Exploration of Quantum Product of Schubert Polynomials

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Abstract

Schubert polynomials, which are polynomials in $x_1, ..., x_n$ determined by a permutation in the symmetric group S_n, have had many interpretations both in algebra and combinatorics since their inception. An important open question in algebraic combinatorics is concerned with the explicit formula for the expansion of the product of two Schubert polynomials into a linear combination of Schubert polynomials, all of which form a basis of S_∞. In this paper, we begin to explore the quantum version of this problem, which was introduced by Sergey Fomin, Sergei Gelfand and Alexander Postnikov in 1997 [6], by exploring multiplication by the quantum Schubert polynomial of special permutations. Our approach will primarily be combinatorial, studying paths in the quantum Bruhat graph via a interpretation introduced by Alexander Postnikov.

1 Introduction

Schubert polynomials $\mathfrak{S}_w$, first introduced in 1982 by Lascoux and Schützenberger [3], have many interpretations in algebra, combinatorics, and geometry. One way that Schubert polynomials arise is as representatives of cohomology classes of the flag variety $\mathrm{H}^*(\mathrm{Fl}_n)$, where

$$\mathrm{Fl}_n = \{0 \subset V_1 \subset ... \subset V_{n-1} \subset \mathbb{C}^n \mid \dim V_i = i \text{ for } i = 1, ..., n-1\}.$$

Another way to view $\mathrm{H}^*(\mathrm{Fl}_n)$ is that it is isomorphic as a graded algebra to the quotient

$$\mathbb{Z}[x_1, x_2, ...]/\langle e_1, e_2, ...\rangle,$$

where $e_k = x_1 + x_2 + ... + x_k$ is the kth elementary symmetric polynomial for all $k = 1, 2, ..., n$ and $\langle e_1, e_2, ...\rangle$ is the ideal generated by the set $\{e_k\}$. Schubert polynomials have a natural generalization to quantum Schubert polynomials σ_w, which can be viewed as representatives of classes of the quantum cohomology $\mathrm{QH}^*(\mathrm{Fl}_n)$ that have certain applications in physics.

Given any two (quantum) Schubert polynomials, their product can always be expressed uniquely as a linear combination of other (quantum) Schubert polynomials with positive integer coefficients. These coefficients are called the (quantum) Schubert Littlewood-Richardson coefficients. One of the biggest open question in algebraic is to provide a combinatorial rule for Schubert Littlewood-Richardson coefficients, which has proven to be a very difficult task.

In the case of quantum Schubert polynomials, even the set of all q-monomials appear as coefficients is a completely unknown, though some progress has been made in the field. Alexander Postnikov, and more recently, Jiyang Gao, Shiliang Gao and Yibo Gao (see [2]) has made some advances to the field.

In this paper, we provide expository background on quantum Schubert polynomials and the quantum Bruhat graph Γ_n from a combinatorial perspective, as well as provide some new results towards finding a formula for multiplication by $\sigma_{t_{i\,i+2}}$ and observations on cycles in Γ_n.

2 Background

2.1 Schubert Polynomials and Pipe Dreams

Let S_n be the symmetric group on n elements generated by the set of *simple transpositions* $S = \{s_i := (i\ i+1) \mid i = 1, ..., n-1\}$, where $s_i = t_{i\,i+1}$ denotes switching the values at positions i and $i+1$. Similarly, define the set of *transpositions* $T = \{t_{ij} := (i\ j) \mid 1 \leq i < j \leq n\}$, where t_{ij} switches positions i and j and are the conjugates of simple transpositions in S. By convention, we will refer to permutation $w = w(1)...w(n)$ simply as $w_1...w_n$. Let $\boldsymbol{a} = a_1...a_l$ denote a *reduced word* of w such that $w = s_{a_1}...s_{a_l}$ and $R(w)$ denote the set of all reduced words of w. Further, denote $l(w)$ as the length of all reduced words in $R(w)$.

Definition 2.1. *Schubert polynomials* $\mathfrak{S}_w$ are polynomials in the variables $x_1, x_2, ...$ indexed by a permutation $w \in S_\infty$, which is the infinite symmetric group of all permutations of $\mathbb{N}$ with a finite set of non-fixed elements. The set $\{\mathfrak{S}_w\}$ forms a basis for the polynomial ring $\mathbb{Z}[x_1, x_2, ...]$ in infinitely many variables.

Definition 2.2. [1] Given a reduced word $\boldsymbol{a} = a_1...a_l \in R(w)$, a tuple $\boldsymbol{\alpha} = (\alpha_1, ..., \alpha_l)$ of strictly positive integers is $\boldsymbol{a}$*-compatible* if

$$\begin{aligned} &\alpha_1 \leq \alpha_2 \leq ... \leq \alpha_l \\ &\alpha_j \leq a_j \qquad \text{for } 1 \leq j \leq l \\ &\alpha_j \leq \alpha_{j+1} \quad \text{if } a_j < a_{j+1}. \end{aligned}$$

Let $C(\boldsymbol{a})$ denote the set of all $\boldsymbol{a}$-compatible sequences.

