

OPTIMAL THRESHOLDS FOR MONOTONE NON-BOOLEAN FUNCTIONS

SABA LEPSVERIDZE AND ALLEN LIN

ABSTRACT. Let $[q] = \{0, 1, \dots, q - 1\}$, let $\Delta[q]$ denote the simplex of probability measures on $[q]$, and let λ denote the Lebesgue measure. We prove that for any symmetric monotone function $f: [q]^n \rightarrow [q]$ and any $a \in [q]$ we have

$$\lambda(\{\mu \in \Delta[q] \mid \Pr_{x \sim \mu^{\otimes n}}[f(x) = a] \in (\varepsilon, 1 - \varepsilon)\}) = O(1/\log n).$$

We also show that this bound is tight. This improves Kalai and Mossel's previous bound of $O(\log \log n / \log n)$ and answers their question completely.

1. INTRODUCTION

It is well known that monotone symmetric Boolean functions undergo sharp thresholds. This property is prevalent and has numerous applications to random graph theory, social choice theory, and percolation theory. In particular, let π_p be the distribution on $\{0, 1\}$ where 1 is chosen with probability p . Friedgut and Kalai [3] proved that for every $0 < \varepsilon < 1/2$, there exists a constant $C(\varepsilon)$ such that for all $n \geq 2$ and all monotone and symmetric functions $f: \{0, 1\}^n \rightarrow \{0, 1\}$, if $\mathbf{E}_{x \sim \pi_p^{\otimes n}}[f(x)] = \varepsilon$ and $\mathbf{E}_{x \sim \pi_q^{\otimes n}}[f(x)] = 1 - \varepsilon$ for some $q > p$, then

$$q - p < \frac{C(\varepsilon)}{\log n}.$$

This result implies that all monotone graph properties undergo sharp thresholds.

Naturally, one might consider functions with more generalized domains and ask if they also undergo sharp thresholds. Let A be a finite set. Here, we consider functions $f: A^n \rightarrow A$ that are monotone and symmetric. We give the definitions below.

1.1. Notation. Following Kalai and Mossel [5], let $S(n)$ denote the group of permutations on elements of the set $\{1, \dots, n\}$. For $\sigma \in S(n)$ and $x \in A^n$, let $y = x_\sigma$ denote the vector satisfying $y_i = x_{\sigma(i)}$ for all $i \in \{1, \dots, n\}$. In other words, y is the vector formed by permuting the bits of x according to σ .

Moreover, for $a \in A$ and $x, y \in A^n$, we define a partial order \leq_a on A^n as follows. We write $x \leq_a y$ if and only if $\{i \mid x_i = a\} \subseteq \{i \mid y_i = a\}$ and for all $i \in \{1, \dots, n\}$ such that $y_i \neq a$ we have $x_i = y_i$. That is, if $x \leq_a y$, then y can be obtained from x by changing the appropriate bits in x to a .

We use the following definitions from Kalai and Mossel [5].

Definition 1. A function $f: A^n \rightarrow A$ is *monotone* if for all $a \in A$ and $x, y \in A^n$ such that $x \leq_a y$, $f(x) = a$ implies $f(y) = a$.

Definition 2. A function $f: A^n \rightarrow A$ is *symmetric* if there exists a transitive group $\Sigma \subseteq S(n)$ such that $f(x_\sigma) = f(x)$ for all $x \in A^n$ and $\sigma \in \Sigma$.

Let $\Delta[A]$ denote the simplex of probability measures on A , and let λ denote the usual Lebesgue measure. Our main result is the following.

Theorem 1. *There exists an absolute constant $C = C(|A|)$ such that if $f: A^n \rightarrow A$ is symmetric and monotone, then*

$$(1) \quad \lambda(\{\mu \in \Delta[A] \mid \varepsilon \leq \Pr_{x \sim \mu^{\otimes n}}[f(x) = a] \leq 1 - \varepsilon\}) \leq \frac{C(\log(1 - \varepsilon) - \log(\varepsilon))}{\log n}$$

for any $a \in A$ and any $0 < \varepsilon < 1/2$.

This improves the threshold result of $O(\log \log n / \log n)$ by Kalai and Mossel [5]. Moreover, for any $\mu \in \Delta[A]$, let $E(\mu)$ be the event that there exists some $a \in [q]$ such that $\Pr_{x \sim \mu^{\otimes n}}[f(x) = a] \in (\varepsilon, 1 - \varepsilon)$. Note that by the union bound, we also have

$$\lambda(\{\mu \in \Delta[A] \mid E(\mu) \text{ is true}\}) \leq \frac{C(\log(1 - \varepsilon) - \log(\varepsilon))}{\log n}.$$

As such, all bounds of applications present in Kalai and Mossel [5] such as monotone graph properties and Condorcet's Jury Theorem are improved to $O(1/\log n)$.

