BINARY X-RAYS OF PERMUTATIONS

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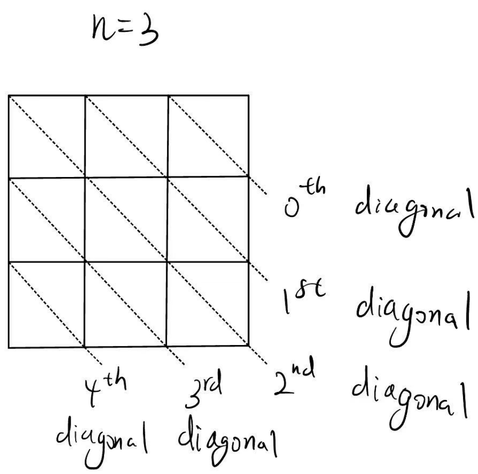
Abstract. The X-ray of a permutation matrix is the summation of elements on each diagonal. Each X-ray satisfies some obvious constraints, but the converse does not hold, i.e., not all integral vectors satisfying the contraints can be realized as the X-ray of a permutation matrix. It has been conjectured that if we add the assumption that the vector is binary, then the converse holds. In this paper, we prove that any binary vector satisfying the constraints can be realized as the X-ray of a doubly stochastic matrix.

1. Introduction

Fix an integer n and consider a permutation π on n elements, which corresponds to a permutation matrix P. Then the X-ray of P is defined to be the sequence $(x_0, x_1, \ldots, x_{2n-2})$, where x_i represents the number of 1's on the i^{th} diagonal, as illustrated by the following diagram:



Similarly, we can define the X-ray for an arbitrary matrix $A \in \mathrm{Mat}_{n\times n}(\mathbb{R})$ to be $(x_0, x_1, \ldots, x_{2n-2})$, where x_i represents the summation of elements of A on the i^{th} diagonal. More concretely,

$$x_k = \sum_{i-j=k-n+1} A_{ij}, 0 \leq k \leq 2n-2.$$

A natural question to ask is that when a sequence of integers $(x_0, x_1, \ldots, x_{2n-2}) \in \mathbb{Z}_{\geq 0}^{2n-1}$ can be realized as the X-ray of a permutation. There are some obvious constraints we need to impose on x_i's, which we will derive in this paper later. However, these constraints are not enough to fully determine

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whether an integer sequence is an X-ray, and even worse, the problem of determine whether a specific sequence is an X-ray is NP-complete by [3] and [4].

However, it was conjectured in [1] that if we impose the additional assumption that $(x_0, x_1, \ldots, x_{2n-2})$ is binary, i.e., $x_i \in \{0, 1\}$, then these constraints are sufficient. In other words, every binary x vector satisfying these constraints can be realized as the X-ray of a permutation matrix. In this paper, we prove a weaker statement: every binary x vector satisfying these constraints can be realized as the X-ray of a doubly stochastic matrix, i.e., a non-negative real matrix having each row and each column sum up to 1.

Here is an overview of our paper. In Section 2, we introduce the s sequence, which is the same as the x sequence but in another perspective. We also provide some obvious constraints on the s sequence. In Section 3, we find the constraints on the x sequence which is equivalent to the constraints on the s sequence, and we will use them as the constraints we want to impose on an x vector. In Section 4, we give more constraints on the x sequence, and show that for binary x vectors, the constraints in Section 3 implies all these constraints. In Section 5, we use the tools we have developed so far to prove the final statement: all x vector that satisfies the specific constraints in Section 3 can be realized as the X-ray of a doubly stochastic matrix.

2. The s sequence

For $a, b \in \mathbb{Z}$, we use $[\![a, b]\!]$ to denote the set $\{m \in \mathbb{Z} : a \leq m \leq b\}$. We use $\mathfrak{S}_n$ to denote the permutation group on n elements. Another way to represent an X-ray is to use the s sequence. Let $Y = \{y \in \mathbb{Z}^n : \{y_1, y_2, \ldots, y_n\} = \{1, 2, \ldots, n\}\}$ with a bijection $\mathfrak{S}_n \xrightarrow{\sim} Y, \pi \mapsto (\pi(1), \pi(2), \ldots, \pi(n))$, then we wonder which $s \in \mathbb{Z}^n$ lies in the Minkowski difference $Y - Y$. There is an action of $\mathfrak{S}_n$ on $\mathbb{Z}^n$ by permuting the entries, and for $\sigma \in \mathfrak{S}_n$ we have commutative diagram

$$\begin{array}{ccc} Y \times Y & \xrightarrow{-} & \mathbb{Z}^n \\ {\scriptstyle \sigma\times\sigma}\downarrow & & \downarrow{\scriptstyle \sigma} \\ Y \times Y & \xrightarrow{-} & \mathbb{Z}^n \end{array}$$

This diagram shows that $s \in \mathbb{Z}^n$ lies in $Y - Y$ if and only if $\sigma(s)$ lies in $Y - Y$, so we can talk about whether an element $\bar{s} \in \mathbb{Z}^n/\mathfrak{S}_n$ lies in $Y - Y$ or not. Also, this diagram tells us that if $s \in Y - Y$, then $\sigma(s) = (1, 2, \ldots, n) - (\pi(1), \ldots, \pi(n))$ for some $\sigma, \pi \in \mathfrak{S}_n$.