Definition 2.3. [1] Given a reduced word $\boldsymbol{a} = a_1...a_l \in R(w)$ and an $\boldsymbol{a}$-compatible sequence $\boldsymbol{\alpha} = (\alpha_1, ..., \alpha_l) \in C(\boldsymbol{a})$, a *reduced-word compatible sequence graph*, or more commonly referred to as a *reduced pipe dream*, is the subset $D(\boldsymbol{a}, \boldsymbol{\alpha}) = \{(\alpha_k, a_k - \alpha_k + 1) \mid 1 \leq k \leq l\} \subset \{1, 2, ..., n\} \times \{1, 2, ..., n\}$, or more simply denoted as D.

Denote $\mathcal{RP}(w)$ as the set of all reduced pipe dreams.

We represent a pipe dream as an upper triangular wiring diagram that consists of two types of tiles: crossing tile and elbow tile.

Theorem 2.4. *[4] A combinatorial interpretation of Schubert polynomials draws a bijection between the set of all monomials of* $\mathfrak{S}_w$ *and the set of all pipe dreams* $D \in \mathcal{RP}(w)$.

$$\mathfrak{S}_w(\boldsymbol{x}) = \sum_{D \in \mathcal{RP}(w)} \boldsymbol{x}^D, \text{ where } \boldsymbol{x}^D = \prod_{(i,j) \in D} x_i$$

where $\boldsymbol{x} = (x_1, ..., x_n)$ *for* $w \in S_n$.

2.2 Quantum Bruhat Graph Γ_n

Definition 2.5. [5] The *quantum Bruhat graph* Γ_n is a weighted directed graph on elements of S_n with the following two types of edges, which hereinafter are informally referred to as *up edges* and *down edges* respectively

$$\text{wt}(w \to wt_{ij}) := \begin{cases} 1 & \text{if } l(wt_{ij}) = l(w) + 1 \\ q_{ij} = q_i...q_{j-1} & \text{if } l(wt_{ij}) = l(w) - l(t_{ij}) \end{cases}$$

where $\text{wt}(w \to wt_{ij}) \in \mathbb{Z}[q_1, ..., q_{n-1}]$. We can also interpret Γ_n as building on the Hasse diagram of the strong Bruhat order, where existing relations $w \lessdot wt_{ij}$ become up edges $w \to wt_{ij}$ with weighted down edges (fig. 1). We reconcile the edge conditions for these two types of edges in the following definition.

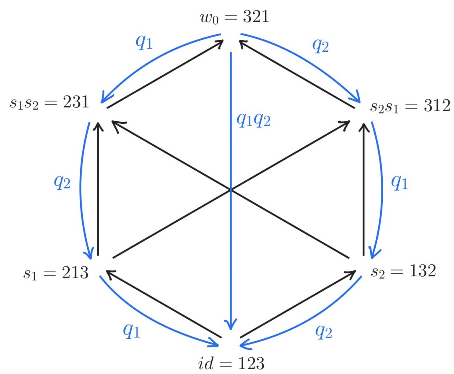

Figure 1: Quantum Bruhat graph Γ_3 (unlabeled edges have weight 1)

Definition 2.6. [2] An *open cyclic interval* from a to b, for $a, b \in [n]$, $a \neq b$, is $(a, b)_c = \{a+1, ..., b-1\}$ if $a < b$ and $(a, b)_c = \{a + 1, ..., n\} \cup \{1, ...b - 1\}$ if $a > b$ where $(a, b)_c = \emptyset$ if a and b are consecutive on the long cycle.

An edge $w \to wt_{ij}$ in Γ_n exists if and only if $w_k \in (w_j, w_i)_c$ for all $i < k < j$.

For example, edge $321 \xrightarrow{t_{13}} 123$ exists because for w =321, $w_2 = 2 \in (w_3, w_1)_c = (1, 3)_c$ while there exists no edge from 123 to 321 because $2 \notin (3, 1)_c$.

Definition 2.7. [6] Let e_i^k denote the ith elementary symmetric polynomial in $x_1, ..., x_k$. The ith *quantum elementary polynomial* E_i^k is the quantum version of e_i^k, where e_i^k equals E_i^k specialized to $q_1 = ... = q_{n-1} = 0$. We offer two definitions of E_i^k, one by matrices and the other by partitions.