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2. PREVIOUS SHARP THRESHOLD RESULTS

2.1. Friedgut and Kalai's Result for $q = 2$. Fundamental to the proofs of these threshold results is the notion of influences. Influences of variables of Boolean functions have been extensively studied and has led to applications in combinatorics, theoretical computer science, and other areas. There are various definitions for influences, all based on dividing the domain into one-dimensional subspaces called *fibres*. For the sake of discussion, we consider general product probability spaces $X = X_1 \times \cdots \times X_n$.

Definition 3. For $x = (x_1, \dots, x_n) \in X$ and for $1 \leq k \leq n$, the *fibre* of x in the k th direction is

$$s_k(x) = \{y \in X \mid y_i = x_i \text{ for all } i \neq k\}.$$

For a function $f: X \rightarrow \{0, 1\}$, the restriction of f to $s_k(x)$ is denoted $f_k^x: X_k \rightarrow \{0, 1\}$ and is defined by

$$f_k^x(t) = f(x_1, \dots, x_{k-1}, t, x_{k+1}, \dots, x_n).$$

The original definition of influence in general product spaces was introduced by Bourgain, Kahn, Kalai, Katznelson, and Linial [2] as follows.

Definition 4. For any function $f: X \rightarrow \{0, 1\}$ and any $1 \leq k \leq n$, the *influence* of the k th variable on f is

$$I_k(f) = \Pr_{x \in X}[f_k^x \text{ is not constant}].$$

In particular, if $f: \{0, 1\}^n \rightarrow \{0, 1\}$ is a function with product measure $\mu^{\otimes n}$, then denote

$$I_k(f; \mu^{\otimes n}) = \Pr_{x \sim \mu^{\otimes n}}[f_k^x \text{ is not constant}].$$

The most well-known theorem regarding influences in product spaces is the BKKKL theorem [2].

Theorem 2 (Bourgain, Kahn, Kalai, Katznelson, and Linal [2]). *There exists a universal constant c such that for any function $f: X \rightarrow \{0, 1\}$, there exists a coordinate k such that*

$$I_k(f) \geq c \mathbf{Var}(f) \log n/n.$$

Note that if $f: \{0, 1\}^n \rightarrow \{0, 1\}$ is symmetric, then all the coordinates have equal influence. This is important as now under the measure $\pi_p^{\otimes n}$ we have

$$\sum_{k=1}^n I_k(f; \pi_p^{\otimes n}) = c \mathbf{Var}_{\pi_p^{\otimes n}}(f) \log n.$$

On the other hand, if $f: \{0, 1\}^n \rightarrow \{0, 1\}$ is monotone, then we can apply a lemma of Russo and Margulis that relates the change of $\mathbf{E}_{x \sim \pi_p^{\otimes n}}[f(x)]$ to the influence of f .

Lemma 1 (Russo [8], Margulis [7]). *Let $f: \{0, 1\}^n \rightarrow \{0, 1\}$. Then under the measure $\pi_p^{\otimes n}$, we have*

$$\frac{d \mathbf{E}_{x \sim \pi_p^{\otimes n}}[f(x)]}{dp} = \sum_{k=1}^n I_k(f; \pi_p^{\otimes n}),$$

This immediately implies for a function $f: \{0, 1\}^n \rightarrow \{0, 1\}$ that is both symmetric and monotone under $\pi_p^{\otimes n}$ that

$$\frac{d \mathbf{E}_{x \sim \pi_p^{\otimes n}}[f(x)]}{dp} = c \mathbf{Var}_{\pi_p^{\otimes n}}(f) \log n = c \mathbf{E}_{x \sim \pi_p^{\otimes n}}[f(x)] \left(1 - \mathbf{E}_{x \sim \pi_p^{\otimes n}}[f(x)]\right) \log n.$$

Friedgut and Kalai's [3] threshold bound of $O(1/\log n)$ quickly follows from the following observation. When $\mathbf{E}_{x \sim \pi_p^{\otimes n}}[f(x)] \leq 1/2$, we have

$$\frac{d \mathbf{E}_{x \sim \pi_p^{\otimes n}}[f(x)]}{dp} \geq \frac{c}{2} \mathbf{E}_{x \sim \pi_p^{\otimes n}}[f(x)] \log n,$$

which implies

$$\frac{d \ln \left(\mathbf{E}_{x \sim \pi_p^{\otimes n}}[f(x)] \right)}{dp} \geq \frac{c}{2} \log n.$$

