Towards a Proof of the Toral Eigenfunction Restriction Theorem in 4D by Spectral Methods

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Abstract

In 2011, Bourgain and Rudnick proved that for a hypersurface Σ of non-vanishing curvature embedded in a 2D or 3D flat torus $\mathbb{T}^d$ (a cube with periodic boundary conditions), the surface L^2-norm of a normalized Laplacian eigenfunction φ_λ (i.e., $\|\varphi_\lambda\|_{L^2(\mathbb{T}^d)} = 1$) will be bounded from above and below by constants that only depend on the surface Σ. In particular, we have $c_\Sigma \leq \|\varphi_\lambda\|_{L^2(\Sigma)} \leq C_\Sigma$ for all λ sufficiently large. As the propagation vector of any toral eigenfunction of eigenvalue $-\lambda^2$ is a lattice point in $\mathbb{Z}^d$ intersected with $\lambda\mathbb{S}^{d-1}$, we must study the asymptotic density of lattice points in spherical caps $\mathbb{Z}^d \cap R\mathbb{S}^{d-1} \cap B_r(b)$ where $R = \lambda$, $r = c^{-1}R^{1-\delta}$, and $b \in R\mathbb{S}^{d-1}$.

When δ is roughly in the range $\left[0, \frac{1}{12}\right]$, Henryk Iwaniec's result for the equidistribution of spherical lattice points suffices; when $\delta \in (\frac{4}{5}, 1]$, a Jarník-type theorem shows that all points will be asymptotically coplanar. In the present paper, we outline our progress towards understanding the density for intermediate values of δ by applying the Kuznetsov trace formula to study the cancellations between Kloosterman sums in Bourgain's argument.

1 The Toral Eigenfunction Restriction Theorem

Theorem 1. *Let $d = 2, 3$ and $\Sigma \subset \mathbb{T}^d$ be a real-analytic hypersurface with nonzero curvature. Then, there exists c_Σ, C_Σ, and Λ_Σ depending only on the hypersurface such that for all $\lambda > \Lambda_\Sigma$*

$$c_\Sigma \|\varphi_\lambda\|_{L^2(\mathbb{T}^d)} \leq \|\varphi_\lambda\|_{L^2(\Sigma)} \leq C_\Sigma \|\varphi_\lambda\|_{L^2(\mathbb{T}^d)}$$

where $-\Delta\varphi_\lambda = \lambda^2\varphi_\lambda$ and $L^2(\Sigma)$ indicates a surface integral over Σ.

Proof. The proof is split across Theorem 8.1 and 9.1 in *Restriction of Toral Eigenfunctions to Hypersurfaces and Nodal Sets* [1]. □

Bourgain notes that $-\Delta\varphi_\lambda = \lambda^2\varphi_\lambda$ forces all of the propagation vectors for the eigenfunction φ_λ to lie on a sphere of radius $R = \lambda$. Let the set of lattice points on the sphere be $\mathcal{E}_\lambda := \mathbb{Z}^d \cap \lambda\mathbb{S}^{d-1}$. Then, φ_λ has the form $\varphi_\lambda(\vec{x}) = \sum_{\vec{z}\in\mathcal{E}_\lambda} a_z e^{i\langle\vec{z},\vec{x}\rangle}$ and can be normalized so that $\|a_z\|_{l^2(\mathcal{E}_\lambda)} = 1$. Next, Bourgain expands the L^2-norm of φ over Σ as an inner product and pulls out the double sum. The terms from the main diagonal contribute $\operatorname{vol}(\Sigma)$ because of the normalization, while the off diagonal terms (i.e., those for which $\vec{z}_1 \neq \vec{z}_2$) decompose dyadically. Let $A_0 = B_1(\vec{v})$ and $A_k(\vec{v}) = B_{2^k}(\vec{v}) \setminus B_{2^{k-1}}(\vec{v})$ be a dyadic spherical shell, so that the norm of the eigenfunction becomes

$$\|\varphi_\lambda\|^2_{L^2(\Sigma)} = \mathrm{vol}(\Sigma) + \sum_{k\geq 0}\sum_{\vec{z}_1\in\mathcal{E}_\lambda}\sum_{\vec{z}_2\in A_k(\vec{z}_1)} \overline{a_{\vec{z}_1}} a_{\vec{z}_2} \int_\Sigma \exp\langle i(\vec{z}_2-\vec{z}_1),\vec{x}\rangle\, d\sigma$$

Using the assumption of nonzero curvature, Bourgain applies the Morse Lemma to isolate critical points and calculate the asymptotics of the integral using a stationary phase analysis. Controlling the contributions of the asymptotics to the final norm follows from bounding $|\mathcal{E}_\lambda \cap A_k(\vec{z}_1)|$, which is connected to the clustering of lattice points on spheres.

Definition 2 (Spherical Cap)**.** A spherical cap $C(\vec{b}, r)$ is a neighborhood around a point b on a sphere of radius $R = |b|$ such that all of its points are within a distance r of b.

$$C(\vec{b}, r) = R\mathbb{S}^{d-1} \cap B_r(\vec{b})$$

We formalize the number of lattice points in a cap as $N_d(\vec{b}, r) = \left|\mathbb{Z}^d \cap C(\vec{b}, r)\right|$ and seek to study $\lim_{R\to\infty} N_d(\vec{b}, r)$ where $|b| = R$ and r is a function of R.

1.1 Bourgain and Rudnick's (B&R) Bounds in $d = 3$

If $r \ll R^{\frac{1}{d+1}}$, then Jarnik's theorem implies that all of the points on $C(\vec{b}; R)$ are coplanar for any choice of $\vec{b}$. Thus, if $r \ll R^{\frac{1}{d+1}}$, we have $\sup_{\vec{b}\in R\mathbb{S}^{d-1}} N_d(\vec{b}, r) \ll r^{d-3+\epsilon}$ for all $\epsilon > 0$.

$$\sup_{\vec{b}\in R\mathbb{S}^{d-1}} N_d(\vec{b}, r) \ll r^{\alpha_1} R^{\alpha_2} + r^{\alpha_3} R^{\alpha_4}$$

When $0 < \epsilon$ and $0 < \eta < \frac{1}{15}$, Bourgain and Rudnick prove the following asymptotics for $\mathbb{S}^{d-1} \subset \mathbb{R}^d$.

d	α_1	α_2	α_3	α_4
3	$1+\eta$	$\epsilon-\eta$	ϵ	1
4	3	$\epsilon-1$	$3/2$	ϵ
≥ 5	$d-1$	$\epsilon-1$	$d-3$	ϵ

1.2 B&R's Progress towards $d = 4$ Bounds

The quantity $N_d(\vec{b}, r)$ counts the number of lattice points inside a ball of radius r centered at $\vec{b}$. Change coordinates so that $\vec{b}$ is the new origin. Now, the condition of lying on the sphere of radius R is $|y|^2 + 2b \cdot y = 0$. Lastly, inscribe the ball inside a cube to conclude

$$N_d(\vec{b}, r) \leq \tilde{N}_d(\vec{b}, r) := \left|\tilde{C}_d(\vec{b}, r)\right| \quad \text{where}$$
$$\tilde{C}_d(\vec{b}, r) := \left\{ y \in \mathbb{Z}^d \cap [-r, r]^d \middle| |y|^2 + 2\vec{b}\cdot y = 0 \right\}$$

1.2.1 Hardy-Littlewood Circle Method

Let $e(x) = \exp(2\pi i x)$ and $e_q(x) = \exp\left(2\pi i \frac{x}{q}\right)$. From $\int_{\mathbb{T}} e(f(y)t)dt = \delta_{f(y)}$, Bourgain dominates the indicator function of $\tilde{C}_d(\vec{b}, r)$ to show

$$\tilde{N}_d(\vec{b}, r) = \sum_{y \in \mathbb{Z}^d \cap [-r,r]^d} \delta_{|y|^2 + 2Rv \cdot y} \leq \int_{\mathbb{T}} \prod_{j=1}^{d} G(t, 2b_j t) dt$$

where

$$G(t, \varphi) = \sum_{y \in \mathbb{Z}} \gamma\Big(\frac{y}{r}\Big) g(t, \varphi; y)$$
$$g(t, \varphi; y) = e\big(y^2 t + y\varphi\big)$$

and $\gamma \in \mathcal{S}(\mathbb{R})$ is an even bump function with $\gamma|_{[-1,1]} \geq 1$. For simplicity, we choose $\gamma(x) = \exp(2 - x^2)$.
Apply Dirichlet's Approximation Theorem to $\mathbb{T}$. Define $\mathbb{Z}_q^\times = \{0 \leq a < q \mid \gcd(a, q) = 1\}$ so that

$$\forall t \in \mathbb{T} \exists q \in \{1, \dots, r-1\} \, \exists a \in \mathbb{Z}_q^\times \left[|qt - a| < r^{-1}\right]$$

Let $\beta = t - \frac{q}{a}$ be the remainder so that $\beta \in \left[-\frac{1}{qr}, \frac{1}{qr}\right]$.

$$\int_{\mathbb{T}} \prod_{j=1}^{d} G(t, 2b_j t) dt \approx \sum_{q=1}^{r} \int_{-\frac{1}{qr}}^{\frac{1}{qr}} \sum_{a \in \mathbb{Z}_q^\times} \prod_{j=1}^{d} G\left(\frac{q}{a} + \beta, 2b_j \left(\frac{q}{a} + t\right)\right) d\beta$$

Here, we use the $\approx$ symbol because the intervals may overlap and the minor arcs are not yet explicity handled.