The quantum elementary polynomial E_i^k is equal to the coefficient of λ^i in the characteristic polynomial $\det(1 + \lambda G_k)$, where matrix G_k is given by

$$G_k = \begin{pmatrix} x_1 & q_1 & 0 & ... & 0 \\ -1 & x_2 & q_2 & ... & 0 \\ 0 & -1 & x_3 & ... & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & ... & x_k \end{pmatrix}.$$

The combinatorial interpretation of E_i^k is as follows. Take the set $K = \{1, ..., k\}$. We will call a subset of one element $\{j\}$ for all $j \in [k]$ a singleton and a subset of two consecutive elements $\{r, r + 1\}$ for all $r \in [k - 1]$ a "dimer". They will correspond to x_j and q_r respectively in a monomial of E_i^k. Then, E_i^k is the sum of all such monomials, which correspond to all possible ways of choosing singletons and dimers from set K such that their disjoint union is a subset of K with i elements.

Definition 2.8. [6] Let h_j^k denote the jth homogeneous symmetric polynomial in $x_1, ..., x_k$ as h_j^k, which is the sum of all monomials of degree j. They are related to e_i^k by the formula

$$h_j^k = \det\left(e_{d-c+1}^{k+j-c}\right)_{c,d=1}^{j}$$

The jth *quantum homogeneous polynomial* H_i^k is the quantum version of h_i^k, where h_j^k equals H_j^k specialized to $q_1 = ... = q_{n-1} = 0$. Denote *complete homogeneous monomial* (CHM) as $h_I = h_{i_1}^1 ... h_{i_{n-1}}^{n-1}$ for $I = (i_1, ..., i_{n-1})$ and $0 \leq i_p < p$.

The combinatorial interpretation for homogeneous symmetric polynomials is the same as for elementary symmetric polynomials, except we now only require that the sum of the sizes of all singleton and dimer sets sum to i, i.e. elements may be repeated.

Definition 2.9. [6] The *standard elementary monomial* (SEM) e_I and the *quantum standard elementary monomial* E_I are the products

$$e_I = e^1_{i_1} e^2_{i_2} ... e^{n-1}_{i_{n-1}}$$

$$E_I = E^1_{i_1} E^2_{i_2} ... E^{n-1}_{i_{n-1}}$$

where $I = (i_1, i_2, ..., i_{n-1})$, $0 \leq i_p < p$.

Lemma 2.10. *[6] The set of elementary symmetric polynomials $\{e_I\}$ forms a $\mathbb{Z}$-linear basis in $\mathbb{Z}[x_1, x_2, ...]$. Both $\{e_I\}$ and $\{E_I\}$ are K-linear bases in $K[x_1, x_2, ...]$ where $K = \mathbb{Z}[q_1, q_2, ...]$.*

Definition 2.11. [6] The K-linear map $\psi : K[x_1, x_2, ...] \to K[x_1, x_2, ...]$ maps

$$\psi : e_I \to E_I \text{ for all } I.$$

In other words, given lemma 2.10, the decomposition of Schubert polynomials $\mathfrak{S}_w$ is unique so we can define *quantum Schubert polynomials* as $\sigma_w = \psi(\mathfrak{S}_w)$.

This map can be analogously defined for $h_I \to H_I$ where $H_I = H^1_{i_1} ... H^{n-1}_{i_{n-1}}$.

Theorem 2.12 (Quantum Monk's formula)**.** *[5]*
For $w \in S_n$ and $1 \leq r < n$, the decomposition of the product of two quantum Schubert polynomials σ_{s_r} and σ_w is given by

$$\sigma_{s_r} * \sigma_w = \sum \sigma_{ws_{ab}} + \sum q_{cd} \sigma_{ws_{cd}}$$

where the first sum is over all transpositions s_{ab} such that $a \leq r < b$ and $l(wt_{ab}) = l(w) + 1$ and the second sum is over all transpositions s_{cd} such that $c \leq r < d$ and $l(wt_{cd}) = l(w) - l(t_{cd}) = l(w) - 2(d - c) + 1$.

Define the operator $\mathcal{T}_{ij}$ as

$$\mathcal{T}_{ij} : \sigma_w \longmapsto \begin{cases} \sigma_{ws_{ij}} & \text{if } l(ws_{ij}) = l(w) + 1 \\ q_{ij}\sigma_{ws_{ij}} & \text{if } l(ws_{ij}) = l(w) - 2(j - i) + 1 \\ 0 & \text{otherwise} \end{cases}$$

Then the quantum Monk's formula (theorem 2.12) is equivalent to

$$\sigma_{s_r} * \sigma_w = \sum_{i \leq r < j} \mathcal{T}_{ij}(\sigma_w)$$

Further, by the quantum Pieri's Formula, operators $\mathcal{E}^{(k)}_i$ and $\mathcal{H}^{(k)}_i$ have expansion

$$\mathcal{E}^{(k)}_i = \sum \mathcal{T}_{a_1 b_1} ... \mathcal{T}_{a_i b_i}$$

$$\mathcal{H}^{(k)}_i = \sum \mathcal{T}_{c_1 d_1} ... \mathcal{T}_{c_i d_i}$$

where the first sum is over $a_1, ..., a_i, b_1, ..., b_i$ such that $a_1, ..., a_i \leq k < b_1 \leq ... \leq b_i$ and $a_1, ..., a_i$ are distinct and the second sum is over $c_1, ..., c_i, d_1, ..., d_i$ such that $c_1 \leq ... \leq c_i \leq k < d_1 ... d_i$ and $d_1, ..., d_i$ are distinct.