Therefore, if $\mathbf{E}_{x \sim \mu_{p_0}^{\otimes n}}[f(x)] = \varepsilon$ and $\mathbf{E}_{x \sim \mu_{p_c}^{\otimes n}}[f(x)] = 1/2$ for some $p_c > p_0 > 0$, then

$$\ln(1/2) - \ln(\varepsilon) = \int_{p=p_0}^{p=p_c} \frac{d \ln \left(\mathbf{E}_{x \sim \pi_p^{\otimes n}}[f(x)] \right)}{dp} dp \geq \frac{c}{2} (p_c - p_0) \log n,$$

which gives

$$p_c - p_0 \leq \frac{2(\ln(1/2) - \ln(\varepsilon))}{c \log n}.$$

Similarly, when $\mathbf{E}_{x \sim \pi_p^{\otimes n}}[f(x)] > 1/2$, then if $\mathbf{E}_{x \sim \mu_{p_c}^{\otimes n}}[f(x)] = 1/2$ and $\mathbf{E}_{x \sim \mu_{p_1}^{\otimes n}}[f(x)] = 1 - \varepsilon$ for some $1 > p_1 > p_c$, then

$$p_1 - p_c \leq \frac{2(\ln(1 - \varepsilon) - \ln(1/2))}{c \log n}.$$

As such,

$$p_1 - p_0 \leq \frac{2(\ln(1 - \varepsilon) - \ln(\varepsilon))}{c \log n},$$

which proves Friedgut and Kalai's result.

2.2. Kalai and Mossel's Result for $q > 2$. In their paper, Kalai and Mossel [5] used another widely known definition of influence.

Definition 5. For any function $f: X \rightarrow \{0, 1\}$ and any $1 \leq k \leq n$, the *influence* of the k th variable on f is

$$\tilde{I}_k(f) = \mathbf{E}_{x \in X}[\mathbf{Var}(f_k^x)].$$

In particular, if $f: A^n \rightarrow \{0, 1\}$ is a function with product measure $\mu^{\otimes n}$ for some $\mu \in \Delta[A]$, then denote

$$\tilde{I}_k(f; \mu^{\otimes n}) = \mathbf{E}_{x \sim \mu^{\otimes n}}[\mathbf{Var}(f_k^x)].$$

In particular, using a hypercontractivity result of Wolff [10], Kalai and Mossel generalized the result of Talagrand [9] on the lower bound of the influence of a function.

Theorem 3 (Kalai and Mossel [5]). *There exists some universal constant c such that for any probability space (X, μ) and any symmetric function $f: X^n \rightarrow \{0, 1\}$, we have*

$$\sum_{k=1}^n \tilde{I}_k(f; \mu^{\otimes n}) \geq \frac{c}{\log(1/\min_{i \in \{1, \dots, n\}} \mu(i))} \mathbf{Var}_\mu(f) \log n.$$

On the other hand, Kalai and Mossel [5] proved a generalization of the Russo-Margulis lemma under a certain class of monotone functions (see Definition 7).

Lemma 2 (Kalai and Mossel [5]). *Let $f: \{0, \dots, q-1\}^n \rightarrow \{0, 1\}$ be a 0-monotone function, and let $\mu \in \Delta[\{0, \dots, q-1\}]$. Write $\mu = (1 - \mu(0))\mu' + \mu(0)\delta_0$ for some $\mu' \in \Delta[\{0, \dots, q-1\}]$ such that $\mu'(0) = 0$, and let $\mu_t = \mu + t\delta_0 - t\mu'$ for $t \in [0, 1 - \mu(0)]$. Then*

$$\frac{d \mathbf{E}_{x \sim \mu_t^{\otimes n}}[f(x)]}{dt} \geq \sum_{i=1}^n \tilde{I}_i(f; \mu^{\otimes n}).$$

Therefore, if $f: \{0, \dots, q-1\}^n \rightarrow \{0, 1\}$ is both symmetric and 0-monotone under $\mu^{\otimes n}$, then

$$\frac{d \mathbf{E}_{x \sim \mu_t^{\otimes n}}[f(x)]}{dt} \geq \frac{c}{\log(1/\min_{i \in \{1, \dots, n\}} \mu(i))} \mathbf{Var}_\mu(f) \log n.$$

Their proof proceeds in the same manner as that of Friedgut and Kalai. Let Γ be the set of measures in $\Delta[\{0, \dots, q-1\}]$ satisfying $\mu(0) = 0$ and $\mu(i) \geq \eta$ for all $i \neq 0$. For any measure $\mu' \in \Gamma$, write $\mu_t = t\delta_0 + (1-t)\mu'$. The contribution from all measures μ_t where $\mu' \notin \Gamma$ or $t < \eta$ is $C(q)\eta$, whereas the contribution from all other measures μ_t where $\mu \in \Gamma$ and $t \geq \eta$ is at most

$$C(q) \left(\frac{(\ln(1-\varepsilon) - \ln(\varepsilon)) \log(1/\eta)}{\log n} \right).$$

Therefore, the total contribution is at most

$$C(q) \left(\eta + \frac{(\ln(1-\varepsilon) - \ln(\varepsilon)) \log(1/\eta)}{\log n} \right),$$

so taking $\eta = (\ln(1-\varepsilon) - \ln(\varepsilon))/\log n$ gives the $O(\log \log n / \log n)$ bound.