1.2.2 Poisson Summing $G(t, \varphi)$

Then, B&R take a coset decomposition and a variable change to apply Poisson summation.

$$\begin{aligned}
G(t, \varphi) &= \sum_{y \in \mathbb{Z}} \gamma\Big(\frac{y}{r}\Big) g\left(\frac{a}{q} + \beta, \varphi; y\right) && \mathbb{Z} = \coprod_{k=0}^{q-1} q\mathbb{Z} + k, \quad y \mapsto qy + k \\
&= \sum_{k=0}^{q-1} e_q\left(k^2 a\right) \sum_{y \in \mathbb{Z}} \gamma\Big(\frac{qy + k}{r}\Big) g\left(\beta, \varphi; qy + k\right) && \text{Factor } a \text{ out of the y-sum.} \\
&= \sum_{k=0}^{q-1} e_q\left(k^2 a\right) \sum_{\xi \in \mathbb{Z}} \int_{\mathbb{R}} e\big(-\xi y\big) \gamma\Big(\frac{qy + k}{r}\Big) g\left(\beta, \varphi; qy + k\right) dy && \text{Poisson summation} \\
&= \sum_{m \in \mathbb{Z}} S(a, m; q) J(\varphi, \beta, m; q) && y \mapsto \frac{y - k}{q}, m = -\xi
\end{aligned}$$

In the final line, $S(a, m; q)$ is a normalized Gauss sum. (The normalization follows from the last u-substitution.) $J(\varphi, \beta, m; q)$ is the integral of a Gaussian wave packet.

$$S(a, m; q) = \frac{1}{q} \sum_{k=0}^{q-1} e_q\left(k^2 a - km\right)$$
$$J(\varphi, \beta, m; q) = \int_{\mathbb{R}} \gamma\left(\frac{y}{r}\right) e\left(y^2 \beta + y\varphi + m\frac{y}{q}\right) dy$$

1.2.3 Producing Kloosterman Sums

The integrand of J is a wave-envelope, so J obeys the following shifting formula when $\varphi = 2b_j\left(\frac{a}{q} + \beta\right)$.

$$J\left(2b_j\left(\frac{a}{q} + \beta\right), \beta, m_j - 2ab_j; q\right) = J\left(2b_j\beta, \beta, m_j; q\right)$$

Take $\varphi = 2b_j t$. Then, using the periodicity of J, shift $m \mapsto m - 2ab$ to simplify the J term and convert the Gauss sums $S(a, m; q)$ into Kloosterman sums $\tilde{K}(m; q, b)$.

$$\int_{\mathbb{T}} \prod_{j=1}^{d} G(t, 2b_j t)\, dt = \sum_{q=1}^{r-1} \sum_{m \in \mathbb{Z}^d} \tilde{J}(m; q, b, (qr)^{-1}) \tilde{K}(m; q, b)$$

where

$$\tilde{J}(m; q, b, \beta_0) = \int_{-\beta_0}^{\beta_0} \prod_{j=1}^{d} J\left(2b_j\beta, \beta, m_j; q\right)\, d\beta$$
$$\tilde{K}(m; q, b) = \sum_{a \in \mathbb{Z}_q^\times} \prod_{j=1}^{d} S(a, m_j - 2ab_j; q)$$

The proof that $\tilde{K}(m; q, b)$ is a Kloosterman sum is quite technical, so we defer it until Section 3.1.2.

1.3 A Strategy to Improve the Bounds

B&R's 4D argument relies on applying the Weil bound to each Kloosterman sum individually. Although the Weil bound is optimal for a single sum, we lose information when applying it to a sum of Kloosterman sums weighted by complex numbers, since we must first use the triangle inequality. (The triangle inequality discards the relative phases of the elements, thereby missing potential cancellations.)

$$\left|\sum_{q=1}^{r-1} c_q \tilde{K}\left(m; q, b\right)\right| \leq \sum_{q=1}^{r-1} |c_q| \left|\tilde{K}\left(m; q, b\right)\right|$$

The Kuznetsov formula expresses a weighted sum of Kloosterman sums in terms of the spectrum of the automorphic Laplacian on hyperbolic space and Fourier coefficients of cusp forms. In the remainder of the paper, we investigate the Kuznetsov formula as a means to account for cancellations between the Kloosterman sums.

2 Refining the Scope to Focus on Shrinking Caps

2.1 B&R's Higher Dimensional Bounds

Bourgain and Rudnick's results about spherical caps can be summarized as follows.

Lemma 3 (Bourgain and Rudnick 2011). *Let C be a cap of size r on the sphere $\{|\vec{x}| = R\} \subset \mathbb{R}^3$. Then for any $0 < \eta < 1/16$,*

$$\#(C \cap \mathbb{Z}^3) \ll R^{\epsilon}\Big(1 + r\Big(\frac{r}{R}\Big)^{\eta}\Big)$$

In higher dimensions, it is only known that

Lemma 4 (Bourgain and Rudnick 2011). *Let $\mathcal{E}_R = \mathbb{Z}^d \cap R\mathbb{S}^{d-1}$ and $\omega(N)$ be the number of prime factors of $N = R^2$, then*

$$|\mathcal{E}_R \cap C_r| \lesssim \begin{cases} \dfrac{r^{d-1}}{R} + r^{d-3} & \textit{if } d \geq 7 \\ \dfrac{r^5}{R} + (\log \omega(N))^2 r^3 & \textit{if } d = 6 \\ \dfrac{r^4}{R} + r^{2+\varepsilon} R^{\varepsilon} & \textit{if } d = 5 \\ \dfrac{r^3}{R}(\log \omega(N))^2 + r^{\frac{3}{2}+\varepsilon} R^{\varepsilon} & \textit{if } d = 4 \end{cases}$$

The exponent in r worsens from $1 + \eta \approx \frac{17}{16}$ to 3, which breaks the later arguments for the upper bound. If the radius of the cap r is small enough that it obeys $r \ll R^{\frac{1}{(d+1)}}$ (where $\ll$ is Vinogradov's asymptotic notation), then a variant of Jarnik's theorem [2] shows that all of the points are asymptotically coplanar. In the next section, we shall handle larger balls with an equidistribution result by Iwaniec.

2.2 Equidistribution of 4D Spherical Lattice Points in Large Balls

On account of the following result proven by Iwaniec, the remaining asymptotic regime is $r = c^{-1} R^{1-\delta}$ for some $c > 0$ and $\delta_0 < \delta < \frac{d}{d+1}$ with $\delta_0 > 0$ a currently unknown lower bound.