We can view the product of the ith quantum elementary polynomial E^k_i on $x_1, x_2, ..., x_k$ and the quantum Schubert polynomial σ_w as the action of the composition a series of $\mathcal{T}_{ij}$ operators on σ_w. In essence, this defines an operator that correspond to each E^k_i. Specifically, the we have $E^k_i * \sigma_w = \mathcal{E}^{(k)}_i(w)$. Similarly, operation on w by the ith quantum complete homogeneous polynomial H^i_k on $x_1, ..., x_k$ is described by $H^k_i * \sigma_w = \mathcal{H}^{(k)}_i(w)$.

Lemma 2.13. *Define long cycles $u_1 = (i+1\ i+2\ ...\ j+1\ i) = s_i s_{i+1} ... s_j$ and $u_2 = (j+1\ i\ i+1\ ...\ j) = s_j s_{j-1} ... s_i$. Then, $\sigma_{u_1} = E^j_{j-i+1}$ and $\sigma_{u_2} = H^i_{j-i}$.*

Proof. The statement follows from $\mathfrak{S}_{u_1} = e_{j-i}^{j-1}$ and $\mathfrak{S}_{u_2} = h_{j-i}^{i}$ as shown by the following pipe dreams using the bijection provided by theorem 2.4. Then, we apply the K-linear map ψ (definition 2.11) to obtain $\mathfrak{S}_{u_1} = \sigma_{u_1}$ and $\mathfrak{S}_{u_2} = \sigma_{u_2}$.

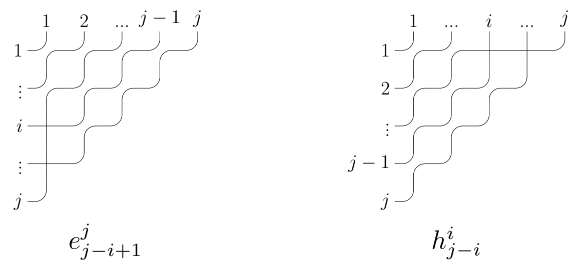

Figure 2: Pipe dreams for Long Cycles w_1 and w_2

∎

Another way to study which q-monomials appear is through directed paths on the quantum Bruhat graph Γ_n. As established by Postnikov [5], a q-monomial $q^d = q_1^{d_1}...q_{n-1}^{d_{n-1}}$ for $d_i \geq 0$ appears in the decomposition of the product of two quantum Schubert polynomials $\sigma_u * \sigma_v$ *if and only if* q^d appears as a path weight from permutation u to $w_0 v$, where $w_0 = (n \;\; n-1 \;\; ... \;\; 1)$ is the longest word in S_n. The weight of a path is the product of all edges in the path, and if there exists no *admissible* paths from u to v, then we assume the path weight equals 1.

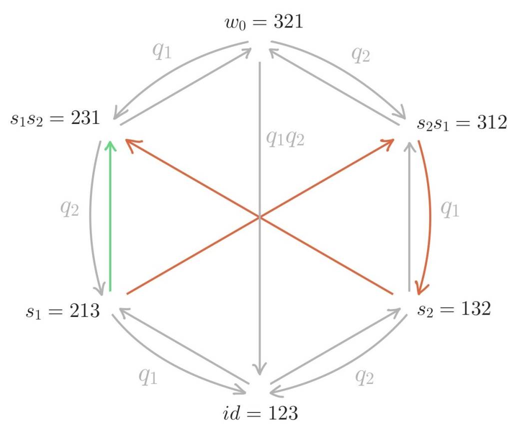

Figure 3: Paths from 213 to 231

Definition 2.14. [5] A path $u_0 \xrightarrow{t_{a_1b_1}} u_1 \xrightarrow{t_{a_2b_2}} ... \xrightarrow{t_{a_lb_l}} u_l$ in Γ_n is *admissible* if there exists k_i for every $i = 1, ..., l$ such that $a_i \leq k_i < b$ and all pairs (k_i, b_i) are distinct.

For example (see fig. 3), there exist two admissible paths from permutation 213 to 231. The first is $213 \xrightarrow{t_{23}} 231$ of path weight 1 where the sequence of (k, b)'s is $(2, 3)$ (drawn in green). The second is $213 \xrightarrow{t_{13}} 312 \xrightarrow{t_{12}} 132 \xrightarrow{t_{13}} 231$ of path weight q_1 where the sequence of (k, b)'s is $(1, 3), (1, 2), (2, 3)$ (drawn in orange).