2.3. General Influences. Following Keller [6], we use a more general definition of influences in general product spaces.

Definition 6. Let $h: [0, 1] \rightarrow \mathbb{R}$. For any $f: X \rightarrow \{0, 1\}$ and any $1 \leq k \leq n$, the h -influence of the k th variable on f is

$$I_k^h(f) = \mathbf{E}_{x \in X} [h(\mathbf{E}[f_k^x])].$$

Note that Definition 4 is obtained by the function $h(t) = \mathbb{I}[t \in (0, 1)]$ and Definition 5 is obtained by the function $h(t) = t(1-t)$.

Keller [6] proved a generalization of the BKKKL theorem for h -influences.

Theorem 4 (Keller [6]). *Denote by $\text{Ent}(t) = -t \log t - (1-t) \log(1-t)$ the entropy function. Let $h: [0, 1] \rightarrow \mathbb{R}$ be such that $h(t) \geq \text{Ent}(t)$ for all $t \in [0, 1]$. Then there exists a universal constant c such that for any function $f: [0, 1]^n \rightarrow \{0, 1\}$, there exists a coordinate k such that*

$$I_k^h(f) \geq c \mathbf{Var}(f) \log n / n.$$

3. PROOF OF THEOREM 1

3.1. Reduction to one value. Assume $A = [q] := \{0, 1, \dots, q-1\}$ without loss of generality. Let $f: [q]^n \rightarrow [q]$ be a symmetric and monotone function. Fix some $a \in [q]$ and some $\varepsilon \in (0, 1/2)$. Define $\tilde{f}: [q]^n \rightarrow \{0, 1\}$ by

$$\tilde{f}(x) = \mathbb{I}[f(x) = a].$$

Note that \tilde{f} inherits the monotonicity of f in the following manner.

Definition 7. We say $f: [q]^n \rightarrow \{0, 1\}$ is a -monotone if $x \leq_a y$ implies $f(x) \leq f(y)$.

Indeed, \tilde{f} is a -monotone. Assume by symmetry that $a = 0$. Therefore, to prove (1), it suffices to prove that there exists some universal constant $C = C(q)$ such that if $f: [q]^n \rightarrow \{0, 1\}$ is symmetric and 0-monotone, then

$$(2) \quad \lambda \left(\left\{ \mu \in \Delta[A] \mid \varepsilon \leq \mathbf{E}_{x \sim \mu^{\otimes n}} [f(x)] \leq 1 - \varepsilon \right\} \right) \leq \frac{C(\log(1-\varepsilon) - \log(\varepsilon))}{\log n}.$$

We sketch a brief outline of the proof. Denote the region under consideration by

$$\mathcal{D} := \{\mu \in \Delta[q] \mid \varepsilon \leq \mathbf{E}_{x \sim \mu^{\otimes n}}[f(x)] \leq 1 - \varepsilon\}.$$

We consider probability measures in $\Gamma := \{\mu \in \Delta[q] \mid \mu(0) = 0\}$, and for each $\mu \in \Gamma$ we write $\mu_t := t\delta_0 + (1-t)\mu$ and consider the set of measures $\{\mu_t \mid t \in [0, 1]\}$. Clearly

$$\bigcup_{\mu \in \Gamma} \{\mu_t \mid t \in [0, 1]\} = \Delta[q],$$

so for each measure $\mu \in \Gamma$ we bound $\lambda(\{t \in [0, 1] \mid \mu_t \in \mathcal{D}\})$. To do this, we show that $\mathbf{E}_{x \sim \mu_t^{\otimes n}}[f(x)]$ varies quickly as a function of t by establishing a lower bound on $d\mathbf{E}_{x \sim \mu_t^{\otimes n}}[f(x)]/dt$. The lower bound on the derivative depends on the second-smallest atom $\alpha = \min_{j \in [q] \setminus \{0\}} \mu(j)$ of μ (since $\mu(0) = 0$ is the smallest). To this end, we partition Γ into $q - 1$ regions $R_i = \{\mu \in \Gamma \mid \mu(i) = \alpha\}$ based on the second-smallest atom and show that $\lambda(\{\mu_t \mid t \in [0, 1]\} \cap \mathcal{D})$ is small for each region, with only a dependence on α . Finally, we remove the dependence on α and show that

$$\lambda(\{\mu_t \mid \mu \in R_i\} \cap \mathcal{D}) = \frac{C(\ln(1 - \varepsilon) - \ln(\varepsilon))}{\log n}$$

for some absolute constant C depending only on q . Since this holds for all regions R_i , this proves (2), which proves the result.

3.2. Generalization of Russo-Margulis. Here, we show a generalization of the Russo-Margulis lemma. This follows along the same lines as in [5], but we include it for the sake of completeness. Define the region $\Gamma := \{\mu \in \Delta[q] \mid \mu(0) = 0\}$.