Thus, from an s sequence $\bar{s} \in (Y - Y)/\mathfrak{S}_n$ we can get an x sequence $(x_0, \ldots, x_{2n-2}) \in \mathbb{Z}_{\geq 0}^{2n-1}$, where $x_i = \#\{j \in [\![1, n]\!] : s_j = i - n + 1\}$, which is the X-ray of the above π. Also, from an x sequence $(x_0, \ldots, x_{2n-2}) \in \mathbb{Z}_{\geq 0}^{2n-1}$ (that is the X-ray of $\pi \in \mathfrak{S}_n$) we can get an s sequence $\bar{s} \in \mathbb{Z}^n/\mathfrak{S}_n$, where $i \in [\![-n + 1, n - 1]\!]$ appears x_{i+n-1} times in s, which lies in $(Y - Y)/\mathfrak{S}_n$ because it is just the image of $(1, 2, \ldots, n) - (\pi(1), \ldots, \pi(n))$ under map $Y - Y \to (Y - Y)/\mathfrak{S}_n$. Moreover, these two maps are inverse to each other.

As a result, talking about whether $x \in \mathbb{Z}_{\geq 0}^{2n-1}$ is an X-ray is the same as talking about whether the corresponding $\bar{s} \in \mathbb{Z}^n/\mathfrak{S}_n$ lies in $(Y - Y)/\mathfrak{S}_n$. For simplicity, when we write an s sequence $(s_1, \ldots, s_n)$ we will omit $\bar{s}$ and assume that the sequence is up to permutation, which allows us to assume $s_1 \leq s_2 \leq \cdots \leq s_n$. Then a binary x vector corresponds to an s sequence with $s_1 < s_2 < \cdots < s_n$.

Now we have some obvious constraints on $s \in Y - Y$. Let $s = (s_1, \dots, s_n) = (\sigma(1), \dots, \sigma(n)) - (\pi(1), \dots, \pi(n))$ with $s_1 < \cdots < s_n$, then

$$\begin{aligned}
s_n &= \sigma(n) - \pi(n) \le n - 1\\
s_n + s_{n-1} &= \sigma(n) + \sigma(n-1) - \pi(n) - \pi(n-1)\\
&\le n + (n-1) - 1 - 2\\
&= (n-2) + ((n-1) - 1)\\
&= 2(n-2)\\
s_n + s_{n-1} + s_{n-2} &= \sigma(n) + \sigma(n-1) + \sigma(n-2) - \pi(n) - \pi(n-1) - \pi(n-2)\\
&\le n + (n-1) + (n-2) - 1 - 2 - 3\\
&= (n-3) + ((n-1) - 2) + ((n-2) - 1)\\
&= 3(n-3)\\
&\vdots
\end{aligned}$$

Thus, $s_n + s_{n-1} + \cdots + s_{n-k+1} \le k(n-k)$. By similar reasoning,

$$\begin{aligned}
s_1 &= \sigma(1) - \pi(1)\\
&\ge -(n-1)\\
s_1 + s_2 &= \sigma(1) + \sigma(2) - \pi(1) - \pi(2)\\
&\ge 1 + 2 - n - (n-1)\\
&= (1 - (n-1)) + (2 - n)\\
&= -2(n-2)\\
&\vdots
\end{aligned}$$

Thus, $s_1 + s_1 + \cdots + s_k \ge -k(n-k)$. We summarize the s inequalities as

$$\begin{aligned}
\text{Forward } s \text{ inequalities: } & s_n + s_{n-1} + \cdots + s_{n-k+1} \le k(n-k), 1 \le k \le n\\
\text{Backward } s \text{ inequalities: } & s_1 + s_2 + \cdots + s_k \ge -k(n-k), 1 \le k \le n.
\end{aligned}$$

Notice that if we plug in $k = n$ into the forward inequalities, we get $s_1 + \cdots + s_n \le 0$, and if we plug in $k = n$ into the backward inequalities, we get $s_1 + \cdots + s_n \ge 0$, so $s_1 + \cdots + s_n = 0$, which is true because $s_1 + \cdots + s_n = \sigma(1) + \cdots + \sigma(n) - (\pi(1) + \cdots + \pi(n)) = n(n+1)/2 - n(n+1)/2 = 0$.