Theorem 5 (Iwaniec 1997 Prop. 11.4). *Let $d \geq 4$. If $f : \mathbb{S}^{d-1} \to \mathbb{C}$ is $\mathcal{C}^{\infty}$, then*

$$r_f(n) := \sum_{m \in \mathbb{Z}^d : |m| = n} f\left(\frac{m}{\sqrt{n}}\right)$$

has the asymptotic for some $\epsilon > 0$

$$r_f(n) = \left(\int_{\mathbb{S}^{d-1}} f(x)dx\right) r_d(n) + O\left(n^{\frac{d-1}{4}+\epsilon}\right)$$

where $r_d(n)$ counts the number of lattice points on a sphere of radius n [3].
The theorem only applies when f is constant in n, not when the support of f is shrinking as $n \to \infty$. To roughly approximate δ_0, let f be a smooth bump function supported on $C(\vec{b}/R, r/R) \subset \mathbb{S}^{d-1}$, where $R = |b|$. (We construct f by making a bump function supported on $C(\vec{b}, r) \subset R\mathbb{S}^{d-1}$ and then linearly scale everything down by multiplying by $\frac{1}{R}$.) Then, by rotational symmetry, we can assume $\vec{b} = e_1$ and define $C(\vec{b}/R, r/R) = \mathbb{S}^{d-1} \cap \{x \in \mathbb{R}^d | x_1 \geq h\}$ for $h = \frac{1}{2}\left(\frac{r}{R}\right)^2$. Also, the spherical cap has an opening angle $\theta = 2\arcsin\left(\frac{r}{2R}\right)$. Recall that the surface area of a $d-1$ dimensional spherical cap is given by the following formulas.

$$A_d(R) = \frac{\pi^{d/2}}{\Gamma(d/2)} R^{d-1}$$

$$I_x(a,b) = \int_0^x t^{a-1}(1-t)^{b-1}\,dt$$

$$\text{vol}\left(C(\vec{b}, r)\right) = A_d(R) = \frac{1}{2} A_d(R) I_{\sin^2\theta}\left(\frac{d-1}{2}, \frac{1}{2}\right)$$

We can perform an easy trigonometric argument to find θ and h in terms of r.

$$\begin{aligned}
&x = R, \quad z = Re^{i\theta} \\
&|x-z| = r \\
&|x-z|^2 = r^2 \\
&R^2|1-e^{i\theta}|^2 = r^2 \\
&|2i\sin(\theta/2)|^2 = |e^{-i\theta/2} - e^{i\theta/2}|^2 = \left(\tfrac{r}{R}\right)^2 \\
&2\sin(\theta/2) = \tfrac{r}{R} \\
&\tfrac{\theta}{2} = \arcsin\left(\tfrac{r}{2R}\right) \\
&\theta = 2\arcsin\left(\tfrac{r}{2R}\right)
\end{aligned}
\qquad
\begin{aligned}
&\tfrac{\theta}{2} + \tfrac{\pi}{2} + \phi = \pi \\
&\phi = \tfrac{\pi}{2} - \tfrac{\theta}{2} \\
&h = r\cos(\phi) \\
&h = r\cos\left(\tfrac{\pi}{2} - \tfrac{\theta}{2}\right) \\
&h = r\cos\left(\tfrac{\pi}{2} - \arcsin\left(\tfrac{r}{2R}\right)\right) \\
&h = r\sin\left(\arcsin\left(\tfrac{r}{2R}\right)\right) \\
&h = \tfrac{r^2}{2R}
\end{aligned}$$

Asymptotically, the surface area integral will behave like

$$\begin{aligned}
\text{vol}(C(\vec{b}, r)) \sim I(\theta) &:= \int_0^{\sin^2(\theta)} t^{1/2}(1-t)^{-1/2}dt \\
&= \int_0^{\theta} \tan(\theta) \cdot 2\sin(\theta)\cos(\theta)d\theta \\
&= \int_0^{\theta} 2\sin^2(\theta)d\theta = \int_0^{\theta}(1-\cos(2\theta))d\theta \\
&= \theta - \frac{1}{2}\sin(2\theta) \sim \theta^3
\end{aligned}$$

Also, $I(\theta)$ can be extended to all of $\mathbb{R}$ by stipulating that it is even and has period π. We can apply Jacobi's 4 square formula and Gronwall's asymptotic formula for the sum of divisors function to conclude that for any $\epsilon > 0$

$$
\begin{aligned}
r_4(n) &= 8\sum_{\substack{m|n\\ 4\nmid m}} m \le 8\sum_{m|n} m = 8\sigma(n)\\
&= 8e^{\gamma} n\ln(\ln(n)) + o(n\ln(\ln(n)))\\
&= O(n^{1+\epsilon})
\end{aligned}
$$

We get the more accurate bound

$$
\begin{aligned}
r_f(R) &= \left(\int_{\mathbb{S}^{d-1}} f(x)dx\right) r_d(R) + O\left(R^{\frac{d-1}{4}+\epsilon}\right)\\
&\sim \mathrm{vol}\left(C(\vec{b}/R, r/R)\right) r_d(R) + O\left(R^{\frac{d-1}{4}+\epsilon}\right)\\
&\sim I\left(2\arcsin\left(\frac{1}{2cR^{\delta}}\right)\right)(8e^{\gamma}R(\ln(\ln(R))) + o(R\ln(\ln(R)))) + O(R^{3/4+\epsilon})
\end{aligned}
$$

but decide to apply $r_4(n) = O(n^{1+\epsilon})$ and $I(\theta) \sim \theta^3$ to conclude the more manageable bound

$$
\begin{aligned}
r_f(R) &\sim R^{-3\delta} r_4(R) + O(R^{3/4+\epsilon})\\
&\sim O(R^{1-3\delta+\epsilon}) + O(R^{3/4+\epsilon})
\end{aligned}
$$

Solving, $1-3\delta < \frac{3}{4}$ yields the constraint that $\delta > \delta_0 = \frac{1}{12}$. Practically speaking, we are most interested in the lines for our computational plot for which $\log(r) \in \log(R)[1/(d+1), 1-\delta_0] = \log(R)\left[\frac{1}{5}, \frac{11}{12}\right]$. Thus, for $d = 4$, we are interested in $\delta \in \left[\frac{1}{12}, \frac{4}{5}\right]$

3 Setting up the Kuznetsov Formula

3.1 A More Detailed Phase Analysis of the Kloosterman Sums

3.1.1 Removing the Linear Phase of the Gauss Sum

Lemma 6. *The quadratic Gauss sum has the following recursive evaluation*

$$
\begin{aligned}
S(a,m;q) &= \frac{1}{q}\sum_{k=0}^{q-1} e_q\big(k^2 a - km\big)\\
&= \begin{cases}
e_q(m^2(2a)_q^{-1}((2)_q^{-1}-1)))S(a,0;q) & 2\nmid q\\
e_q(-(m^2/4)(a)_q^{-1})S(a,0;q) & 2\mid m, 2\mid q\\
e_q\left(\phi(q/2)m^2(a)_q^{-1}\right)2S(2a,0;q/2) & 2\nmid m, 2\parallel q\\
0 & 2\nmid m, 4\mid q
\end{cases}
\end{aligned}
$$

where $\phi(q/2) = 2(4)_{q/2}^{-1}\left((2)_{q/2}^{-1}-1\right)$. *For* $2\nmid m, 2\parallel q$, *we can also write to simplify the factor appearing in front of* m^2

$$
S(a,m;q) = e_q(-(a)_q^{-1}((m^2-1)/4 - \phi(q/2)))2S(2a,0;q/2)
$$

Proof. We consider translations $k \mapsto k + d$ for various values of d

$$S(a,m;q) = \frac{1}{q}\sum_{k=0}^{q-1} e_q\big(k^2a + k(2da - m) + (d^2a - dm)\big)$$
$$= e_q(d^2a - dm)S(a, -(2da - m); q)$$

Case 1 (q is odd): $2a$ will be invertible modulo q, so letting $d = m(2a)_q^{-1}$ will show that

$$S(a,m;q) = e_q(m^2(2a)_q^{-2}a - m^2(2a)_q^{-1}))S(a,0;q)$$
$$= e_q(m^2(2a)_q^{-1}((2)_q^{-1} - 1)))S(a,0;q)$$

Case 2 (q and m are even): Let $m = 2\mu$, so that $2da - 2\mu = 0$ becomes the desired formula. $d = \mu(a)_q^{-1}$ suffices. Then, $d^2a - dm = \mu^2(a)_q^{-2}a - \mu(a)_q^{-1}m = \mu(a)_q^{-1}(\mu - m) = -\mu^2(a)_q^{-1}$.

$$S(a,m;q) = e_q(-\mu^2(a)_q^{-1})S(a,0;q)$$

Case 3 (q is even and m is odd):
Case 3a ($4 \nmid q$): As $2 \mid\mid q$, 2 and $q/2$ are coprime to one another. We can apply a factorization

$$S(a,m;q) = S(2a,m;q/2)S\big(aq/2,m;2\big)$$

to reduce to the first case (since $q/2$ is odd)

$$S(2a,m;q/2) = S(2a,0;q/2)\, e_{q/2}\left(m^2(4a)_{q/2}^{-1}\left((2)_{q/2}^{-1} - 1\right)\right)$$
$$= S(2a,0;q/2)\, e_q\left(2m^2(4a)_{q/2}^{-1}\left((2)_{q/2}^{-1} - 1\right)\right)$$
$$= S(2a,0;q/2)\, e_q\left(\phi(q/2)m^2(a)_q^{-1}\right)$$

Finally, we have that

$$S(a,m;q) = S(2a,m;q/2)S\big(aq/2,m;2\big)$$
$$= S(2a,m;q/2)S\big(1,1;2\big) \qquad \text{S(a,b;c) depends only on } a \text{ and } b \bmod c$$
$$= 2S(2a,0;q/2)\, e_q\left(\phi(q/2)m^2(a)_q^{-1}\right)$$