Lemma 3. *Let $f: [q] \rightarrow \{0, 1\}$ be a 0-monotone function. Let $\mu \in \Gamma$ be a probability measure and define $\mu_t = t\delta_0 + (1-t)\mu$ for $t \in [0, 1]$. Then*

$$\frac{d\mathbf{E}_{x \sim \mu_t}[f(x)]}{dt} = \frac{\mathbb{I}[f \text{ is not constant}] \mathbf{E}_{x \sim \mu_t}[1 - f(x)]}{1 - t}.$$

Proof. Note $d\mathbf{E}_{x \sim \mu_t}[f(x)]/dt = f(0) - \mathbf{E}_{x \sim \mu}[f(x)]$. This expression is 0 when f is constant. If f is not constant, then $f(0) = 1$ since f is 0-monotone and the expression becomes $\mathbf{E}_{x \sim \mu}[1 - f(x)]$. But $\mu = (\mu_t - t\delta_0)/(1 - t)$ and $\mathbf{E}_{x \sim \delta_0}[1 - f(x)] = 0$ since $f(0) = 1$, so

$$\mathbf{E}_{x \sim \mu}[1 - f(x)] = \frac{\mathbf{E}_{x \sim \mu_t}[1 - f(x)] - t\mathbf{E}_{x \sim \delta_0}[1 - f(x)]}{1 - t} = \frac{\mathbf{E}_{x \sim \mu_t}[1 - f(x)]}{1 - t},$$

as required. \square

Lemma 4. *Let $f: [q]^n \rightarrow \{0, 1\}$ be a 0-monotone function. Let $\mu \in \Gamma$ be a probability measure and define $\mu_t = t\delta_0 + (1-t)\mu$ for $t \in [0, 1]$. Then*

$$\frac{d\mathbf{E}_{x \sim \mu_t^{\otimes n}}[f(x)]}{dt} = \frac{1}{1 - t} \sum_{k=1}^n \mathbf{E}_{x \sim \mu_t^{\otimes n}}[\mathbb{I}[f_k^x \text{ is not constant}] \mathbf{E}_{x_k \sim \mu_t}[1 - f_k^x(x_k)]].$$

Proof. By the product rule and Lemma 3,

$$\begin{aligned}
\frac{d\mathbf{E}_{x \sim \mu_t^{\otimes n}}[f(x)]}{dt} &= \sum_{x \in [q]^n} \sum_{k=1}^n \frac{d\mu_t(x_k)}{dt} \prod_{\ell \neq k} \mu_t(x_\ell) f(x) \\
&= \sum_{k=1}^n \sum_{x_k \in [q]} \mathbf{E}_{x \sim \mu_t^{\otimes n}} \left[\frac{d\mu_t(x_k)}{dt} f_k^x(x_k) \right] \\
&= \sum_{k=1}^n \mathbf{E}_{x \sim \mu_t^{\otimes n}} \left[\frac{d\mathbf{E}_{x_k \sim \mu_t}[f_k^x(x_k)]}{dt} \right] \\
&= \frac{1}{1-t} \sum_{k=1}^n \mathbf{E}_{x \sim \mu_t^{\otimes n}} [\mathbb{I}[f_k^x \text{ is not constant}] \mathbf{E}_{x_k \sim \mu_t}[1 - f_k^x(x_k)]],
\end{aligned}$$

as required. \square

3.3. A Lower Bound on the Derivative. We assume that f is not constant. In the sequel, C denotes different constants at different lines, depending on q only.

In this section, we prove the following.

Proposition 1. *Let $f: [q]^n \rightarrow \{0, 1\}$ be a 0-monotone and symmetric function and let $\mu \in \Gamma$ with second-smallest atom $\alpha = \min_{j \in [q] \setminus \{0\}} \mu(j)$. Define $\mu_t = t\delta_0 + (1-t)\mu$ for $t \in [0, 1]$. There exists an absolute constant $C = C(q)$ such that*

$$\frac{d\mathbf{E}_{x \sim \mu_t^{\otimes n}}[f(x)]}{dt} \geq \frac{C \mathbf{E}_{x \sim \mu_t^{\otimes n}}[f(x)](1 - \mathbf{E}_{x \sim \mu_t^{\otimes n}}[f(x)]) \log n}{\log(1/\alpha)}.$$

Before we prove the proposition, we need the following lemma.

Lemma 5. *Let $f: [q]^n \rightarrow \{0, 1\}$ be a 0-monotone function and let $\mu \in \Gamma$ with second-smallest atom $\alpha = \min_{j \in [q] \setminus \{0\}} \mu(j)$. Define $\mu_t = t\delta_0 + (1-t)\mu$ for $t \in [0, 1]$. If on any fibre $s_k(x)$ the function $f_k^x: [q] \rightarrow \{0, 1\}$ is not constant, then*

$$\alpha(1-t) \leq \mathbf{E}_{x_k \sim \mu_t}[1 - f_k^x(x_k)].$$

Proof. Fix $\mu \in \Gamma$ and any fibre $s_k(x)$. Note that $f_k^x(0) = 1$ by 0-monotonicity and $f_k^x(j) = 0$ for some $j \in [q] \setminus \{0\}$ since f_k^x is not constant. Therefore,

$$\mathbf{E}_{x_k \sim \mu_t}[1 - f_k^x(x_k)] \geq \mu_t(0)(1 - f_k^x(0)) + \mu_t(j)(1 - f_k^x(j)) \geq \alpha(1-t),$$

as required. \square

Now we prove Proposition 1.