3. Equivalence of constraints

Recall we have the s inequalities: $s_n \le n-1, s_n + s_{n-1} \le 2(n-2), s_n + s_{n-1} + s_{n-2} \le 3(n-3), \dots, s_1 \ge -(n-1), s_1 + s_2 \ge -2(n-2), \dots$ We want to transform these inequalities into inequalities in $x_0, x_1, \dots, x_{2n-2}$. Note that if we pick x_{2n-2} many $(n-1)$'s, x_{2n-3} many $(n-2)$'s, up to x_{2n-1-k} many $(n-k)$'s, then the forward s inequality becomes

$$\begin{aligned}
&(n-1)x_{2n-2} + \cdots + (n-k)x_{2n-1-k} \le (x_{2n-2} + \cdots + x_{2n-1-k})(n - x_{2n-2} - \cdots - x_{2n-1-k})\\
\Leftrightarrow &(x_{2n-2} + \cdots + x_{2n-1-k})^2 \le x_{2n-2} + 2x_{2n-3} + \cdots + kx_{2n-1-k}.
\end{aligned}$$

Now we claim that these inequalities are equivalent to the forward s inequalities: $s_n + \cdots + s_{n-i} \ge i(n-i)$. Our previous discussions show that from the forward s inequalities we can get (the above)

x inequalities. Now from the above x inequalities, in order to get the s inequalities, we only need to prove the above inequality where x_{2n-1-k} is replaced by $0 \le x \le x_{2n-1-k}$. Let

$$g(x) = (x_{2n-2} + \cdots + x_{2n-k} + x)^2 - (x_{2n-2} + 2x_{2n-3} + \cdots + (k-1)x_{2n-k} + kx).$$

Then we have $g(0) \le 0, g(x_{2n-1-k}) \le 0$, so because g is a quadratic polynomial in x and has leading coefficient 1 we see $g(x) \le 0$ for all $0 \le x \le x_{2n-1-k}$, as desired.

Also, there exists n many s_i is equivalent to $x_0 + x_1 + \cdots + x_{2n-2} = n$. Thus, we get the equivalent x constraints:

$$\begin{aligned} x_i \in \mathbb{Z}_{\ge 0}, 0 \le i \le 2n-2 \\ x_0 + x_1 + \cdots + x_{2n-2} = n \\ (x_{2n-2} + x_{2n-3} + \cdots + x_{2n-1-k})^2 \le x_{2n-2} + 2x_{2n-3} + \cdots + kx_{2n-1-k}, 1 \le k \le 2n-1 \\ (x_0 + x_1 + \cdots + x_{k-1})^2 \le x_0 + 2x_1 + \cdots + kx_{k-1}, 1 \le k \le 2n-1. \end{aligned}$$

We now claim that these constraints are equivalent to the following alternative x constraints:

$$\begin{aligned} x_i \in \mathbb{Z}_{\ge 0}, -(n-1) \le i \le n-1 \\ x_0 + x_1 + \cdots + x_{2n-2} = n \\ 2kx_{2n-2} + (2k-1)x_{2n-3} + \cdots + x_{2n-1-2k} \le k(k+1), 1 \le k \le n \\ 2kx_0 + (2k-1)x_1 + \cdots + x_{2k-1} \le k(k+1), 1 \le k \le n, \end{aligned}$$

where we agree that $x_{2n-1} = x_{-1} = 0$.

We first show that the original x constraints imply the alternative x constraints. Because the original x constraints is equivalent to the s constraints, it suffices to show that the s constraints imply the alternative x constraints. For a s sequence $(s_1, s_2, \ldots, s_n)$, a *shrinking operation* is to specify $i, j \in [\![1, n]\!]$ such that $s_i + 2 \le s_j$, then replace s_i by $s_i + 1$ and replace s_j by $s_j - 1$.

We claim that every s sequence can be obtained by the "root" s sequence $(-n+1, -n+3, \ldots, n-3, n-1)$ via shrinking operations (up to permutation). To prove this claim, consider a tournament among n players, where each game has three outcomes: win, draw, or lose. A player get 1 score from a win, 0 score from a draw, and -1 score from a lose. Then the s sequence corresponds to a score sequence in this tournament by [2]. Now pick two players p_i, p_j such that $s_i \le s_j$, i.e., the score of p_i is less or equal to the score of p_j. If p_i wins p_j, then after changing the result to a draw, we would have $s_i^{\text{new}} = s_i - 1$ and $s_j^{\text{new}} = s_j + 1$, which corresponds to an inverse shrinking operation. By the same argument, if p_i draws with p_j, then after changing the result to lose, we perform an inverse shrinking operation.

