proof of Proposition 4.5.

Acknowledgments

This project is part of the UROP+ program at MIT. I would like to thank my mentor, Vasily Krylov, for providing assistance and insights throughout this project. I am grateful of Professor Roman Bezrukavnikov for suggesting this project, and of MIT Department of Mathematics for organizing this program.

2 Properties of Gr

2.1 Affine Grassmannian

Let G be a linear algebraic group over the complex numbers. Let $\mathcal{O} = \mathbb{C}[[z]]$ and $K = \mathbb{C}((z)) = \mathrm{Frac}(\mathcal{O})$. The *affine Grassmannian* is $\mathrm{Gr}_G = G(K)/G(\mathcal{O})$. When the context is clear, we often write $\mathrm{Gr} = \mathrm{Gr}_G$.

Example 2.1. For $G = \mathrm{GL}_n$, $\mathrm{Gr}_{\mathrm{GL}_n} = \mathrm{GL}_n(K)/\mathrm{GL}_n(\mathcal{O})$. Since $K = \mathrm{Frac}(\mathcal{O})$, an isomorphism $K^n \to K^n$ is determined by its restriction to $\mathcal{O}^n$. Thus, the points in $\mathrm{GL}_n(K) = \mathrm{Aut}(K^n)$ correspond to embeddings (of $\mathcal{O}$-modules) $\mathcal{O}^n \to K^n$. Multiplying by $\mathrm{GL}_n(\mathcal{O}) = \mathrm{Aut}(\mathcal{O}^n)$ on the right does not change the image of this map, so we have a bijection

$$\mathrm{GL}_n(K)/\mathrm{GL}_n(\mathcal{O}) \to \{\text{sub-}\mathcal{O}\text{-modules of } K^n \text{ isomorphic to } \mathcal{O}^n\},$$
$$g \mapsto \mathrm{im}(g|_{\mathcal{O}^n}).$$

In short, $\mathrm{Gr}_{\mathrm{GL}_n}$ can be identified with the space of $\mathcal{O}$-lattices in K^n.

Gr is infinite dimensional as a complex variety. It is (the $\mathbb{C}$-points of) an *ind-scheme*. An ind-scheme X is a union of subsets X_i indexed by a direct system $i \in I$, such that X_i are schemes, and for each arrow $i \to j$, X_i is a closed subscheme of X_j. Say X is ind-projective if all X_i are projective.

Remark. In literature Gr is an ind-scheme. For our purposes, it is enough to consider the geometry of the $\mathbb{C}$-points. Our Gr is actually the $\mathbb{C}$-points of the scheme-theoretic Gr in literature.

Proposition 2.2. $\mathrm{Gr}_{\mathrm{GL}_n}$ *is an ind-projective scheme.*

Proof. We will use the lattice description of $\mathrm{Gr}_{\mathrm{GL}_n}$ in Example 2.1. Let $\Lambda_0 = \mathcal{O}^n \subseteq K^n$ be the standard lattice. $\mathrm{Gr}_{\mathrm{GL}_n}$ is a union of subspaces $\mathrm{Gr}^{(N)}$, where

$$\mathrm{Gr}^{(N)} = \{\Lambda \mid z^N\Lambda_0 \subseteq \Lambda \subseteq z^{-N}\Lambda_0\}.$$

It suffices to show that $\mathrm{Gr}^{(N)}$ is projective.

$z^{-N}\Lambda_0/z^N\Lambda_0$ is a $2Nn$-dimensional vector space, and $\Lambda/z^N\Lambda_0$ is its subspace. This gives an embedding of $\mathrm{Gr}^{(N)}$ into the usual Grassmannian $\mathrm{Gr}(2Nn) := \coprod_k \mathrm{Gr}(k, 2Nn)$. Note that $z^{-N}\Lambda_0/z^N\Lambda_0$ is equipped with a nilpotent operator z. A subspace correspond to some $\Lambda \in \mathrm{Gr}^{(N)}$ if and only if it is stable under z. Therefore $\mathrm{Gr}^{(N)}$ is a closed subscheme of $\mathrm{Gr}(2Nn)$ and is thus projective. □

Remark. In general, Gr_G is ind-projective for any reductive group G. This is shown by chosing a suitable embedding $G \to \mathrm{GL}_n$, see [11, Prop. 1.2.6].

Another important example is Gr_T when T is a torus. First, note that $\mathrm{Gr}_{\mathbb{G}_m} = K^\times/\mathcal{O}^\times = \{z^a \mid a \in \mathbb{Z}\} \cong \mathbb{Z}$. For a torus T, if we fix an isomorphism $T \cong \mathbb{G}_m^n$, then $\mathrm{Gr}_T \cong \mathbb{Z}^n$. A more natural description is given in the following proposition.

Proposition 2.3. *Let T be a torus. Then Gr_T is isomorphic to the coweight lattice $X_*(T) = \mathrm{Hom}(\mathbb{G}_m, T)$, endowed with the discrete topology.*

Proof. Let's define a map $\mathrm{Hom}(\mathbb{G}_m, T) \to T(K)$. For a morphism $\mathrm{Spec}\,\mathbb{C}[z, z^{-1}] = \mathbb{G}_m \to T$, composing with $\mathrm{Spec}\,K \to \mathrm{Spec}\,\mathbb{C}[z, z^{-1}]$ gives you a K-point of T. To check that this gives an isomorphism $\mathrm{Hom}(\mathbb{G}_m, T) \to \mathrm{Gr}_T$, fix an isomorphism $T \cong \mathbb{G}_m^n$ and compute directly. □

For a coweight λ, we will denote the corresponding point in Gr_T by z^λ.