Proof of Proposition 1. Let $\alpha_t = \alpha(1-t)$. By Lemma 5 it follows that

$$(3) \quad \log\left(\frac{1}{\alpha_t}\right) \geq \log\left(\frac{1}{\mathbf{E}_{x_k \sim \mu_t}[1 - f_k^x(x_k)]}\right).$$

Let

$$\Phi_k(\mu_t) = \mathbf{E}_{x \sim \mu_t^{\otimes n}} [\mathbb{I}[f_k^x \text{ is not constant}] \mathbf{E}_{x_k \sim \mu_t}[1 - f_k^x(x_k)]].$$

Then by (3) we obtain

$$\begin{aligned} \left(2 + 2 \log \left(\frac{1}{\alpha_t}\right)\right) \Phi_k(\mu_t) &= \mathbf{E}_{x \sim \mu_t^{\otimes n}} \left[2\mathbb{I}[f_k^x \text{ is not constant}] \left(\mathbf{E}_{x_k \sim \mu_t} [1 - f_k^x(x_k)] \right. \right. \\ &\quad \left. \left. + \log \left(\frac{1}{\alpha_t}\right) \mathbf{E}_{x_k \sim \mu_t} [1 - f_k^x(x_k)] \right) \right] \\ &\geq \mathbf{E}_{x \sim \mu_t^{\otimes n}} \left[2\mathbb{I}[f_k^x \text{ is not constant}] \left(\mathbf{E}_{x_k \sim \mu_t} [1 - f_k^x(x_k)] \right. \right. \\ &\quad \left. \left. + \log \left(\frac{1}{\mathbf{E}_{x_k \sim \mu_t} [1 - f_k^x(x_k)]}\right) \mathbf{E}_{x_k \sim \mu_t} [1 - f_k^x(x_k)] \right) \right]. \end{aligned}$$

Under the measure $\mu_t^{\otimes n}$ we naturally transform f to a function $F: [0, 1]^n \rightarrow \{0, 1\}$ as follows. Define $G: [0, 1] \rightarrow [q]$ by

$$G(x) = i \quad \text{if } x \in \left[\sum_{\ell < i} \mu_t(\ell), \sum_{\ell \leq i} \mu_t(\ell) \right) \quad \text{and} \quad G(1) = q - 1.$$

Then define $F: [0, 1]^n \rightarrow \{0, 1\}$ by

$$F(x_1, \dots, x_n) = f(G(x_1), \dots, G(x_n)).$$

It follows that

$$\begin{aligned} \left(2 + 2 \log \left(\frac{1}{\alpha_t}\right)\right) \Phi_k(\mu_t) &\geq \mathbf{E}_{x \in [0, 1]^n} \left[2\mathbb{I}[F_k^x \text{ is not constant}] \left(\mathbf{E}_{x_k \in [0, 1]} [1 - F_k^x(x_k)] \right. \right. \\ &\quad \left. \left. + \log \left(\frac{1}{\mathbf{E}_{x_k \in [0, 1]} [1 - F_k^x(x_k)]}\right) \mathbf{E}_{x_k \in [0, 1]} [1 - F_k^x(x_k)] \right) \right]. \end{aligned}$$

Define the function $h: [0, 1] \rightarrow \mathbb{R}$ by

$$h(t) = 2\mathbb{I}[t \in (0, 1)] (1 - t) (1 - \log(1 - t)).$$

One easily verifies that $h(t) \geq \text{Ent}(t)$ for all $t \in [0, 1]$. Therefore, by Theorem 4 and the symmetry of F there exists some universal constant C such that

$$\begin{aligned} \left(2 + 2 \log \left(\frac{1}{\alpha_t}\right)\right) \frac{d \mathbf{E}_{x \sim \mu_t^{\otimes n}} [f(x)]}{dt} &= \frac{1}{1 - t} \sum_{k=1}^n \left(2 + 2 \log \left(\frac{1}{\alpha_t}\right)\right) \Phi_k(\mu_t) \\ &\geq \frac{1}{1 - t} \sum_{k=1}^n I_k^h(F) \\ &\geq \frac{C \text{Var}(F) \log n}{1 - t} \\ &= \frac{C \text{Var}_{\mu_t^{\otimes n}}(f) \log n}{1 - t}. \end{aligned}$$

As such,

$$\begin{aligned} \frac{d \mathbf{E}_{x \sim \mu_t^{\otimes n}}[f(x)]}{dt} &\geq \frac{C \mathbf{E}_{x \sim \mu_t^{\otimes n}}[f(x)](1 - \mathbf{E}_{x \sim \mu_t^{\otimes n}}[f(x)]) \log n}{(1-t)(1 + \log(1/\alpha_t))} \\ &\geq \frac{C \mathbf{E}_{x \sim \mu_t^{\otimes n}}[f(x)](1 - \mathbf{E}_{x \sim \mu_t^{\otimes n}}[f(x)]) \log n}{\log(1/\alpha)}, \end{aligned}$$

where in the last inequality we use the facts that $\alpha \leq 1/2$, so $1 \leq \log(1/\alpha_t)$ for all $t \in [0, 1]$, and that there exists some constant c such that $(1-t) \log(1/\alpha_t) \leq c \log(1/\alpha)$ for all $t \in [0, 1]$. \square