Case 3a Aliter (Simpler m^2 coefficient): Let $m = 2\mu + 1$, we wish to find d such that $2da - m \equiv_q 1$. Taking $da = \mu + 1$, we have that $d = (a)_q^{-1}(\mu + 1)$ where $\mu = \lfloor m/2 \rfloor$.

$$S(a,m;q) = e_q\left(d\left(da - m\right)\right)S(a, -(2da - m); q)$$
$$= e_q((a)_q^{-1}(\mu + 1)((\mu + 1) - (2\mu + 1)))S(a,-1;q)$$
$$= e_q(-(a)_q^{-1}\mu(\mu + 1))S(a,-1;q)$$

As $4 \nmid q$, $q/2$ is odd. $\gcd(a,q)=1$ will imply $\gcd(a,q/2)=1$ allowing us to conclude

$$\begin{aligned} S(a,-1;q) &= S(qa/2,-1;2)\cdot S(2a,-1;q/2) \\ &= 2S(2a,-1;q/2) \end{aligned}$$

Here, we have used the fact that $qa/2$ and 1 are both odd, so that $(qa/2)k^2+k \equiv_2 0$ for $k=0,1$.
By case 1,

$$\begin{aligned} S(2a,-1;q/2) &= e_{q/2}((-1)^2(4a)^{-1}_{q/2}((2)^{-1}_{q/2}-1)))S(2a,0;q/2) \\ &= e_{q/2}((4a)^{-1}_{q/2}((2)^{-1}_{q/2}-1)))S(2a,0;q/2) \end{aligned}$$

qa is even, but $\gcd(q,a)=1$ implies $4 \nmid qa$. $2 \mid qa$, so

$$S(a,0;q) = S(2a,0;q/2)S(qa/2,0;2) = S(2a,0;q/2)((-1)^{\frac{qa}{2}}+1) = 0$$

preventing us from expressing the identity as a phase times $S(a,0;q)$. As a result, $2S(2a,1;q/2)$ shall take the role of $S(a,0;q)$. When we collect our terms, the identity becomes

$$\begin{aligned} S(a,m;q) &= e_q(-(a)^{-1}_q\mu(\mu+1)+2(4a)^{-1}_{q/2}((2)^{-1}_{q/2}-1)))2S(2a,0;q/2) \\ &= e_q(-(a)^{-1}_q(\mu(\mu+1)-2(4)^{-1}_{q/2}((2)^{-1}_{q/2}-1)))2S(2a,0;q/2) \end{aligned}$$

as $2(a)^{-1}_{q/2}a \equiv_q 2$ and a is invertible modulo q. Another way to see this is that $(a)^{-1}_{q/2}a \equiv_{q/2} 1$, so $(a)^{-1}_{q/2}a \equiv_q jq/2+1$ for $j=0,1$ depending on which coset we choose. Multiplying by 2 forgets our choice. Lastly, we see that $m=2\mu+1$ implies that $\mu(\mu+1)=(m^2-1)/4$, so we can express the equation as

$$\begin{aligned} S(a,m;q) &= e_q(-(a)^{-1}_q((m^2-1)/4-2(4)^{-1}_{q/2}((2)^{-1}_{q/2}-1)))2S(2a,0;q/2) \\ &= e_q(-(a)^{-1}_q((m^2-1)/4-\phi(q/2))2S(2a,0;q/2) \\ &= e_q\left(\left(\frac{(q/2)^2-1}{4}-\frac{m^2-1}{4}\right)(a)^{-1}_q\right)2S(2a,0;q/2) \end{aligned}$$

Case 3b ($4 \mid q$)*:* Let $q=2^n\tilde{q}$ for an odd $\tilde{q}$. Then, $S(a,m;q)=S(\tilde{q}a,m;2^n)\cdot S(2^na,m;\tilde{q})=0$ as $S(\tilde{q}a,m;2^n)=0$. □

3.1.2 Creating and Simplifying the Kloosterman Sums

Lemma 7. *Let $4 \mid d$. The product of Gauss sums has the following phases depending on the residue of $q \pmod 4$ and parity of m.*

$$\begin{aligned}\tilde{K}(m;q,b) &= \prod_{j=1}^{d} S(a, m_j - 2ab_j; q)\\ &= q^{-d/2} C_q^d e_q(A_q a + B_q (a)_q^{-1} + m\cdot b)\end{aligned}$$

$$A_q, B_q, C_q = \begin{cases} -|b|^2, \left(\frac{q^2-1}{4}\right)|m|^2, 1 & 2\nmid q, m\in\mathbb{Z}^d\\ -|b|^2, -\frac{1}{4}|m|^2, (-1)^{d/4}\sqrt{2} & 4\mid q, m\in(2\mathbb{Z})^d\\ |b|^2, \frac{1}{4}\left(d(q/2)^2 - |m|^2\right), 2\sqrt{2} & 2\mid\mid q, m\in(2\mathbb{Z}+1)^d\\ 0 & \text{otherwise}\end{cases}$$

Proof. We combine the results of Lemmas 1 and 2 to find
$S(a,0;q) = \varepsilon_q \frac{1}{\sqrt{q}} \left(a \mid q\right)$

$$e_q\left(2m^2(a)_q^{-1}(4)_{q/2}^{-1}\left((2)_{q/2}^{-1} - 1\right)\right) 2S(2a,0;q/2) = 2e_q\left(2m^2(a)_q^{-1}(4)_{q/2}^{-1}\left((2)_{q/2}^{-1}-1\right)\right)\varepsilon_{q/2}\frac{1}{\sqrt{q/2}}\left(2a \mid q/2\right)$$

$$S(a,m;q) = \frac{1}{\sqrt{q}}\begin{cases} e_q\left(m^2(2a)_q^{-1}((2)_q^{-1}-1))\right)\varepsilon_q\left(a\mid q\right) & 2\nmid q\\ \sqrt{2}e_q\left(-(m/2)^2(a)_q^{-1} + \frac{q}{8}\right)\varepsilon_a^{-1}\left(q\mid a\right) & 2\mid m, 4\mid q\\ 0 & 2\nmid m, 4\mid q\\ 2\sqrt{2}e_q\left(2(4)_{q/2}^{-1}\left((2)_{q/2}^{-1}-1\right)m^2(a)_q^{-1}\right)\varepsilon_{q/2}\left(2a\mid q/2\right) & 2\nmid m, 2\mid\mid q\\ 0 & 2\mid m, 2\mid\mid q\end{cases}$$

Note that when $2\nmid q$, $S(a,m;q)$ never vanishes. When $2\mid q$, the product depends on the parity of the components of $m - 2ab \equiv_2 m$ (for vectors we have the convention that $\equiv$ is interpreted component-wise).

$$\prod_{j=1}^{d} S(a, m_j - 2ab_j; q) \neq 0 \implies (4\mid q\wedge m\in(2\mathbb{Z})^d)\vee(2\mid\mid q\wedge m\in(2\mathbb{Z}+1)^d)$$

Thus, we need only consider the three cases $2\nmid q$, $4\mid q\wedge m\in(2\mathbb{Z})^d$, and $2\mid\mid q\wedge m\in(2\mathbb{Z}+1)^d$. ε_q, ε_a^{-1}, $\varepsilon_{q/2}$, $(a\mid q)$, $(q\mid a)$, and $(2a\mid q/2)$ all have orders dividing 4. By the restrictions on parity of m and q, we have *no mixing* of the three nonzero cases, so $4\mid d$ implies that all terms involving ε and the Jacobi symbols will disappear. For notational simplicity, let

$$\prod_{j=1}^{d} S(a, m_j - 2ab_j; q) = q^{-d/2} C_q^d e_q(\Phi(a,m,b;q))$$

Case 1 ($2\nmid q$): $C_q = 1$

$$
\begin{aligned}
\Phi(a,m,b;q) &= ((2)_q^{-1}-1))\sum_{j=1}^{d}(2a)_q^{-1}(m_j-2ab_j)^2 \\
&= \left(\frac{q+1}{2}-1\right)\sum_{j=1}^{d}(2a)_q^{-1}(m_j^2-4am_jb_j+4a^2b_j^2) \\
&= \left(\frac{q-1}{2}\right)((2a)_q^{-1}|m|^2-2m\cdot b+2a|b|^2) \\
&\equiv_q \left(\frac{q-1}{2}\right)\left(\frac{q+1}{2}\right)(a)_q^{-1}|m|^2+m\cdot b-a|b|^2 \\
&\equiv_q -a|b|^2+\left(\frac{q^2-1}{4}\right)|m|^2(a)_q^{-1}+m\cdot b
\end{aligned}
$$