2.2 Affine Grassmannian of Borel subgroup

Let G be a reductive group with maximal torus and Borel subgroup $T \subseteq B \subseteq G$. In the last section, we have described the structure of Gr_T. In this section, we will describe Gr_B, which will be useful in proving certain properties of Gr_G.

Lemma 2.4. *The map* $\mathrm{Gr}_B \to \mathrm{Gr}_G$ *is a bijection of sets (but not a homeomorphism!).*

Proof. We need $B(K)/B(\mathcal{O}) \to G(K)/G(\mathcal{O})$ to be surjective, or $B(\mathcal{O})\backslash G(\mathcal{O}) \to B(K)\backslash G(K)$ to be surjective. Let $\mathcal{B} = B\backslash G$.

First we will show that $B(K)\backslash G(K) = \mathcal{B}(K)$. Note that $G(K) = G(\overline{K})^{\mathrm{Gal}(\overline{K}/K)}$, so we have an exact sequence

$$1 \to B(K) \to G(K) \to \mathcal{B}(K) \to H^1(\mathrm{Gal}(\overline{K}/K), B(\overline{K})).$$

By the additive and multiplicative version of Hilbert's Theorem 90, and the fact that B has a filtration with successive quotients $\mathbb{G}_a$ or $\mathbb{G}_m$, we have $H^1(\mathrm{Gal}(\overline{K}/K), B(\overline{K})) = 0$. Thus $B(K)\backslash G(K) = \mathcal{B}(K)$. $B(\mathcal{O})\backslash G(\mathcal{O})$ is the image of $G(\mathcal{O})$ in $B(K)\backslash G(K) = \mathcal{B}(K)$, so $B(\mathcal{O})\backslash G(\mathcal{O}) = \mathcal{B}(\mathcal{O})$. By the valuative criterion for properness, $\mathcal{B}(\mathcal{O}) \to \mathcal{B}(K)$ is a bijection, so $B(\mathcal{O})\backslash G(\mathcal{O}) \to B(K)\backslash G(K)$ is a bijection. □

The structure of Gr_B can be explicitly described. Let U be the unipotent radical of B.

Proposition 2.5. $\mathrm{Gr}_B \cong \mathrm{Gr}_T \times \mathrm{Gr}_U$. *Furthermore* $\mathrm{Gr}_U \cong U[z^{-1}]_1 := \ker(U[z^{-1}] \to U)$.

Proof. This is just a consequence of $B = T \ltimes U$. We have

$$B(K) = T(K)U(K) = \coprod z^\lambda T(\mathcal{O})U(K) = \coprod z^\lambda U(K)T(\mathcal{O})$$

so $B(K)/B(\mathcal{O}) = \coprod z^\lambda U(K)/U(\mathcal{O})$.

To understand $U(K)/U(\mathcal{O})$, notice that U has a filtration with successive quotients isomorphic to $\mathbb{G}_a$. Similarly as above, this reduces the computation to $\mathbb{G}_a$, and $\mathbb{G}_a(K)/\mathbb{G}_a(\mathcal{O}) = \mathbb{C}[z^{-1}]_1 = \bigoplus_{n\geq 1} \mathbb{C}z^{-n}$. □

With Lemma 2.4, many set-wise properties of Gr_G can be checked on Gr_B.

Theorem 2.6. *T acts on $G(K)$ by left multiplication, inducing an action on* Gr_G. *The T-fixed points of* Gr_G *are* Gr_T.

Proof. Since the map $\mathrm{Gr}_B \to \mathrm{Gr}_G$ is T-equivariant, it suffices to find the T-fixed points of Gr_B. Suppose $x \in \mathrm{Gr}_B^T$. Then x can be uniquely represented as $z^\lambda y$ where $y \in U[z^{-1}]_1$. For $t \in T$,

$$tx = tz^\lambda y = z^\lambda ty = z^\lambda tyt^{-1}$$

(remember we quotient by $B(\mathcal{O})$). Since $tyt^{-1} \in U[z^{-1}]_1$, by uniqueness we get $tyt^{-1} = y$ for all t, so $y = 1$. Thus $x = z^\lambda$ which proves $\mathrm{Gr}_B^T \subseteq \mathrm{Gr}_T$. Clearly $\mathrm{Gr}_T \subseteq \mathrm{Gr}_B^T$. Thus $\mathrm{Gr}_T = \mathrm{Gr}_B^T = \mathrm{Gr}_G^T$. □

2.3 Stratification with Gr^λ

In this section, we will discuss the orbits of $G(\mathcal{O})$ acting on Gr by multiplying on the left. These are the double cosets $G(\mathcal{O})\backslash G(K)/G(\mathcal{O})$. We have the Cartan decomposition (see [10, §3.3.3])

$$G(K) = \bigsqcup_{\lambda \in X_*(T)^+} G(\mathcal{O}) z^\lambda G(\mathcal{O}).$$

Here, λ is a positive coweight. As a result, we have

Theorem 2.7. *The orbits of the $G(\mathcal{O})$-action on* Gr *are* $\mathrm{Gr}^\lambda = G(\mathcal{O}) \cdot z^\lambda$ *for dominant coweights* λ. $\mathrm{Gr} = \bigsqcup_{\lambda \in X_*(T)^+} \mathrm{Gr}^\lambda$.

Example 2.8. Assume $G = \mathrm{GL}_n$. Recall that the points in $\mathrm{Gr}_{\mathrm{GL}_n}$ are identified with $\mathcal{O}$-lattices in K^n. For a dominant coweight $\lambda = (\lambda_1, \ldots, \lambda_n)$, $\lambda_1 \geq \cdots \geq \lambda_n$, z^λ corresponds to the lattice with basis $z^{\lambda_1} e_1, \ldots, z^{\lambda_n} e_n$.