3 Down Step Patterns in Γ_n

The path from permutation u to v on Γ_n consists of a sequence of up and down edges, which mapped to edge weights is a sequence of 1 and q_{ij}.

Lemma 3.1. *Consecutive down steps cannot have the same q_{ij} weight.*

Proof. Assume for the sake of contradiction that there exists some subsequence in the path from u to v that is has the following weights

$$q_{ij}, \underbrace{1, ..., 1}_{\text{some 1's}}, q_{ij}.$$

Denote $u' \xrightarrow{q_{ij}} u'' \to ... \to v' \xrightarrow{q_{ij}} v''$ as the starting and ending permutations of the subsequence. For the down edges to exist, u' and v' must satisfy definition 2.6. Therefore, values u'_j and u'_i must switch with up steps from u'' to v'. This will result in an invalid (k, b) sequence so no admissible path exists.

$$\begin{array}{cccc} \text{index} & i & ... & j \\ \text{value} & u'_i & ... & u'_j \end{array} \xrightarrow{t_{ij}} \begin{array}{ccc} i & ... & j \\ u'_j & ... & u'_i \end{array} \to ... \to \begin{array}{ccc} i & ... & j \\ u'_i & ... & u'_j \end{array} \xrightarrow{t_{ij}} \begin{array}{ccc} i & ... & j \\ u'_j & ... & u'_i \end{array}$$

□

Theorem 3.2. *The next down step followed by a down step $t_{i\,i+1}$ or $t_{i\,i+2}$ must be in the form $t_{k_1\,k_2}$ such that (k_2, k_1) is strictly lexicographically larger than $(i+1, i)$ or $(i+2, i)$ respectively for the path to be admissible.*

Proof. Accounting for lemma 3.1, assume for the sake of contradiction that (k_1, k_2) is lexicographically smaller than the previous down step of that form. We will refer to down edges by their weights for the rest of the proof. Note that if $k_2 \leq i$, then the path is clearly not admissible.

Case 1: Weights $q_i, 1, ..., 1, q_k$ where $k < i$. Then, the (k, b) sequence is necessarily $(i, i+1), ..., (k, k+1)$. However, $k < i$ so the path is not admissible $\Rightarrow$ Contradiction.

Case 2: Weights $q_i q_{i+1}, 1, ..., 1, q_{i-1} q_i$. Then, the (k, b) sequence must be $(i, i+2), ..., (i, i+1)$. The following shows change on the permutation by the first transposition of the path $t_{i\,i+2}$

$$\begin{array}{ccccccc} \text{index} & ... & i-1 & i & i+1 & i+2 & ... \\ \text{value} & ... & w_{i-1} & w_i & w_{i+1} & w_{i+2} & ... \end{array} \xrightarrow{t_{i\,i+2}} \begin{array}{ccccccc} \text{index} & ... & i-1 & i & i+1 & i+2 & ... \\ \text{value} & ... & w_{i-1} & w_{i+2} & w_{i+1} & w_i & ... \end{array}$$

such that $w_i > w_{i+2}$ and $w_{i+1} \in (w_{i+2}, w_i)_c$. It is safe to assume that $w_{i-1} > w_{i+1}$ as the opposite case would involve more up steps which takes up available (k, b) pairs. However, since $w_{i+2} \notin (w_{i+1}, w_{i-1})_c$, changes by up steps must be made to the permutation so that the last transposition $t_{i-1\,i+1}$ is a valid down step. Since $1 \leq w_{i+2} < w_{i+1} < w_{i-1} \leq n$, the only choice is to increase the value at index i to be greater than w_{i+1}, which involves the transposition $t_{i\,i+1}$, resulting in two $(i, i+1)$ tuples in (k, b) sequence that makes the path not admissible. □

4 Cycles in Γ_n

4.1 Description of Cycle Weights

In accordance to the admissible paths condition (definition 2.14), cycles can count as valid paths for some permutations in S_n. This corresponds to studying the q-monomials in the decomposition of $\sigma_w * \sigma_{w_o w}$. From generated data, we try to describe the possible q-monomials that can appear.

Remark 4.1. The sum of lengths of all up transpositions equals the sum of lengths of all down transpositions.

Conjecture 4.2. *Denote the path weight of a cycle as $q_1^{d_1} ... q_{n-1}^{d_{n-1}}$. Then, $d_i \leq 1$ for all $i = 1, ..., n-1$, or in other words, all down steps in the path have disjoint q-terms.*

Conjecture 4.3. *The weights of cycle paths are always in the form $q_i ... q_j$ such that $i + 2 \leq j$.*

If we assume conjecture 4.2 to be true, the proof becomes eliminating q-monomials in the form q_i and $q_i q_{i+1}$. This can be easily shown.