3.4. Line Segments Have Short Length. Proposition 1 shows that the derivative $d \mathbf{E}_{x \sim \mu_t^{\otimes n}}[f(x)]/dt$ is large with a dependence on the second-smallest atom α of μ . We partition Γ into $q - 1$ regions

$$R_i = \left\{ \mu \in \Delta[q] \mid \mu(i) = \min_{j \in [q] \setminus \{0\}} \mu(j) \right\}$$

for $i \in \{1, \dots, q-1\}$. In other words, R_i is the set of all measures μ in $\Delta[q]$ such that the second-smallest atom of μ is $\mu(i)$. Now fix a region R_i and suppose that $\mu \in R_i$. By definition, $\alpha = \mu(i)$. Recall that

$$\mathcal{D} := \{ \mu \in \Delta[q] \mid \varepsilon \leq \mathbf{E}_{x \sim \mu^{\otimes n}}[f(x)] \leq 1 - \varepsilon \}.$$

In this section we show the following. Again, C denotes different constants at different lines, depending on q only.

Proposition 2. *Let $f: [q]^n \rightarrow \{0, 1\}$ be a 0-monotone and symmetric function, and let $\mu \in R_i$. Define $\mu_t = t\delta_0 + (1-t)\mu$ for $t \in [0, 1]$. There exists an absolute constant $C = C(q)$ such that*

$$\lambda(\{\mu_t \mid t \in [0, 1]\} \cap \mathcal{D}) \leq \frac{C(\ln(1-\varepsilon) - \ln(\varepsilon)) \log(1/\mu(i))}{\log n}$$

Proof. Fix $\mu \in R_i$. By Proposition 1, there exists an absolute constant C such that

$$(4) \quad \frac{d \mathbf{E}_{x \sim \mu_t^{\otimes n}}[f(x)]}{dt} \geq \frac{C \mathbf{E}_{x \sim \mu_t^{\otimes n}}[f(x)](1 - \mathbf{E}_{x \sim \mu_t^{\otimes n}}[f(x)]) \log n}{\log(1/\mu(i))}.$$

When $\mathbf{E}_{x \sim \mu_t^{\otimes n}}[f(x)] \leq 1/2$, (4) implies

$$(5) \quad \frac{d \ln \left(\mathbf{E}_{x \sim \mu_t^{\otimes n}}[f(x)] \right)}{dt} \geq \frac{C \log n}{\log(1/\mu(i))}.$$

Suppose that $\mathbf{E}_{x \in \mu_p^{\otimes n}}[f(x)] = \varepsilon$ and $\mathbf{E}_{x \in \mu_r^{\otimes n}}[f(x)] = 1/2$ for some $r > p > 0$. Then

$$\ln(1/2) - \ln(\varepsilon) = \int_{t=p}^{t=r} \frac{d \ln \left(\mathbf{E}_{x \sim \mu_t^{\otimes n}}[f(x)] \right)}{dt} dt \geq \frac{C \log n}{\log(1/\mu(i))} (r - p),$$

which implies

$$r - p \leq \frac{C(\ln(1/2) - \ln(\varepsilon))}{\log n} \log(1/\mu(i)).$$

Similarly, (5) holds as well when $\mathbf{E}_{x \sim \mu_t^{\otimes n}}[f(x)] \geq 1/2$. As such, if $\mathbf{E}_{x \sim \mu_r^{\otimes n}}[f(x)] = 1/2$ and $\mathbf{E}_{x \sim \mu_q^{\otimes n}}[f(x)] = 1 - \varepsilon$ for some $1 > q > r$, then

$$q - r \leq \frac{C(\ln(1 - \varepsilon) - \ln(\varepsilon))}{\log n} \log(1/\mu(i)).$$

Therefore,

$$q - p \leq \frac{C(\ln(1 - \varepsilon) - \ln(\varepsilon))}{\log n} \log(1/\mu(i)),$$

which proves the proposition. \square

3.5. Cross Sections Have Small Lebesgue Measure. Proposition 2 shows that for a probability measure $\mu \in R_i$, the set of all probability measures μ_t that intersects \mathcal{D} contributes a length of $O(\log(1/\mu(i))/\log(n))$. We now show that the total contribution of $\{\mu_t \mid \mu \in R_i \text{ and } t \in [0, 1]\}$ that intersects \mathcal{D} is $O(1/\log n)$. To do this, we define the central measure $\mu^* \in \Gamma$ by $\mu^*(i) = 1/(q - 1)$ for all $i \in [q] \setminus \{0\}$ and define the boundaries $\Gamma_i := \{\mu \in R_i \mid \mu(i) = 0\}$. For any measure $\mu \in \Gamma_i$, we consider the two-dimensional cross section comprising measures of the form

$$\mu_{s,t} := t\delta_0 + s(1 - t)\delta_i + (1 - t)(1 - s)\mu \quad \text{for } (s, t) \in [0, 1]^2.$$

For example, Figure 1 depicts Γ when $q = 4$, so $\mu^*(1) = \mu^*(2) = \mu^*(3) = 1/3$ and $\mu^*(0) = 0$. For any $\mu \in \Gamma_i$, we see that $\mu_{0,0} = \mu$, $\mu_{1,0} = \mu^*$, and $\mu_{0,1} = \mu_{1,1} = \delta_0$.