To see that $\mathrm{Gr} = \bigsqcup G(\mathcal{O}) z^\lambda$, we need to show that any lattice can be transformed to some z^λ via action of $G(\mathcal{O})$. For a lattice with basis $v_1, \ldots, v_n$, write each v_i in coordinates

$$v_i = v_{i1} e_1 + \cdots + v_{in} e_n.$$

Let a_{ij} be the z-adic valuation of v_{ij} (i.e. the lowest degree of z in v_{ij}), and let $a = a_{i_0 j_0}$ be the lowest among all a_{ij}. Then there is a transformation in $G(\mathcal{O})$ which sends v_{i_0} to $z^a e_n$, and eliminates the z^a terms in other v_j. Repeating this process turns our basis into a basis of the form $z^{\lambda_1} e_1, \ldots, z^{\lambda_n} e_n$.

Theorem 2.9. Gr^λ *is a smooth, quasi-projective variety of dimension* $2\langle \rho, \lambda \rangle$*, where* ρ *is the half-sum of positive roots.*

Proof. The orbit space $\mathrm{Gr}^\lambda = G(\mathcal{O}) \cdot z^\lambda$ is a quotient of $G(\mathcal{O})$ by the stabilizer of z^λ. Thus, the tangent space of Gr^λ at z^λ is isomorphic to $\mathfrak{g}(\mathcal{O}) / \mathrm{stab}_{\mathfrak{g}(\mathcal{O})}(z^\lambda)$.

We have

$$\mathrm{stab}_{\mathfrak{g}(\mathcal{O})}(z^\lambda) = \{X \in \mathfrak{g}(\mathcal{O}) \mid \mathrm{Ad}\, z^\lambda (X) \in \mathfrak{g}(\mathcal{O})\}.$$

Write $\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha$, then $\mathfrak{t}(K)$ and $\mathfrak{g}_\alpha(K)$ are $\mathrm{Ad}\, z^\lambda$-invariant. Note that

$$\mathfrak{g}_\alpha(K) = \bigoplus_{n \in \mathbb{Z}} z^n \mathfrak{g}_\alpha,$$
$$\mathfrak{g}_\alpha(\mathcal{O}) = \bigoplus_{n \geq 0} z^n \mathfrak{g}_\alpha.$$

$\mathrm{Ad}\, z^\lambda$ fixes $\mathfrak{t}(K)$ and acts on $\mathfrak{g}_\alpha(K)$ by multiplying $z^{\langle \alpha, \lambda \rangle}$. Thus, $X \in \mathrm{stab}_{\mathfrak{g}(\mathcal{O})}(z^\lambda)$ iff the projection of X to $\mathfrak{g}(K)$ is contained in $\bigoplus_{n \geq 0, n \geq -\langle \alpha, \lambda \rangle} z^n \mathfrak{g}_\alpha$. Therefore

$$\mathfrak{g}(\mathcal{O}) / \mathrm{stab}_{\mathfrak{g}(\mathcal{O})}(z^\lambda) \cong \bigoplus_{\alpha : \langle \alpha, \lambda \rangle < 0} \mathfrak{g}_\alpha \oplus z \mathfrak{g}_\alpha \oplus \cdots \oplus z^{-\langle \alpha, \lambda \rangle - 1} \mathfrak{g}_\alpha.$$

Its dimension is

$$\sum_{\alpha \in \Delta^-} -\langle \alpha, \lambda \rangle = \sum_{\alpha \in \Delta^+} \langle \alpha, \lambda \rangle = 2\langle \rho, \lambda \rangle.$$

Notice that for sufficiently large n, $z^n\mathfrak{g}_\alpha$ is always contained in the stabilizer. This shows in fact $\mathrm{Gr}^\lambda = G(\mathbb{C}[z]/z^N)\cdot z^\lambda$, where $N > \langle \alpha, \lambda\rangle$ for all α. Thus Gr^λ has a transitive action of a finite-dimensional algebraic group, so it is smooth and quasi-projective. □

Let $\overline{\mathrm{Gr}^\lambda}$ be the closure of Gr^λ in Gr. Then $\overline{\mathrm{Gr}^\lambda}$ is projective. We also have $\overline{\mathrm{Gr}^\lambda} = \bigsqcup_{\mu\le\lambda}\mathrm{Gr}^\mu$, see [11, Prop. 2.1.5].

Consider the $\mathbb{C}^\times$-action on Gr induced by the $\mathbb{C}^\times$-action on K and $\mathcal{O}$, $z\mapsto tz$. Then the action on $\mathcal{O}$ has limit in $\mathbb{C}$ as $t\to 0$. Thus, $\mathrm{Gr}^\lambda = G(\mathcal{O})z^\lambda$ has limit in Gz^λ as $t\to 0$. We obtain a corollary of Theorem 2.7,

Corollary 2.10. *The $\mathbb{C}^\times$ fixed points of* Gr *are $G\cdot z^\lambda$ for positive coweights λ.*

The $\mathbb{C}^\times$-action allows us to describe the geometry of Gr^λ further. Gr^λ is non-singular with a $\mathbb{C}^\times$-action that contracts it to $G\cdot z^\lambda$. By Bialynicki-Birula theorem [3, Thm. 4.1], $\mathrm{Gr}^\lambda \to G\cdot z^\lambda$ is an affine fibration. Now, notice that $\mathrm{stab}_G(z^\lambda)$ is a parabolic subgroup, so $G\cdot z^\lambda$ is a partial flag variety.