Case 2 ($4 \mid q \wedge m \in (2\mathbb{Z})^d$): $C_q = \sqrt{2}$

$$
\begin{aligned}
\Phi(a,m,b;q) &= \frac{dq}{8}-\frac{1}{4}\sum_{j=1}^{d}(m_j^2-4am_jb_j+4a^2b_j^2)(a)_q^{-1} \\
&= \frac{dq}{8}-\frac{1}{4}(|m|^2(a)_q^{-1}-4m\cdot b+4a|b|^2) \\
&= -a|b|^2-\frac{1}{4}|m|^2(a)_q^{-1}+m\cdot b+\frac{dq}{8}
\end{aligned}
$$

Case 3 ($2 \mid\mid q \wedge m \in (2\mathbb{Z}+1)^d$): $C_q = 2\sqrt{2}$
Note that $2(4)_{q/2}^{-1} \equiv_q \frac{q/2+1}{2}$.

$$
\begin{aligned}
\Phi(a,m,b;q) &= 2(4)_{q/2}^{-1}\left((2)_{q/2}^{-1}-1\right)\sum_{j=1}^{d}(m_j-2ab_j)^2(a)_q^{-1} \\
&\equiv_q 2(4)_{q/2}^{-1}\left((2)_{q/2}^{-1}-1\right)\left(|m|^2(a)_q^{-1}-4m\cdot b+4a|b|^2\right) \\
&= \frac{(q/2)^2-1}{4}\left(|m|^2(a)_q^{-1}-4m\cdot b+4a|b|^2\right)
\end{aligned}
$$

Case 3 Aliter:

$$
S(a,m;q) = e_q\left(\left(\frac{(q/2)^2-1}{4}-\frac{m^2-1}{4}\right)(a)_q^{-1}\right)2S(2a,0;q/2)
$$

$$
2S(2a,0;q/2) = \frac{2\sqrt{2}}{\sqrt{q}}\varepsilon_{q/2}\left(2a \mid q/2\right)
$$

$$
S(a,m;q) = \frac{2\sqrt{2}}{\sqrt{q}}e_q\left(\left(\frac{(q/2)^2-1}{4}-\frac{m^2-1}{4}\right)(a)_q^{-1}\right)\varepsilon_{q/2}\left(2a \mid q/2\right)
$$

$C_q = 2\sqrt{2}$
Once again, $\left(\varepsilon_{q/2}\left(2a \mid q/2\right)\right)^d = 1$ can be neglected.

$$\begin{aligned}
\Phi(a,m,b;q) &= \left(d\frac{(q/2)^2-1}{4} - \sum_{j=1}^{d}\frac{\left(m_j-2ab_j\right)^2-1}{4}\right)(a)_q^{-1} \\
&= \frac{1}{4}\left(d(q/2)^2 - \sum_{j=1}^{d}\left(m_j-2ab_j\right)^2\right)(a)_q^{-1} \\
&= \frac{1}{4}\left(d(q/2)^2 - |m|^2 + 4ab\cdot m + 4a^2|b|^2\right)(a)_q^{-1} \\
&= \left(|b|^2 a + \frac{1}{4}\left(d(q/2)^2 - |m|^2\right)(a)_q^{-1}\right) + m\cdot b
\end{aligned}$$

□

Lemma 8. *Let $d = 4$. The Kloosterman term becomes*

$$\tilde{K}(m;q,b) = q^{-2}e_q(b\cdot m)\begin{cases} K((|b|/2)^2, |m|^2; q) & 2 \nmid q, 2 \mid |b| \\ -4K\left(|b|^2, (|m|/2)^2; q\right) & 4 \mid q, m \in (2\mathbb{Z})^d \\ -64K\left(|b|^2, -\left(|m|/2\right)^2; q\right) & 4 \mid q, m \in (2\mathbb{Z}+1)^d \end{cases}$$

For the case $2 \nmid q$, there may be a way of keeping the $\frac{1}{2}$ factor on the m-term instead of the b term. As $m \in \mathbb{Z}^d$ may not be divisible by 2, more delicate casework will be required.

Proof. Case 1 ($2 \nmid q$):

$$\begin{aligned}
\tilde{K}(m;q,b) &= q^{-d/2}e_q(b\cdot m)K(-|b|^2, |m|^2\varphi(q); q) \\
&= q^{-d/2}e_q(b\cdot m)K(-\varphi(q)|b|^2, |m|^2; q) && \gcd(\phi_0(q), q) = 1 \\
&= q^{-d/2}e_q(b\cdot m)K((|b|/2)^2, |m|^2; q) && -\phi_0(q)|b|^2 \equiv_q (|b|/2)^2
\end{aligned}$$

Case 2 ($4 \mid q \wedge m \in (2\mathbb{Z})^d$):

$$\begin{aligned}
\tilde{K}(m;q,b) &= (-1)^{d/4}q^{-d/2}e_q(b\cdot m)K\left(-|b|^2, \frac{-|m|^2}{4}; q\right) \\
&= (-1)^{d/4}q^{-d/2}e_q(b\cdot m)K\left(|b|^2, \frac{|m|^2}{4}; q\right) && \text{Symmetry of the Kloosterman sum}
\end{aligned}$$

Case 3 ($2 \mid\mid q \wedge m \in (2\mathbb{Z}+1)^d$):

$$\Phi(a,m,b;q) = \left(|b|^2 a + \frac{1}{4}\left(d(q/2)^2 - |m|^2\right)(a)_q^{-1}\right) + m\cdot b$$

$$\tilde{K}(m;q,b) = q^{-d/2}e_q(b\cdot m)K\left(|b|^2, (q/2)^2 - \frac{|m|^2}{4}; q\right)$$

As $m \in (2\mathbb{Z}+1)^d$, $4 \mid |m|^2$ for $4 \mid d$. As always, $(q/2)^2 \equiv_{q/2} q/2 \equiv_{q/2} 0$, so $(q/2)^2 \equiv_q q/2$ or $(q/2)^2 \equiv_q 0$. If $(q/2)^2 \equiv_q 0$, then $q \mid \frac{q^2}{4}$, $4q \mid q^2$, and $4 \mid q$. By assumption, $2 \mid\mid q$, so $(q/2)^2 \equiv_q 0$ is a contradiction. Thus, $(q/2)^2 \equiv_q q/2$. Also, $\gcd(a,q) = 1$, so $2 \nmid a$ and $e_q\left((q/2)(a)_q^{-1}\right) = (-1)^{(a)_q^{-1}} = -1$.

$$\begin{aligned} K\left(|b|^2, (q/2)^2 - \frac{|m|^2}{4}; q\right) &= \sum_{a \in \mathbb{Z}_q^\times} e_q\left(|b|^2 a - \frac{|m|^2}{4}(a)_q^{-1}\right) e_q\left((q/2)^2 (a)_q^{-1}\right) \\ &= -K\left(|b|^2, -\frac{|m|^2}{4}; q\right) \end{aligned}$$

□

3.1.3 The Final Weight Function

Theorem 9. *The final weight function occurring on the arithmetic side of the Kuznetsov formula is*

$$\phi\left(\frac{2\pi|m||b|}{q}\right) = q^{-1}\chi_{[1,r]}(q) A(m; q, b, \beta_0)$$

where $\chi_{[1,r]}(q)$ *is a smooth approximation of the indicator function on* $[1,r]$ *and* $A(m;q,b,\beta_0) = e_q(b \cdot m)\tilde{J}(m;q,b,\beta_0)$

3.2 Circumventing Artificial Oscillatory Terms in the Weight Function

3.2.1 A Naive Trigonometric Mask

The weight function depends on the residue of $q \pmod 4$. One may consider introducing periodic functions such as sin and cos to select different residue classes. Unfortunately, on the spectral side of the Kuznetsov formula, the parameter q is exchanged for $\tilde{x} \propto q^{-1}$, which introduces oscillatory terms roughly of the form $\sin(\tilde{x}^{-1})$ and $\cos(\tilde{x}^{-1})$. These terms become highly oscillatory and damage the precision of the bounds in the later analysis. From a high-level standpoint, this oscillation is undesirable because it does not naturally arise from facts about the original counting problem or spectral geometry in hyperbolic space. Instead, we can use subgroups of $SL_2(\mathbb{Z})$ to place the necessary constraints on the residue of $q \pmod 4$ without introducing artificial sines and cosines.