Case 1: Cycle path with weight q_i. The (k, b) pair $(i, i+1)$ cannot appear twice in any admissible path $\Rightarrow$ Contradiction.

Case 2: Cycle path with weight $q_i q_{i+1}$. Then, by remark 4.1, the weights of edges in the path is either some permutation of the set $\{q_i, q_{i+1}, 1, 1\}$ or of the set $\{q_i q_{i+1}, 1, 1, 1\}$, none of which result in admissible paths.

Conjecture 4.4. *For permutation $w \in S_n$, if there exists a non-empty path from w to w, then there exists a path with weight $q_1 ... q_{n-1}$.*

Note that conjecture 4.2, conjecture 4.3, and conjecture 4.4 are verified for S_4 and S_5, where no cycles exist in S_3 because $\mathbb{Z}[q_1, q_2]$ cannot satisfy conjecture 4.3. Further, since $q_1 q_2 q_3 \in \mathbb{Z}[q_1, q_2, q_3]$ is the only q-monomial that satisfies conjecture 4.3, all permutations that have cycles in S_4 have only the maximum cycle weight.

4.2 Generating Permutations with Non-empty Cycle Paths

We also consider which permutations have cycle paths and which do not.

Lemma 4.5. *If permutation $w \in S_n$ has cycle paths, then $w_0 w$ has cycles with identical path weights.*

Proof. By commutative property of multiplication and $w_0^2 = e$, $\sigma_w * \sigma_{w_0 w} = \sigma_{w_0 w} * \sigma_w = \sigma_{w_0 w} * \sigma_{w_0(w_0 w)}$. □

The question remains as to how to identify which permutations have cycles and which do not. It seems that the number of permutations with cycles in S_n increases as n increases: 8 permutations have cycles in S_4 while 80 permutations have cycles in S_5.

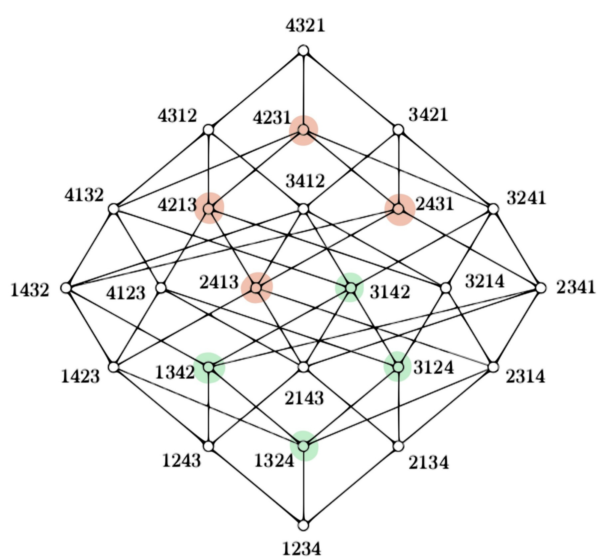

Figure 4: Permutations with Cycles in S_4 (orange nodes map to green nodes by w_0)

5 Explicit Form of $t_{i\,i+2}$

5.1 Repeated Application of Monk's Rule

Corollary 5.1. *For $w \in S_n$, a direct result from the quantum Monk's Formula is the explicit formula for*

$$(\sigma_{s_{r_1}} * ... * \sigma_{s_{r_l}}) * \sigma_w = \sum \left(\prod_{i=1}^{l} q'_{a_i b_i} \right) \cdot \sigma_{w t_{a_1 b_1} ... t_{a_l b_l}}$$

where $a_i \leq r_i < b_i$ for all $i = 1, ..., l$ and $q'_{a_j b_j}$ is the weight of the edge $w t_{a_1 b_1} ... t_{a_{j-1} b_{j-1}} \xrightarrow{t_{a_j b_j}} w t_{a_1 b_1} ... t_{a_j b_j}$ for all $j = 2, ..., l$.

As we know, the explicit form for multiplication by $\sigma_{t_{i\,i+1}}$ is given by the quantum Monk's formula(theorem 2.12). We try to generalize the rule by finding the explicit form for multiplication by $\sigma_{t_{i\,i+2}}$. By repeated application of Monk's formula (corollary 5.1), we have the following expansion:

$$\sigma_{t_{i\,i+2}} = \sigma_{s_i} * \sigma_{s_{i+1}} * \sigma_{s_i} - \left(\sigma_{s_{i+1} s_{i-1} s_i} + \sigma_{s_{i+2} s_{i+1} s_i} + \sigma_{s_{i-1} s_i s_{i+1}} + q_i \sigma_{s_{i+1}}\right) \tag{1}$$

From lemma 2.13 and Monk's, we know that all terms in the above expansion has an explicit form except $\sigma_{s_{i+1} s_{i-1} s_i}$. This is further confirmed by Woodruff's theorems [7], which characterized the permutations whose Schubert polynomials can be written as a single SEM e_I or CHM h_I. Note that the pattern formed by $s_{i+1} s_{i-1} s_i$ is 2413, which contains both pattern 312 and 231.