2.4 Description of Gr as G-torsors

In this section, we find an alternative way to define Gr as G-torsors on a curve together with a trivialization away from a point. We will see that this gives a partition of Gr into its $G[z^{-1}]$-orbits.

Let $D = \operatorname{Spec}\mathcal{O}$ and $\mathring{D} = \operatorname{Spec} K$. D is known as the formal disk. Denote by $\mathcal{E}^0$ the trivial G-torsor over D. The following theorem provides an alternative way to define Gr.

Theorem 2.11. Gr *corresponds to the set of $(\mathcal{E},\sigma)$ where $\mathcal{E}$ is a G-torsor over D, and σ is a trivialization of $\mathcal{E}$ over $\mathring{D}$ ($\sigma : \mathcal{E}|_{\mathring{D}} \cong \mathcal{E}^0|_{\mathring{D}}$).*

To prove this we need a lemma.

Lemma 2.12. *All G-torsors on D are trivial.*

Proof. Let $\mathcal{E}\to D$ be a G-torsor. Since G is smooth, the morphism $\mathcal{E}\to D$ is also smooth. Let $D_n = \operatorname{Spec}\mathbb{C}[z]/z^n$. Then we have embeddings

$$\mathsf{pt} = D_0 \subseteq D_1 \subseteq D_2 \subseteq \cdots \subseteq D$$

and $D = \varinjlim D_n$. Clearly $\mathcal{E}|_{D_0}$ is trivial. Pick a section $D_0\to\mathcal{E}$. Since $\mathcal{E}\to D$ is smooth, by the infinitesimal lifting property (see [7, Ex. II.8.6]), a section $D_n\to\mathcal{E}$ extends to a section $D_{n+1}\to\mathcal{E}$. We obtain compatible sections $D_n\to\mathcal{E}$ which give a section $D\to\mathcal{E}$. Therefore the torsor $\mathcal{E}$ is trivial. □

Proof of Theorem 2.11. By lemma, there is a trivialization $\phi : \mathcal{E}\to\mathcal{E}^0$. Then

$$\sigma\phi^{-1} \in \mathrm{Aut}(\mathcal{E}^0|_{\mathring{D}}) = G(K).$$

Note that ϕ can be chosen up to $\mathrm{Aut}(\mathcal{E}^0) = G(\mathcal{O})$, so $(\mathcal{E},\sigma)$ determines a well-defined element in $G(K)/G(\mathcal{O}) = \mathrm{Gr}$. Conversely, an element of Gr also determines $(\mathcal{E},\sigma)$. □

Theorem 2.13. *Let X be a smooth projective curve with a point $x\in X$.* Gr *corresponds to the set of $(\mathcal{E},\sigma)$ where $\mathcal{E}$ is a G-torsor over X, and σ is a trivialization of $\mathcal{E}$ over $X\setminus\{x\}$.*

Proof. There is a map $D \to X$ whose reduced image is the point $\{x\}$. Explicitly, the embedding $\{x\} \hookrightarrow X$ is isomorphic to $\{0\} \hookrightarrow \mathbb{A}^1$ locally, and the map $D \to \mathbb{A}^1$ corresponds to $\mathbb{C}[t] \to \mathbb{C}[[z]]$, $t \mapsto z$.

A point in Gr corresponds to $(\mathcal{E}_D, \sigma)$, a G-torsor on D together with a trivialization on $\mathring{D}$. We would like to extend this to a torsor on X. $X \setminus \{x\} \hookrightarrow X$ and $D \to X$ are a fpqc cover. The trivial G-torsor on $X \setminus x$, $\mathcal{E}_D$ on D, and the gluing function σ together define a G-torsor on X by fpqc descent (see [9, Section 023R]). σ extends to a trivialization on $X \setminus \{x\}$. $\square$

Consider the case $X = \mathbb{P}^1$. Take $(\mathcal{E}, \sigma)$ where σ is a trivialization of $\mathcal{E}$ on $\mathbb{P}^1 \setminus 0$. Choose a trivialization $\phi : \mathcal{E}|_{\mathbb{P}^1\setminus\infty} \to \mathcal{E}^0|_{\mathbb{P}^1\setminus\infty}$. Then $\sigma\phi^{-1}|_{\mathbb{P}^1\setminus\{0,\infty\}}$ determines an element of

$$\mathrm{Aut}(\mathcal{E}^0|_{\mathbb{P}^1\setminus\{0,\infty\}}) = G[z, z^{-1}].$$

Here, ϕ can be choosed in $\mathrm{Aut}(\mathcal{E}^0|_{\mathbb{P}^1\setminus\infty}) = G[z]$, so we obtain a map

$$\mathrm{Gr} \to G[z, z^{-1}]/G[z].$$

Similar as before, this is an isomorphism.

From Theorem 2.13, we immediately obtain a map $\mathrm{Gr} \to \mathrm{Bun}_G(X)$, $(\mathcal{E}, \sigma) \mapsto \mathcal{E}$. This leads us to the decomposition of Gr into its $G[z^{-1}]$-orbits.

Theorem 2.14. $\mathrm{Gr} = \bigsqcup_{\lambda \in X_*(T)^+} G[z^{-1}] \cdot z^\lambda$. *The subvarieties* $G[z^{-1}] \cdot z^\lambda$ *are fibers of the map* $\mathrm{Gr} \to \mathrm{Bun}_G(\mathbb{P}^1)$*,* $(\mathcal{E}, \sigma) \mapsto \mathcal{E}$*.*

Proof. For each $\mathcal{E} \in \mathrm{Bun}_G(\mathbb{P}^1)$, the choice of σ can be changed by $\mathrm{Aut}(\mathcal{E}^0|_{\mathbb{P}^1\setminus 0}) = G[z^{-1}]$. Following the previous discussion, $\mathcal{E}$ is obtained by gluing trivial torsors on $\mathbb{P}^1 \setminus 0$ and $\mathbb{P}^1 \setminus \infty$ with the gluing function $\sigma\phi^{-1}$. The action of $G[z^{-1}]$ multiplies σ by an automorphism in $\mathrm{Aut}(\mathcal{E}^0|_{\mathbb{P}^1\setminus 0})$, which does not change the isomorphism class of $\mathcal{E}$. Therefore the orbits of $G[z^{-1}]$ correspond to $\mathrm{Bun}_G(\mathbb{P}^1)$.