3.2.2 Simplifying the Weight Function with $\Gamma_0(q) \leq SL_2(\mathbb{Z})$

Consider the surjection from $SL_2(\mathbb{Z})$ to the set of matrices on the integers modulo n that still have determinant one. Concretely, this surjection is given by sending each entry of a matrix to its residue modulo q.

$$
\begin{aligned}
&\pi : SL_2(\mathbb{Z}) \to SL_2(\mathbb{Z}/q\mathbb{Z}) \\
&\pi \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a \mod q & b \mod q \\ c \mod q & d \mod q \end{pmatrix}
\end{aligned}
$$

Using the projection π, we can define three principal subgroups of $SL_2(\mathbb{Z})$: $\Gamma(q)$, $\Gamma_0(q)$, and $\Gamma_1(q)$.

$$
\begin{aligned}
\Gamma(q) &= \mathrm{Ker}(\pi) \\
\Gamma_0(q) &= \pi^{-1}\left(\left\{\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} : a, b, d \in \mathbb{Z}/q\mathbb{Z}, ad = 1\right\}\right) \\
\Gamma_1(q) &= \pi^{-1}\left(\left\{\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} : b \in \mathbb{Z}/q\mathbb{Z}\right\}\right)
\end{aligned}
$$

Proposition 10 (Double Coset Decompositions)**.** *For two cusps $\mathfrak{a}$, $\mathfrak{b}$ of a finite volume (though not necessarily co-compact) subgroup $\Gamma \le SL_2(\mathbb{Z})$, we have the following double coset decomposition.*

$$
\sigma_{\mathfrak{a}}^{-1}\Gamma\sigma_{\mathfrak{b}} = \delta_{\mathfrak{ab}}\Omega_\infty \cup \bigcup_{c>0} \bigcup_{d \in \mathbb{Z}/c\mathbb{Z}} \Omega_{d/c}
$$

Here, we have defined

$$
\begin{aligned}
\Omega_\infty &= B\omega_\infty B = B \\
\Omega_{d/c} &= B\omega_{d/c}B
\end{aligned}
$$

for

$$
\omega_\infty = \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \in \Gamma \quad \textit{and} \quad \omega_{d/c} = \begin{pmatrix} * & * \\ c & d \end{pmatrix} \in \Gamma
$$

$$
B = \left\{\begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} : z \in \mathbb{Z}\right\}
$$

In the union over $c > 0$ and $d \in \mathbb{Z}/c\mathbb{Z}$, it is understood that $\Omega_{d/c} = \emptyset$ if $\nexists \omega_{d/c} \in \Gamma$.

Definition 11 (Kloosterman Sums)**.** The generalized Kloosterman sum $S_{\mathfrak{ab}}(m, n; c)$ for frequencies m and n and modular basis c is given by a convolution of the additive characters $\psi_{d/c}(m) = e_c(md)$ and $\psi_{-a/c}(n) = e_c(-na)$.

$$
S_{\mathfrak{ab}}(m, n; c) = \sum_{\begin{pmatrix} a & * \\ c & d \end{pmatrix} \in B\backslash\sigma_{\mathfrak{a}}^{-1}\Gamma\sigma_{\mathfrak{b}}/B} e_c(dm + na)
$$

where $B\backslash\sigma_{\mathfrak{a}}^{-1}\Gamma\sigma_{\mathfrak{b}}/B$ simply denotes the set of B-double cosets of $\sigma_{\mathfrak{a}}^{-1}\Gamma\sigma_{\mathfrak{b}}$ [4].

Note that this definition is well posed, since every double B-coset uniquely determines c for all of its representatives and determines a and d up to integer multiples of c. These integer multiples belong to the kernel of the homomorphism $e_c : (\mathbb{Z}/c\mathbb{Z}, +) \to \mathbb{S}^1 \subset \mathbb{C}$, so they are forgotten.

Also,

$$\mathcal{C}_{\mathfrak{ab}}(\Gamma) = \left\{ c > 0 : \begin{pmatrix} * & * \\ c & * \end{pmatrix} \in \sigma_{\mathfrak{a}}^{-1}\Gamma\sigma_{\mathfrak{b}} \right\}$$

is the set of c for which the Kloosterman sum $S_{\mathfrak{ab}}(m, n; c)$ is defined.

Example 12 (Kloosterman Sums $\Gamma_0(q) \le SL_2(\mathbb{Z})$)**.** We wish to find a Γ such that $\mathcal{C}_{\mathfrak{ab}}(\Gamma)$ imposes a constraint on the residue of $c \mod 4$ to avoid having to introduce the unnatural oscillatory term given roughly by $\sin(q)$, which would degrade the spectral information recoverable from the Kuznetsov trace formula.
As $\Gamma_0(q)$ enforces that $c \equiv_q 0$, $\Gamma = \Gamma_0(q)$ is a natural candidate for this purpose. To further simplify things, we consider the case when both cusps $\mathfrak{a}$ and $\mathfrak{b}$ are the point at ∞.

$$\mathfrak{a} = \mathfrak{b} = \infty, \quad \text{and} \quad \sigma_{\mathfrak{a}} = \sigma_{\mathfrak{b}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

The double coset decomposition has a particularly simple form.

$$\begin{aligned} \sigma_{\mathfrak{a}}^{-1}\Gamma\sigma_{\mathfrak{b}} &= \delta_{\mathfrak{ab}}\Omega_\infty \cup \bigcup_{c>0} \bigcup_{d \in \mathbb{Z}/c\mathbb{Z}} \Omega_{d/c} \\ &= B \cup \bigcup_{c>0} \bigcup_{d \in \mathbb{Z}/c\mathbb{Z}} \Omega_{d/c} \\ &= B \cup \bigcup_{c \in q\mathbb{N}} \bigcup_{d \in (\mathbb{Z}/c\mathbb{Z})^\times} \Omega_{d/c} \end{aligned}$$

where $\Omega_{d/c}$ is given by

$$\begin{aligned} \Omega_{d/c} &= B\omega_{d/c}B \\ &= \left\{ \begin{pmatrix} a + cz_1 & * \\ c & d + cz_2 \end{pmatrix} \mid ad \equiv_q 1, c \equiv_q 0, z_1, z_2 \in \mathbb{Z} \right\} \end{aligned}$$

As hinted earlier, the set of c on which the Kloosterman sum is defined enforces that $c \equiv_q 0$.

$$\begin{aligned} \mathcal{C}_{\infty\infty}(\Gamma_0(N)) &= \left\{ c > 0 : \begin{pmatrix} * & * \\ c & * \end{pmatrix} \in \Gamma_0(q) \right\} \\ &= \{c > 0 : c \equiv_q 0\} \\ &= q\mathbb{N} \end{aligned}$$

Finally, the Kloosterman sum $S_{\mathfrak{ab}} : \mathbb{Z} \times \mathbb{Z} \times \mathcal{C}_{\mathfrak{ab}}(\Gamma) \to \mathbb{R}$ simplifies to

$$\begin{aligned} S_{\mathfrak{ab}}(m,n;c) &= \sum_{\begin{pmatrix} a & * \\ c & d \end{pmatrix} \in B \backslash \sigma_{\mathfrak{a}}^{-1} \Gamma \sigma_{\mathfrak{b}} / B} e_c(dm + na) \\ &= \sum_{d \in (\mathbb{Z}/c\mathbb{Z})^{\times}} e_c(md + n\,(d)_c^{-1}) \end{aligned}$$

Theorem 13 (Deshouillers and Iwaniec 1982)**.** *Let $m, n \in \mathbb{N}$, and let $\phi \in C^3((0,\infty))$ be compactly supported. Let $\mathfrak{a}, \mathfrak{b}$ be two cusps of $\Gamma = \Gamma_0(q)$. Denote by $\sum^{\gamma \in \Gamma}$ a summation over the positive real numbers γ for which $S_{\mathfrak{ab}}(m,n;\gamma)$ is defined. Let $D^{\Gamma}_{\mathfrak{ab}}(\gamma)$ be defined by*

$$D^{\Gamma}_{\mathfrak{ab}}(\gamma) = \left\{ \delta \mod \gamma\mathbb{Z} \middle| \exists \alpha, \beta \in \mathbb{Z} \left[\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \sigma_{\mathfrak{a}}^{-1} \Gamma \sigma_{\mathfrak{b}} \right] \right\}$$