Theorem 5.2. *[7] For $w \in S_n$, Schubert polynomial $\mathfrak{S}_w$ is a single SEM e_I if and only if w avoids the patterns* 312 *and* 1432*.*

Theorem 5.3. *[7] For $w \in S_n$, Schubert polynomial $\mathfrak{S}_w$ is a single CHM h_I if and only if w avoids the patterns* 321 *and* 231*.*

Special Case. We consider the very special case of $t_{13} = 3214...$, which simplifies to the above formula to

$$\sigma_{t_{13}} = \sigma_{s_1} * \sigma_{s_2} * \sigma_{s_1} - \left(\sigma_{s_3 s_2 s_1} + q_1 \sigma_{s_2}\right)$$

such that all terms containing s_{i-1} goes to 0. Also notice that t_{13} satisfies the pattern avoidance conditions from theorem 5.2. Further, $\mathfrak{S}_{t_{13}} = e_1^1 e_2^2$ can indeed be expressed as a single SEM term.

However, an explicit formula for $\sigma_{s_{i+1} s_{i-1} s_i}$ is not necessary to obtain an explicit formula for $\sigma_{t_{i\,i+2}}$. First, note that $t_{i\,i+2}$ has two different reduced words, $t_{i\,i+2} = s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$, where repeatedly applying Monk's Formula results in a different decomposition

$$\sigma_{t_{i\,i+2}} = \sigma_{s_{i+1}} * \sigma_{s_i} * \sigma_{s_{i+1}} - \left(\sigma_{s_{i+2} s_i s_{i+1}} + \sigma_{s_{i+2} s_{i+1} s_i} + \sigma_{s_{i-1} s_i s_{i+1}} + q_i \sigma_{s_i}\right). \tag{2}$$

Although the explicit formula $\sigma_{s_{i+2} s_i s_{i+1}}$ is again not known, we can cancel it by subtracting eq. (2) for the $i-1$th case from eq. (1) for the ith case.

Theorem 5.4. *The difference of quantum Schubert polynomials indexed by permutation $t_{i\,i+2}$ for two consecutive i's is given by*

$$\begin{aligned} \sigma_{t_{i\,i+2}} - \sigma_{t_{i-1\,i+1}} = &\left(\sigma_{s_i} * \sigma_{s_{i+1}} * \sigma_{s_i} - \sigma_{s_i} * \sigma_{s_{i-1}} * \sigma_{s_i}\right) \\ &+ \left(\sigma_{s_{i+1} s_i s_{i-1}} - \sigma_{s_{i+2} s_{i+1} s_i}\right) + \left(\sigma_{s_{i-2} s_{i-1} s_i} - \sigma_{s_{i-1} s_i s_{i+1}}\right) \\ &+ \left(q_{i-1} \sigma_{s_{i-1}} - q_i \sigma_{s_{i+1}}\right) \end{aligned}$$

where $i \geq 2$. Given that $\sigma_{t_{13}}$ has an explicit formula as a sum of quantum Schubert polynomials with known operators, we can iteratively get t_{24}, t_{35}, ... by this difference formula.

5.2 Further Thoughts and Observations

A common representation of the symmetric group S_4 is a *permutahedron* (see fig. 5). Permutation $w_0 = 4321$ is incident to three faces, where the two hexagonal faces are symmetric about the octagonal cross-section created by permutations 4321, 4231, 3142, 2143, 1234, 1324, 2413, and 3412. A possible conjecture from here is that paths for any permutation $w \in S_4$ to $w_0 t_{13}$ can be mapped to paths from $w' \in S_4$ to $w_0 t_{24}$, where w' is the image of w reflected across the octagonal plane.

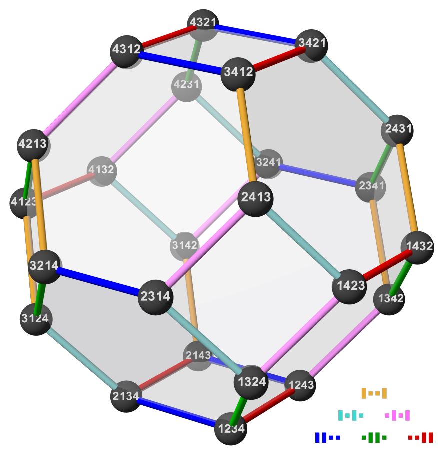

Figure 5: Permutahedron of S_4

Conjecture 5.5. *For $w, w' \in S_4$ where w' is the reflectional image of w over the plane determined by w_0, id, and 3412, the q-monomial $q_1^{d_1} q_2^{d_2} ... q_{n-1}^{d_{n-1}}$ exists as a path weight from w to $w_0 t_{13}$ if and only if there exists a path with weight $q_1^{d'_1} q_2^{d'_2} ... q_{n-1}^{d'_{n-1}}$ from w' to $w_0 t_{24}$ such that the map $\theta : q_1^{d_1} ... q_{n-1}^{d_{n-1}} \to q_1^{d'_1} ... q_{n-1}^{d'_{n-1}}$ is the permutation on q indices, specifically $(1\,2\,...\,n-1) \to (n-1\,...\,2, 1)$.*