It suffices to show that isomorphism classes of G-torsors on $\mathbb{P}^1$ are indexed by dominant coweights. This is proved in [8, §4.2]. $\square$

Example 2.15. For $G = \mathrm{GL}_n$, the G-torsors are rank n vector bundles. We have $X_*(T) \cong \mathbb{Z}^n$. Write $\lambda = (\lambda_1, \dots, \lambda_n)$ where $\lambda_1 \geq \cdots \geq \lambda_n$. The orbit $G[z^{-1}]z^\lambda$ correspond to the vector bundle $\mathcal{O}(\lambda_1) \oplus \cdots \oplus \mathcal{O}(\lambda_n)$. By Birkhoff-Grothendieck theorem, all vector bundles on $\mathbb{P}^1$ are of this form.

3 Equivariant cohomology and localization theorem

3.1 Equivariant cohomology of algebraic groups

In this section I will review some basic facts about equivariant cohomology. For a group G, there is an ind-variety EG on which G acts freely. Let $BG = EG/G$, then $EG \to BG$ is the universal G-torsor. Let G act on a space X. The equivariant cohomology is $H^*_G(X) = H^*(X \times^G EG)$. We will write $H^*_G := H^*_G(\mathsf{pt}) = H^*(BG)$. The projection $X \to \mathsf{pt}$ is a G-equivariant map, so it induces $H^*_G \to H^*_G(X)$, making $H^*_G(X)$ a module over H^*_G.

For a torus T of rank r, a character can be seen as a T-action on a line. That gives a line bundle on BT. It's first Chern class is an element of $H^2(BT) = H^2_T$. This defines a map $X^*(T) \to H^2_T$, which is an isomorphism after $\otimes_{\mathbb{Z}}\mathbb{C}$. H^*_T is generated by H^2_T, so $H^*_T = \mathbb{C}[\mathsf{t}]$ is a polynomial algebra with r generators in degree 2.

For a reductive group G with maximal torus T and Weyl group W, There is an embedding $H_G^* \to H_T^*$ induced by $BT \to BG$. In fact, we have $H_G^* = \mathbb{C}[\mathfrak{t}]^W$. By Chevalley-Shephard-Todd theorem, this is a polynomial ring, and thus

$$H_G^* = \mathbb{C}[\mathfrak{t}]^W = \mathbb{C}[\mathfrak{t}/W] \cong \mathbb{C}[x_1, \ldots, x_r]$$

where the degrees of x_i are even. Let $\deg(x_i) = 2d_i$. The numbers d_i have important properties, for example, $H^*(G)$ is an exterior algebra with generators in degree $2d_i - 1$, see [5, §46.1].

3.2 Change of group

This lemma will be used in Section 4.

Lemma 3.1. *Suppose G acts on X, and $H \subseteq G$ is a subgroup. Further suppose that $H_G^* \to H_H^*$ is flat. Then $H_H^*(X) = H_G^*(X) \otimes_{H_G^*} H_H^*$.*

Proof. Since H acts on EG freely, we can assume $EH = EG$. There is a homotopy pullback diagram

$$\begin{array}{ccc} X \times^H EG & \longrightarrow & BH \\ \downarrow & & \downarrow \\ X \times^G EG & \longrightarrow & BG \end{array}$$

The Eilenberg-Moore spectral sequence of this diagram gives a spectral sequence

$$E_2^{*,*} = \mathrm{Tor}^{*,*}_{H_G^*}(H_G^*(X), H_H^*) \Rightarrow H_H^*(X).$$

When H_H^* is flat over H_G^*, all Tor terms vanish, so $H_H^*(X) = H_G^*(X) \otimes_{H_G^*} H_H^*$. □

3.3 Atiyah-Bott localization theorem

Theorem 3.2. *Let a torus T act on a quasi-projective variety X. The restriction map $H_T^*(X) \to H_T^*(X^T)$ is an isomorphism after inverting non-zero elements of H_T^*. In other words, the kernel and cokernel are torsion H_T^*-modules.*

Proof. See [4].

□

4 Equivariant cohomology of Gr

The equivariant cohomology $H^*_{G(\mathcal{O})\rtimes\mathbb{C}^\times}(\mathrm{Gr})$ has been determined by Bezrukavnikov and Finkelberg [2]. In this section I will present their result.

Theorem 4.1. *There is an isomorphism $\mathbb{C}[N_{\mathfrak{t}/W\times\mathfrak{t}/W}\Delta] \xrightarrow{\alpha^*} H^*_{G(\mathcal{O})\rtimes\mathbb{C}^\times}(\mathrm{Gr})$.*

Here, $N_{\mathfrak{t}/W\times\mathfrak{t}/W}\Delta$ is the deformation to the normal cone of the diagonal embedding $\mathfrak{t}/W \xrightarrow{\Delta} \mathfrak{t}/W \times \mathfrak{t}/W$. This will be defined in Section 4.1. The map α^* will be defined in Section 4.3.