For frequencies $m, n > 0$, the standard Kuznetsov trace formula is

$$\begin{aligned} &\sum^{\gamma \in \Gamma} \frac{1}{\gamma} S_{\mathfrak{ab}}(m,n;\gamma)\, \varphi\left(\frac{4\pi\sqrt{mn}}{\gamma}\right) = \\ &\frac{1}{2\pi} \sum_{k \equiv_2 0} \sum_{1 \le j \le \theta_k(q)} \frac{i^k (k-1)!}{(4\pi\sqrt{mn})^{k-1}}\, \overline{\psi_{jk}(\mathfrak{a}, m)}\, \psi_{jk}(\mathfrak{b}, n)\, \tilde{\varphi}(k-1) + \\ &\sum_{j \ge 1} \frac{\overline{\rho_{j\mathfrak{a}}(m)}\, \rho_{j\mathfrak{b}}(n)}{\cosh(\pi\kappa_j)}\, \hat{\varphi}(\kappa_j) + \frac{1}{\pi} \sum_{\mathfrak{c}} \int_{-\infty}^{\infty} \left(\frac{m}{n}\right)^{-ir} \overline{\varphi_{\mathfrak{c}\mathfrak{a}m}\left(\tfrac{1}{2} + ir\right)}\, \varphi_{\mathfrak{c}\mathfrak{b}n}\left(\tfrac{1}{2} + ir\right)\, \hat{\varphi}(r)\, dr \end{aligned}$$

where $\kappa_j = \sqrt{\lambda_j - \frac{1}{4}}$ for the j^{th} eigenvalue λ_j of the Laplacian, $\rho_j(n)$ is the n^{th} Fourier coefficient associated with the cusp form corresponding to λ_j, and

$$\varphi_{\mathfrak{ab}n}(s) = \sum^{\gamma \in \Gamma} \gamma^{-2s} \sum_{\delta \in D^{\Gamma}_{\mathfrak{ab}}(\gamma)} e\left(n\frac{\delta}{\gamma}\right)$$

The opposite sign case is simpler

$$\begin{aligned} &\sum^{\gamma \in \Gamma} \frac{1}{\gamma} S_{\mathfrak{ab}}(m,-n;\gamma), \varphi\left(\frac{4\pi\sqrt{mn}}{\gamma}\right) = \\ &\sum_{j \ge 1} \rho_{j\mathfrak{a}}(m) \rho_{j\mathfrak{b}}(n) \frac{\check{\varphi}(\kappa_j)}{\cosh(\pi\kappa_j)} + \\ &\frac{1}{\pi} \sum_{\mathfrak{c}} \int_{-\infty}^{\infty} (mn)^{ir} \varphi_{\mathfrak{c}\mathfrak{a}m}\left(\frac{1}{2} + ir\right) \varphi_{\mathfrak{c}\mathfrak{b}n}\left(\frac{1}{2} + ir\right) \check{\varphi}(r), dr \end{aligned}$$

In the above formulae, the integral transforms $\tilde{\varphi}$, $\hat{\varphi}$, and $\check{\varphi}$ are defined by

$$\tilde{\varphi}(l) = \int_0^\infty J_l(y)\varphi(y)\frac{dy}{y}$$
$$\hat{\varphi}(r) = \frac{\pi}{\sinh(\pi r)}\int_0^\infty \frac{J_{2ir}(x) - J_{-2ir}(x)}{2i}\varphi(x)\frac{dx}{x}$$
$$\check{\varphi}(r) = \frac{4}{\pi}\cosh(\pi r)\int_0^\infty K_{2ir}(x)\varphi(x)\frac{dx}{x}$$

Proof. See Theorem 1 in the paper [5]. □

When $\mathfrak{a} = \mathfrak{b} = \infty$ and $\Gamma = \Gamma_0(q)$, the left hand sides of the Kuznetsov trace formulae simplify as

$$\begin{aligned}\sum^{\gamma\in\Gamma} \frac{1}{\gamma} S_{\mathfrak{ab}}(m, \pm n; \gamma)\,\phi\left(\frac{4\pi\sqrt{mn}}{\gamma}\right) &= \sum_{\gamma\in q\mathbb{N}} \frac{1}{\gamma}\phi\left(\frac{4\pi\sqrt{mn}}{\gamma}\right) \sum_{d\in(\mathbb{Z}/\gamma\mathbb{Z})^\times} e_\gamma\left(md \pm n\,(d)_\gamma^{-1}\right)\\ &= \sum_{\gamma\in q\mathbb{N}} \frac{1}{\gamma}\phi\left(\frac{4\pi\sqrt{mn}}{\gamma}\right) S_{\infty\infty}(m, \pm n; \gamma)\end{aligned}$$

and $D^\Gamma_{\mathfrak{ab}}(\gamma)$ becomes

$$\begin{aligned}D^{\Gamma_0(q)}_{\infty\infty}(\gamma) &= \left\{\delta \mod \gamma\mathbb{Z} \middle| \exists \alpha, \beta \in \mathbb{Z}\left[\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \sigma_\infty^{-1}\Gamma_0(q)\sigma_\infty = \Gamma_0(q)\right]\right\}\\ &= \{\delta \mod \gamma\mathbb{Z} | \exists \alpha, \beta \in \mathbb{Z}\,[\alpha\delta \equiv_q 1 \wedge \alpha\delta - \beta\gamma = 1 \wedge \gamma \equiv_q 0]\}\\ &= \{\delta \mod \gamma\mathbb{Z} | \gcd(\delta, q) = \gcd(\delta, \gamma) = 1\}\end{aligned}$$

Recalling that the Kloosterman sum term for our problem is

$$\tilde{K}(m; q, b) = e_q(b \cdot m)K_0(m; q, b)$$

where

$$K_0(m; q, b) = q^{-2}\begin{cases} K((|b|/2)^2, |m|^2; q) & 2 \nmid q, 2 \mid |b| \\ -4K\left(|b|^2, (|m|/2)^2; q\right) & 4 \mid q, m \in (2\mathbb{Z})^d \\ -64K\left(|b|^2, -(|m|/2)^2; q\right) & 2 \mid\mid q, m \in (2\mathbb{Z}+1)^d \end{cases}$$

If $2 \mid |b|$, then

$$
\begin{aligned}
&\sum_{m\in\mathbb{Z}^d}\sum_{q=1}^{r-1}\tilde{J}(m;q,b,(qr)^{-1})\tilde{K}(m;q,b) = \\
&\sum_{m\in\mathbb{Z}^d}\sum_{q=1}^{r-1}K_0(m;q,b)A(m;q,b,\beta_0) = \\
&\sum_{m\in\mathbb{Z}^d}\sum_{q\in\mathbb{N}}S_1(q;m,b) - \sum_{m\in\mathbb{Z}^d}\sum_{q\in2\mathbb{N}}S_1(q;m,b) \\
&-4\sum_{m\in(2\mathbb{Z})^d}\sum_{q\in4\mathbb{N}}S_2(q;m,b) \\
&-64\sum_{m\in(2\mathbb{Z}+1)^d}\sum_{q\in2\mathbb{N}}S_3(q;m,b) + 64\sum_{m\in(2\mathbb{Z}+1)^d}\sum_{q\in4\mathbb{N}}S_3(q;m,b)
\end{aligned}
$$

where

$$
\begin{aligned}
S_1(q;m,b) &= q^{-1}\phi\left(\frac{2\pi|m||b|}{q};m,b,\beta_0=(qr)^{-1}\right)K((|b|/2)^2,|m|^2;q) \\
S_2(q;m,b) &= q^{-1}\phi\left(\frac{2\pi|m||b|}{q};m,b,\beta_0=(qr)^{-1}\right)K\left(|b|^2,(|m|/2)^2;q\right) \\
S_3(q;m,b) &= q^{-1}\phi\left(\frac{2\pi|m||b|}{q};m,b,\beta_0=(qr)^{-1}\right)K\left(|b|^2,-(|m|/2)^2;q\right) \\
\phi\left(\frac{2\pi|m||b|}{q};m,b,\beta_0\right) &= q^{-1}\mathbb{1}_{[1,r]}(q)\,A(m;q,b,\beta_0) \\
A(m;q,b,\beta_0) &= e_q(b\cdot m)\tilde{J}(m;q,b,\beta_0)
\end{aligned}
$$

Thus, we can apply the generalized Kuznetsov formula to each q-sum by noting that $K(m,n;q) = S_{\infty\infty}(m,n;q)$.
As a note, when $2\nmid|b|$, the odd q have a more complicated phase shift. Blomer and Milicevic [6] have developed a technique that leverages Mobius inversion to peel out dependencies similar to the $\left(\frac{q^2-1}{4}\right)$ term occuring when q is odd.