For example, paths from $w = 1432$ to $w_0 t_{13}$ have weights q_2, $q_1 q_2$, $q_2 q_3$, $q_1 q_2 q_3$ and paths from $w' = 3214$ to $w_0 t_{24}$ have weights q_2, $q_1 q_2$, $q_2 q_3$, $q_1 q_2 q_3$, which satisfy the map θ.

Further, we can generalize conjecture 5.5 to S_n. This can give us an easier way to calculate quantum coefficients of $\sigma_w * \sigma_{t_{i\,i+2}}$. However, note that while the reflection plane in S_4 does not pass through any of the hexagonal faces incident to w_0, this is not true for S_5, or more generally, for S_{2k+1}. Nevertheless, the map θ still applies.

Conjecture 5.6. ***(Generalized to S_n)*** *For symmetric group S_n, $n = 4, 5, ...$, permutation $w \in S_n$ and its image permutation w' reflected across the plane determined by w_0, id, and $s_1 s_{n-1}$ satisfies the following. A path from w to $w_0 t_{i\,i+2}$ has weight $q_1^{d_1} q_2^{d_2} ... q_{n-1}^{d_{n-1}}$ if and only if there exists a path with weight $q_1^{d'_1} q_2^{d'_2} ... q_{n-1}^{d'_{n-1}}$ from w' to $w_0 t_{(n-1-i)\,(n+1-i)}$ where the two q-monomials are related by map θ as defined above, for all $1 \le i \le n-2$.*

For example, paths from $w = w_0 t_{13} = 34521$ to $w_0 t_{24}$ in S_5 have weights $q_2 q_3$, $q_2 q_3 q_4$, and $q_2 q_3^2 q_4$ and paths from $w' = w_0 t_{35} = 54123$ have weights $q_2 q_3$, $q_1 q_2 q_3$, and $q_1 q_2^2 q_3$.

6 Examples

The following computer-generated data are all possible q-monomials that appear as weights of paths from permutation w to $w_0 v$, or equivalently, in the decomposition of quantum product $\sigma_w * \sigma_v$.

$n = 4$

w	$v = t_{13}$	$v = t_{24}$
1234	1	1
1243	1	$1,\ q_3,\ q_2q_3$
1324	$1,\ q_2,\ q_1q_2$	$1,\ q_2,\ q_2q_3$
1342	1	$q_3,\ q_2q_3$
1423	q_2, q_1q_2	$1,\ q_2,\ q_2q_3$
1432	$q_2,\ q_1q_2,\ q_2q_3,\ q_1q_2q_3$	$q_3,\ q_2q_3$
2134	$1,\ q_1,\ q_1q_2$	1
2143	$1,\ q_1,\ q_1q_2$	$1,\ q_3,\ q_2q_3$
2314	$1,\ q_2,\ q_1q_2$	$q_2,\ q_2q_3$
2341	1	q_2q_3
2413	$q_2,\ q_1q_2$	$q_2,\ q_2q_3$
2431	$q_2,\ q_1q_2,\ q_2q_3,\ q_1q_2q_3$	q_2q_3
3124	$q_1,\ q_1q_2$	1
3142	$q_1,\ q_1q_2,\ q_1q_2q_3$	$q_3,\ q_2q_3,\ q_1q_2q_3$
3214	$q_1,\ q_1q_2$	$q_2,\ q_1q_2,\ q_2q_3,\ q_1q_2q_3$
3241	$q_1,\ q_1q_2,\ q_1q_2q_3$	$q_2q_3,\ q_1q_2q_3$
3412	q_1q_2	q_2q_3
3421	$q_1q_2,\ q_1q_2q_3,\ q_1q_2^2q_3$	q_2q_3
4123	q_1q_2	1
4132	$q_1q_2,\ q_1q_2q_3$	$q_3,\ q_2q_3,\ q_1q_2q_3$
4213	q_1q_2	$q_2,\ q_1q_2,\ q_2q_3,\ q_1q_2q_3$
4231	$q_1q_2,\ q_1q_2q_3$	$q_2q_3,\ q_1q_2q_3$
4312	q_1q_2	$q_2q_3,\ q_1q_2q_3,\ q_1q_2^2q_3$
4321	$q_1q_2,\ q_1q_2q_3,\ q_1q_2^2q_3$	$q_2q_3,\ q_1q_2q_3,\ q_1q_2^2q_3$

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