4.1 Deformation to the normal cone

For a closed subscheme $Z \subseteq X$ with ideal sheaf $\mathcal{I}$, the normal cone can be defined purely algebraically. $C_X Z = \operatorname{Spec}_X \bigoplus_{n\geq 0} \mathcal{I}^n/\mathcal{I}^{n+1}$. This is similar to the tubular neighborhood in topology. However, unlike the tubular neighborhood, the normal cone does not embed in X. To compensate for this, we define an object called the deformation to the normal cone.

The *deformation to the normal cone* $N_X Z$ is defined as

$$\operatorname{Spec}_X \bigoplus_{n=-\infty}^{\infty} \mathcal{I}^{-n} t^n.$$

Here, $\mathcal{I}^n = \mathcal{O}_X$ for $n \leq 0$.

Note the map $N_X Z \to X \times \mathbb{A}^1$. Over 0, this is the normal cone $C_X Z \to X$, and away from 0 this is an isomorphism. Another way to describe the deformation to the normal cone is the affine open subset of $\mathrm{Bl}_{X\times\mathbb{A}^1}(Z \times \{0\}) = \operatorname{Proj}_X \bigoplus_{n=0}^{\infty} (\mathcal{I}, t)^n$ where $t \neq 0$.

This particular case for affine spaces is important to us.

Proposition 4.2. *Suppose $V \cong \mathbb{A}^n$ and $W \cong \mathbb{A}^m$, then*

$$N_{V\times W} W = \{(x, y, t, z) \in V \times W \times \mathbb{A}^1 \times V \mid tz = x\}.$$

The map $N_{V\times W} W \to V \times W \times \mathbb{A}^1$ is $(x, y, t, z) \mapsto (x, y, t)$.

Proof. Let $V = \operatorname{Spec} \mathbb{C}[x_1, \dots, x_n]$ and $W = \operatorname{Spec} \mathbb{C}[y_1, \dots, y_m]$. Then $W \subseteq V \times W$ is given by the ideal $I = (x_1, \dots, x_n)$. Recall that we have

$$\bigoplus_{n\geq 0} I^n = \mathbb{C}[x_1, \dots, x_n, y_1, \dots, y_m, z_1, \dots, z_n]/(x_i z_j - x_j z_i)$$

as a graded algebra, where $\deg(z_i) = 1$. Thus

$$\bigoplus_{n=-\infty}^{\infty} I^{-n} t^n = \mathbb{C}[x_1, \dots, x_n, y_1, \dots, y_m, z_1, \dots, z_n, t]/(x_i z_j - x_j z_i, t z_i - x_i).$$

The relations $x_i z_j - x_j z_i$ are redundant. The Spec of this algebra is the scheme above. ◻

Corollary 4.3. *Suppose $V \cong \mathbb{A}^n$ is the affine space, and $V \xrightarrow{\Delta} V \times V$ is the diagonal embedding. Then*

$$N_{V\times V} \Delta = \{(x, y, t, z) \in V \times V \times \mathbb{A}^1 \times V \mid tz = x - y\}.$$

The map $N_{V\times V} \Delta \to V \times V \times \mathbb{A}^1$ is $(x, y, t, z) \mapsto (x, y, t)$.

The following corollary will be useful in Section 4.

Corollary 4.4. *If $f, g : X \to V$ and $p : X \to \mathbb{A}^1$ satisfy $f(x) = g(x)$ when $p(x) = 0$, then the map $(f, g, p) : X \to V \times V \times \mathbb{A}^1$ factors as $X \to N_{V\times V} \Delta \to V \times V \times \mathbb{A}^1$.*

Proof. We need to find $h : X \to V$ such that $ph = f - g$.

We can assume X is affine. By choosing a coordinate for V and working with each coordinate, we can assume $V \cong \mathbb{A}^1$. Then $\operatorname{Hom}(X, \mathbb{A}^1) \cong \mathcal{O}_X$. The statement translates to the following. We have $p, f - g \in \mathcal{O}_X$ such that the zero locus of p is contained in the zero locus of $f - g$, then there exists $h \in \mathcal{O}_X$ such that $ph = f - g$. This is clearly true as $f - g \in (p)$. ◻

4.2 Graded dimension

Recall that $H^*(G)$ is an exterior algebra with generators in degrees $2d_i - 1$.

Proposition 4.5. *The graded dimension of $H^*(\mathrm{Gr})$ is the same as that of $\mathbb{C}[t_1, \ldots, t_r]$ where $\deg t_i = 2d_i - 2$.*

Remark. The following proof does not determine the ring structure of $H^*(\mathrm{Gr})$. In fact $H^*(\mathrm{Gr})$ is isomorphic to the polynomial ring $\mathbb{C}[t_1, \ldots, t_r]$, see Ginzburg.

Proof. Recall that Gr is the loop space of the maximal compact subgroup of G. Up to homotopy types, it may as well be the loop space of G. Consider the Eilenberg-Moore spectral sequence for the path-loop fibration,

$$E_2^{*,*} = \mathrm{Tor}_{*,*}^{H^*(G)}(\mathbb{C}, \mathbb{C}) \Rightarrow H^*(\mathrm{Gr}).$$

Write $H^*(G) = \Lambda^\bullet V$, where the basis elements of V are in degrees $2d_1 - 1, \ldots, 2d_r - 1$, respectively. To compute the Tor in question, notice that the following is a resolution of $\mathbb{C}$ with projective $\Lambda^\bullet V$-modules (this is due to the Koszul duality of the exterior algebra and the symmetric algebra, see [1])

$$\cdots \to \Lambda^\bullet V \otimes S^n V \to \cdots \to \Lambda^\bullet V \otimes S^2 V \to \Lambda^\bullet V \otimes V \to \Lambda^\bullet V,$$

$$d : (x_1 \wedge \cdots \wedge x_k) \otimes (y_1 \cdots y_i) \mapsto \sum_{j=1}^{i} (x_1 \wedge \cdots \wedge x_k \wedge y_j) \otimes (y_1 \cdots \widehat{y_j} \cdots y_i).$$