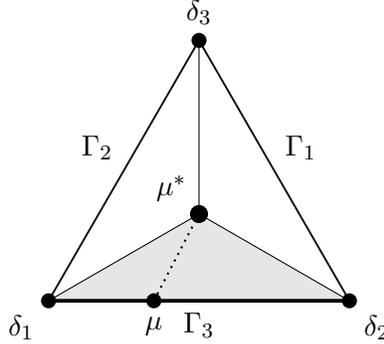


FIGURE 1. The region Γ when $q = 4$.

Again, C denotes different constants at different lines, depending on q only.

Proposition 3. *Let $f: [q]^n \rightarrow \{0, 1\}$ be a 0-monotone and symmetric function. There exists an absolute constant $C = C(q)$ such that for any region R_i , we have*

$$\lambda \left(\bigcup_{\mu \in R_i} \{\mu_t \mid t \in [0, 1]\} \cap \mathcal{D} \right) \leq \frac{C(\ln(1 - \varepsilon) - \ln(\varepsilon))}{\log n}.$$

Proof. Fix a measure $\mu \in \Gamma_i$. By construction, $\mu_{s,t}(i) = s(1-t)$. By Proposition 2, there exists some constant C such that

$$\begin{aligned} \lambda(\{\mu_{s,t} \mid (s,t) \in [0,1]^2\} \cap \mathcal{D}) &\leq C \int_{s=0}^{s=1} \lambda(\{\mu_{s,t} \mid t \in [0,1]\}) ds \\ &\leq C \int_{s=0}^{s=1} \frac{(\ln(1-\varepsilon) - \ln(\varepsilon)) \log(1/s)}{\log n} ds \\ &\leq \frac{C(\ln(1-\varepsilon) - \ln(\varepsilon))}{\log n}, \end{aligned}$$

where in the last line we use the fact that there exists some constant C such that

$$\int_{s=0}^{s=1} \log\left(\frac{1}{s}\right) ds \leq C.$$

Therefore,

$$\begin{aligned} \lambda\left(\bigcup_{\mu \in R_i} \{\mu_t \mid t \in [0,1]\} \cap \mathcal{D}\right) &\leq C \int_{\mu \in \Gamma_i} \lambda(\{\mu_{s,t} \mid (s,t) \in [0,1]^2\} \cap \mathcal{D}) \lambda(d\mu) \\ &\leq \frac{C(\ln(1-\varepsilon) - \ln(\varepsilon))}{\log n}, \end{aligned}$$

as required. \square

Remark 1. Theorem 1 is sharp up to multiplicative constants. Ben-Or and Linial [1] constructed the tribes function that showed the $O(\log n/n)$ bound on influences presented by Kahn, Kalai, and Linial [4] is sharp. We present a variant of the tribes function $f: [q]^n \rightarrow [q]$ such that

$$\lambda(\{\mu \in \Delta[q] \mid \varepsilon \leq \Pr_{x \sim \mu^{\otimes n}}[f(x) = a] \leq 1 - \varepsilon\}) = \Theta(1/\log n)$$

for all $a \in [q]$. Consider $[q]^n$ endowed with the measure $\mu^{\otimes n}$ for some $\mu \in \Delta[q]$. Partition the sets $\{1, 2, \dots, n\}$ into sets $\{T_1, T_2, \dots, T_{n/r}\}$ of size

$$r = \left\lfloor \frac{\log n - \log \log n + \log \log(1/\mu(0))}{\log(1/\mu(0))} \right\rfloor$$

each. Define $f: [q]^n \rightarrow [q]$ by setting $f(x) = 0$ if there exists some $i \in \{1, \dots, n/r\}$ such that $x_j = 0$ for all $j \in T_i$, and otherwise setting $f(x) = x_i$ where $i \in \{1, \dots, n\}$ is the smallest index where $x_i \neq 0$.

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MASSACHUSETTS INSTITUTE OF TECHNOLOGY, 77 MASSACHUSETTS AVE, CAMBRIDGE, MA 02139-4301

Email address: `sabal@mit.edu`

MASSACHUSETTS INSTITUTE OF TECHNOLOGY, 77 MASSACHUSETTS AVE, CAMBRIDGE, MA 02139-4301

Email address: `allenees@mit.edu`