4 Next Steps

4.0.1 Studying the Oscillatory Integrals with Varieties

For the standard Kuzntesov trace formula, we must solve the following oscillatory integrals defined in terms of Bessel functions of the first kind.

$$\mathcal{J}(\phi)(s) = \frac{i\pi}{2\sinh(\pi s)} \int_0^\infty (J_{2is}(x) - J_{-2is}(x))\phi(x)\frac{dx}{x}$$
$$J_\alpha(x) = \sum_{m=0}^\infty \frac{(-1)^m}{m!\Gamma(\alpha+m+1)} \left(\frac{x}{2}\right)^{2m+\alpha}$$
$$I_0(m,\alpha) = \int_0^\infty x^{2m-1+\alpha}\phi(x)dx$$
$$I(m,\alpha) = \frac{(-1)^m}{m!\Gamma(\alpha+m+1)} 2^{-(2m+\alpha)} I_0(m,\alpha)$$

After expanding the Bessel functions with their power series the integral transform of the weight function ϕ becomes.

$$\mathcal{J}(\phi)(s) = \frac{i\pi}{2\sinh(\pi s)} \sum_{m=0}^\infty [I(m,2is) - I(m,-2is)]$$

Recall the following definitions

$$\phi\left(\frac{2\pi|m||b|}{q}\right) = q^{-1}\chi_{[1,r]}(q)A(m;q,b,\beta_0)$$
$$A(m;q,b,\beta_0) = e_q(b\cdot m)\tilde{J}(m;q,b,\beta_0)$$
$$\tilde{J}(m;q,b,\beta_0) = \int_{-\beta_0}^{\beta_0} \prod_{j=1}^d J(2b_j\beta,\beta,m;q)d\beta$$
$$J(\varphi,\beta,m;q) = \int_{\mathbb{R}} \gamma\left(\frac{y}{r}\right) e\left(y^2\beta + y\varphi + m\frac{y}{q}\right)dy$$

Let $\chi_{[1/r,1]}$ be a smooth bump function. We can expand the term by term integration of the power series as follows.

$$\begin{aligned}
I_0(m,\alpha) &= \int_0^\infty x^{2m-1+\alpha}\phi(x)dx \\
&= \int_0^\infty x^{2m-1+\alpha}\tilde{x}\chi_{[1/r,1]}(\tilde{x})A(m;\tilde{x}^{-1},b,\tilde{x}/r)dx \\
&= \int_0^\infty x^{2m-1+\alpha}\tilde{x}\chi_{[1/r,1]}(\tilde{x})e(b\cdot m\tilde{x}) \int_{-\beta_0}^{\beta_0}\int_{\mathbb{R}^d} \chi_{[-r,r]^d}(y)e\left(|y|^2\beta + 2\beta b\cdot y + m\cdot y\tilde{x}\right)dyd\beta dx
\end{aligned}$$

From our definitions in previous sections, we have $\frac{\tilde{x}}{r} = \frac{x}{2\pi Rr|m|}$ and $\tilde{x}m = \frac{x}{2\pi R}\hat{m}$. Let $\tilde{\chi}$ be defined by

$$\chi_{[1/r,1]}(\tilde{x}) = \chi_{[1/r,1]}\left(\frac{x}{2\pi|m|R}\right) = \chi_{[2\pi|m|\frac{R}{r},2\pi|m|R]}(x) =: \tilde{\chi}(x)$$

For notational simplicity, introduce a constant $\mathcal{N} = (2\pi|m|)^{2m+\alpha}R^d r^{-1}(1-r^{-1})$.

$$
\begin{aligned}
I_0(m,\alpha) &= \frac{1}{2\pi|m|R}\int_{\mathbb{R}}\tilde{\chi}(x)\int_{-\frac{x}{2\pi|m|Rr}}^{\frac{x}{2\pi|m|Rr}}\int_{\mathbb{R}^d} x^{2m+\alpha}e(b\cdot m\tilde{x}) \\
&\times \chi_{[-r,r]^d}(y)R^d e\left(R^2|y|^2\beta + 2R\beta b\cdot y + Ry\cdot m\tilde{x}\right) dy\, d\beta\, dx \qquad y\mapsto Ry,\ \beta\mapsto \frac{\beta}{2\pi|m|Rr} \\
&= \int_{\mathbb{R}}\chi_{[r^{-1},1]}(\tilde{x})\int_{-\tilde{x}/r}^{\tilde{x}/r}\int_{\mathbb{R}^d}(2\pi|m|R\tilde{x})^{2m+\alpha}e(b\cdot m\tilde{x})\chi_{[-r,r]^d}(y) \quad x\mapsto 2\pi|m|R\tilde{x} \\
&\times e(|y|^2\beta + 2\beta b\cdot y + y\cdot m\tilde{x})\, dy\, d\beta\, d\tilde{x} \\
&= \mathcal{N}\int_0^1\int_{-1}^1\int_{\mathbb{R}^d}\tilde{x}^{2m+\alpha+1}\chi_{[-1,1]^d}\left(cR^\delta y\right) \qquad \tilde{x} = (1-t)r^{-1}+t \\
&\times e\left(\left[(|y|^2 + 2\hat{b}\cdot y)\tilde{\beta}\tilde{x}cR^{1+\delta} + R(y+\hat{b})\cdot m\tilde{x}\right]\right) dy\, d\tilde{\beta}\, dt \qquad \beta\mapsto\tilde{\beta}\frac{x}{r}
\end{aligned}
$$

Finally, $I_0(m,\alpha)$ can be written as an oscillatory integral in three dimensional space. We can draw an analogy to a wave packet in physics. There are multiple oscillating functions $\varrho^2\tilde{\beta}\tilde{x}$, $\hat{b}\cdot m\tilde{x}$, $J_1(2\pi|z|\varrho)$ whose frequencies grow at different rates as $R\to\infty$. The term with the slowest scaling in R will play the role of the wave envelope.

$$
\begin{aligned}
I_0(m,\alpha) = \mathcal{N}\int_0^1\int_{-1}^1\int_0^\infty &\tilde{x}^{2m+\alpha+1}\chi_{[-1,1]^d}\left(cR^\delta y\right) \\
&e\left(\varrho^2\tilde{\beta}\tilde{x}cR^{1+\delta} + R\hat{b}\cdot m\tilde{x}\right) \\
&2\pi\varrho^{-1}|z|^{-1}J_1(2\pi|z|\varrho)d\varrho d\tilde{\beta}dt
\end{aligned}
$$

where $z = 2cR^{1+\delta}\tilde{\beta}\tilde{x}\hat{b} + R\tilde{x}m$. $\Phi_1(\varrho,\tilde{\beta},\tilde{x})$ will have the highest associated frequency $R^{1+\delta}$, and its Hessian is nondegenerate along the normal bundle of the variety which encourages us to use the co-area formula and the clean version of the Morse-Bott stationary phase formula discussed in *Semi-classical Analysis* by Guillemin and Sternberg [7].

$$
\begin{aligned}
\Phi_1(\varrho,\tilde{\beta},\tilde{x}) &= \varrho^2\tilde{\beta}\tilde{x} \\
\nabla\Phi_1(\varrho,\tilde{\beta},\tilde{x}) &= \begin{pmatrix} 2\varrho\tilde{\beta}\tilde{x} \\ \varrho^2\tilde{x} \\ \varrho^2\tilde{\beta}\end{pmatrix} \\
\nabla^2\Phi_1(\varrho,\tilde{\beta},\tilde{x}) &= \begin{pmatrix} 2\tilde{\beta}\tilde{x} & 2\varrho\tilde{x} & 2\varrho\tilde{\beta} \\ 2\varrho\tilde{x} & 0 & \varrho^2 \\ 2\varrho\tilde{\beta} & \varrho^2 & 0\end{pmatrix}
\end{aligned}
$$

In particular, the variety on which the gradient of Φ_1 vanishes is given by

$$
V = \left\{(\varrho,\tilde{\beta},\tilde{x})\in\mathbb{R}^3|\nabla\Phi_1(\varrho,\tilde{\beta},\tilde{x}) = 0\right\} = \left\{\left(\varrho,\tilde{\beta},0\right)|\varrho,\tilde{\beta}\in\mathbb{R}^2\right\}\cup\{(0,0,\tilde{x})\,|\tilde{x}\in\mathbb{R}\}
$$

The only point where the Hessian of Φ_1 is completely degenerate in all three variables is the origin, which we expect would provide the principle asymptotic of the integral. We will split the integral into different regimes where the Bessel function can be well approximated by a low order polynomial or a cosine function. The regime where the Bessel function is linear contains a neighborhood of the origin, so we can "zoom in" and apply stationary phase to Φ_1 while treating J_1 like part of the amplitude. The geometry of $|z|\varrho = C$ for a constant $C > 0$ is rather complicated, but we may be able to approximate the boundary in dyadic spherical or cubical shells.

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