Thus $\mathrm{Tor}_p^{\Lambda^\bullet V}(\mathbb{C}, \mathbb{C}) = S^p V$. Note that V has generators in odd degrees, so $S^p V$ are in the odd degrees when p is odd, and even degrees when p is even. This shows there cannot be differentials on E_2. We have $H^k(\mathrm{Gr}) \cong \bigoplus_{q-p=k} \mathrm{Tor}_{p,q}^{\Lambda^\bullet V}(\mathbb{C}, \mathbb{C})$, so

$$H^*(\mathrm{Gr}) \cong \bigoplus_p S^p V[p] = S^\bullet(V[1]).$$

∎

From here, we can deduce the graded dimension of $H^*_{G(\mathcal{O}) \rtimes \mathbb{C}^\times}(\mathrm{Gr})$.

Proposition 4.6. *The graded dimension of $H^*_{G(\mathcal{O}) \rtimes \mathbb{C}^\times}(\mathrm{Gr})$ coincides with the graded dimension of $\mathbb{C}[x_1, \ldots, x_r, y_1, \ldots, y_r, \hbar]$ with $\deg x_i = 2d_i$, $\deg y_i = 2d_i - 2$, and $\deg \hbar = 2$.*

Proof. Note that $H^*_{G(\mathcal{O}) \rtimes \mathbb{C}^\times} \cong H^*_{G \times \mathbb{C}^\times} = H^*_G \otimes H^*_{\mathbb{C}^\times}$. Next, notice that the spectral sequence $H^*(\mathrm{Gr}) \otimes H^*_{G(\mathcal{O}) \rtimes \mathbb{C}^\times} \Rightarrow H^*_{G(\mathcal{O}) \rtimes \mathbb{C}^\times}(\mathrm{Gr})$ collapses as everything is in even degree, so this follows from Proposition 4.5. ∎

4.3 Proof of Theorem 4.1

We have a map $f : H^*_{G(\mathcal{O})} \to H^*_{G(\mathcal{O}) \rtimes \mathbb{C}^\times}(\mathrm{Gr})$. There is another map $g : H^*_{G(\mathcal{O})} \to H^*_{G(\mathcal{O}) \rtimes \mathbb{C}^\times}(\mathrm{Gr})$ coming from the right action of $G(\mathcal{O})$ on $G(K)$. To see the symmetry of f and g, notice that we can write

$$H^*_{G(\mathcal{O}) \rtimes \mathbb{C}^\times}(\mathrm{Gr}) = H^*_{(G(\mathcal{O}) \times G(\mathcal{O})) \rtimes \mathbb{C}^\times}(G(K)),$$

where $G(\mathcal{O}) \times G(\mathcal{O})$ acts on $G(K)$ by $(x,y) : a \mapsto xay^{-1}$. The two projections $\mathsf{pr}_1, \mathsf{pr}_2 : (G(\mathcal{O}) \times G(\mathcal{O})) \rtimes \mathbb{C}^\times \to G(\mathcal{O})$ induce the maps f and g.

Let $p : \mathbb{C}[\hbar] = H^*_{\mathbb{C}^\times} \to H^*_{G(\mathcal{O})\rtimes\mathbb{C}^\times}(\mathrm{Gr})$. Notice that $H^*_{G(\mathcal{O})\rtimes\mathbb{C}^\times}(\mathrm{Gr})$ is a commutative algebra as it is nonzero only in even degrees, so we can let $X = \operatorname{Spec} H^*_{G(\mathcal{O})\rtimes\mathbb{C}^\times}(\mathrm{Gr})$. Also note that $H^*_{G(\mathcal{O})} = H^*_G = \mathbb{C}[\mathfrak{t}/W]$. Denote by $f^\sharp, g^\sharp, p^\sharp$ also the maps corresponding to f, g, p on affine schemes, then we get

$$(f^\sharp, g^\sharp, p^\sharp) : X \to \mathfrak{t}/W \times \mathfrak{t}/W \times \mathbb{A}^1.$$

Consider the composition of (f, g, p) with

$$\phi_\lambda : H^*_{G(\mathcal{O})\rtimes\mathbb{C}^\times}(\mathrm{Gr}) \xrightarrow{r} H^*_{T\times\mathbb{C}^\times}(\mathrm{Gr}) \xrightarrow{i^*_\lambda} H^*_{T\times\mathbb{C}^\times}$$

where i^*_λ is the restriction to the point z^λ.

$\phi_\lambda f$ is the map $H^*_{G(\mathcal{O})} \to H^*_{T\times\mathbb{C}^\times}$ induced by $T \times \mathbb{C}^\times \hookrightarrow G(\mathcal{O}) \rtimes \mathbb{C}^\times \to G(\mathcal{O})$. This composition is the same as $T \times \mathbb{C}^\times \to T \hookrightarrow G(\mathcal{O})$, so in terms of schemes, $\phi_\lambda f$ corresponds to the projection $\mathfrak{t} \times \mathbb{A}^1 \to \mathfrak{t} \to \mathfrak{t}/W$.

Next consider $\phi_\lambda g : H^*_{G(\mathcal{O})} \to H^*_{T\times\mathbb{C}^\times}$. Under the action of $T \times \mathbb{C}^\times$ on $G(K)$, z^λ is not a fixed point, but is transformed to a point in $z^\lambda G(\mathcal{O})$. Specifically, for $(x, a) \in T \times \mathbb{C}^\times$, one can check that $(x, a) \cdot z^\lambda = z^\lambda x\lambda(a)$, where $x\lambda(a) \in G(\mathcal{O})$. Therefore the map $\phi_\lambda g$ is induced by $T \times \mathbb{C}^\times \to G(\mathcal{O})$, $(x, a) \mapsto x\lambda(a)$. The corresponding map of schemes is

$$\begin{aligned} \mathfrak{t} \times \mathbb{A}^1 &\to \mathfrak{t}/W, \\ (x, a) &\mapsto x + a\lambda. \end{aligned}$$

Finally, $\phi_\lambda p : \mathbb{C}[\hbar] \to H^*_{T\times\mathbb{C}^\times}$ corresponds to the map of schemes $\mathfrak{t} \times \mathbb{A}^1 \to \mathbb{A}^1$. Let

$$\phi = \prod \phi_\lambda : H^*_{G(\mathcal{O})\rtimes\mathbb{C}^\times}(\mathrm{Gr}) \xrightarrow{r} H^*_{T\times\mathbb{C}^\times}(\mathrm{Gr}) \xrightarrow{i^*} \prod_{\lambda \in X_*(T)} H^*_{T\times\mathbb{C}^\times}.$$

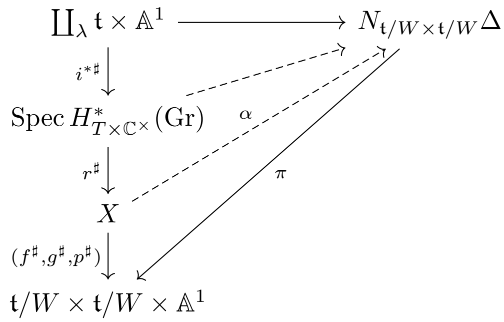
By the above computation, the following diagram commutes,

$$\begin{array}{ccc} \coprod_\lambda \mathfrak{t} \times \mathbb{A}^1 & \xrightarrow{\phi^\sharp} & X \\ {\scriptstyle m}\downarrow & & \downarrow{\scriptstyle (f^\sharp, g^\sharp, p^\sharp)} \\ \mathfrak{t} \times \mathfrak{t} \times \mathbb{A}^1 & \longrightarrow & \mathfrak{t}/W \times \mathfrak{t}/W \times \mathbb{A}^1 \end{array}$$

where $m_\lambda : \mathfrak{t} \times \mathbb{A}^1 \to \mathfrak{t} \times \mathfrak{t} \times \mathbb{A}^1$ is given by $(x, a) \mapsto (x, x + a\lambda, a)$. In particular m is dominant as $X_*(T)$ is dense in $\mathfrak{t}$ in the Zarisky topology. As a result $(f^\sharp, g^\sharp, p^\sharp) \circ \phi^\sharp$ is also dominant.

Notice that m, as well as $(f^\sharp, g^\sharp, p^\sharp) \circ \phi^\sharp$ satisfy the condition of Corollary 4.4. Thus, $(f^\sharp, g^\sharp, p^\sharp) \circ$

$\phi^\sharp$ factors through $\pi : N_{\mathfrak{t}/W\times\mathfrak{t}/W}\Delta \to \mathfrak{t}/W \times \mathfrak{t}/W \times \mathbb{A}^1$, as in the following diagram.



By Proposition 2.6, the $T \times \mathbb{C}^\times$-fixed points of Gr are z^λ for coweights λ. By Atiyah-Bott localization theorem (Theorem 3.2), i^* has torsion kernel and cokernel. Therefore there is a lift drawn as the dashed arrow $\operatorname{Spec} H^*_{T\times\mathbb{C}^\times}(\mathrm{Gr}) \to N_{\mathfrak{t}/W\times\mathfrak{t}/W}\Delta$. Next, r is injective as it is just the map

$$H^*_{G(\mathcal{O})\rtimes\mathbb{C}^\times}(\mathrm{Gr}) \to H^*_{G(\mathcal{O})\rtimes\mathbb{C}^\times}(\mathrm{Gr}) \otimes_{H^*_{G(\mathcal{O})\rtimes\mathbb{C}^\times}} H^*_{T\times\mathbb{C}^\times}$$

by Proposition 3.1 (in fact, $H^*_{G(\mathcal{O})\rtimes\mathbb{C}^\times} \to H^*_{T\times\mathbb{C}^\times}$ is flat as the latter is a free module over the former). Therefore, $r^\sharp$ is surjective, so $(f^\sharp, g^\sharp, p^\sharp)$ also satisfy the conditions of Corollary 4.4. Thus, there is a map $\alpha : X \to N_{\mathfrak{t}/W\times\mathfrak{t}/W}\Delta$ commuting with the rest of the diagram.

The morphism $(f^\sharp, g^\sharp, p^\sharp)$ is dominant, and π is an isomorphism away from 0, so α is dominant. Thus, the induced map on rings $\alpha^* : \mathbb{C}[N_{\mathfrak{t}/W\times\mathfrak{t}/W}\Delta] \to H^*_{G(\mathcal{O})\rtimes\mathbb{C}^\times}(\mathrm{Gr})$ is injective. By Corollary 4.3 and Proposition 4.6, the graded dimensions of both algebras coincide with that of $\mathbb{C}[x_1, \ldots, x_r, y_1, \ldots, y_r, \hbar]$ where $\deg x_i = 2d_i$, $\deg y_i = 2d_i - 2$, and $\deg \hbar = 2$. We conclude that α^* is an isomorphism.

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