Now we reorder the players such that $s_1 \le s_2 \le \cdots \le s_n$, and changing the results of the tournament such that p_i loses p_j if $i < j$, then the score sequence becomes $(-n+1, -n+3, \ldots, n-1)$. Note that each time we only need to change one match from win to draw, or from draw to lose, so we see after some inverse shrinking operations we can get $(-n+1, -n+3, \ldots, n-1)$. Therefore, the original s sequence can be obtained by the root s sequence via some shrinking operations.

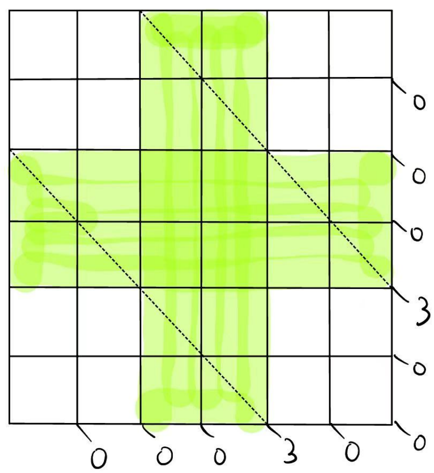
Finally, note that the root s sequence satisfies the alternative x constraints, and if a s sequence satisfies the alternative x constraints, then after a shrinking operation it still satisfies the alternative x constraints. Thus, the s constraints implies the alternative x constraints.

Now we show the other direction. Assume that the alternative x constraints hold. Fix a k and consider $t = x_{2n-2} + \cdots + x_{2n-1-k}$. Then

$$\begin{aligned}
&2tx_{2n-2} + \cdots + (2t+1-k)x_{2n-1-k} \leq 2tx_{2n-2} + \cdots + x_{2n-1-2t} \leq t(t+1) \\
\Leftrightarrow\, &2t(x_{2n-2} + \cdots x_{2n-1-k}) + (x_{2n-1} + \cdots + x_{2n-1-k}) - x_{2n-2} - 2x_{2n-3} - \cdots - kx_{2n-1-k} \leq t(t+1) \\
\Leftrightarrow\, &(x_{2n-2} + \cdots + x_{2n-1-k})^2 \leq x_{2n-2} + 2x_{2n-3} + \cdots + kx_{2n-1-k},
\end{aligned}$$

which is 1-side of the original x constraints. The other side of the constraints is completely symmetric.

Now let us illustrate the fact that these constraints cannot fully determine whether $x \in \mathbb{Z}_{\geq 0}^{2n-1}$ can be realized as the X-ray of a permutation matrix. Let $n = 6, x_3 = x_7 = 3$, and all other $x_i = 0$, as depicted below:



One can verify that this x sequence satisfies the alternative x constraints. If this x sequence corresponds to a permutation matrix, then since the green region can be covered by 2 rows and 2 columns we see there can only be at most four 1's in the green region. However, each of the two diagonals depicted in the diagram contains three 1's, so we see that the green region contains at least six 1's, which gives a contradiction.

4. Inequality Induced by Weights

4.1. **Inequalities.** For simplicity, in this section we will work with board with 1 size larger, which is the same as the original problem if we replace n by $n-1$. If an x sequence $(x_0, \ldots, x_{2n}) \in \mathbb{R}_{\geq 0}^{2n+1}$ is the X-ray of some doubly stochastic matrix A (the indices will run from 0 to n), then we have

$$\begin{aligned}
\text{Non-negative: } & A_{ij} \geq 0, 0 \leq i, j \leq n \\
\text{Equation } R_i\text{: } & \sum_j A_{ij} = 1, 0 \leq i \leq n \\
\text{Equation } C_j\text{: } & \sum_i A_{ij} = 1, 0 \leq j \leq n \\
\text{Equation } D_k\text{: } & \sum_{i-j=k-n} A_{ij} = x_k, 0 \leq k \leq 2n.
\end{aligned}$$

We now pick $(w_0, \dots, w_n) \in \mathbb{R}_{\geq 0}^{n+1}$. For $k \in [\![0, 2n]\!]$, let

$$b_k = \min_{\substack{0 \leq i,j \leq n \\ i+j=k}} (w_i + w_j),$$

then after adding w_i copies of equation R_i, w_i copies of equation C_i, and minus b_k copies of equation D_k, we get $\sum_{i,j} \beta_{ij} A_{ij} = 2\sum_{i=0}^{n} w_i - \sum_{k=0}^{2n} b_k$, where $\beta_{ij} \geq 0$ for all i, j. Thus, we see we have

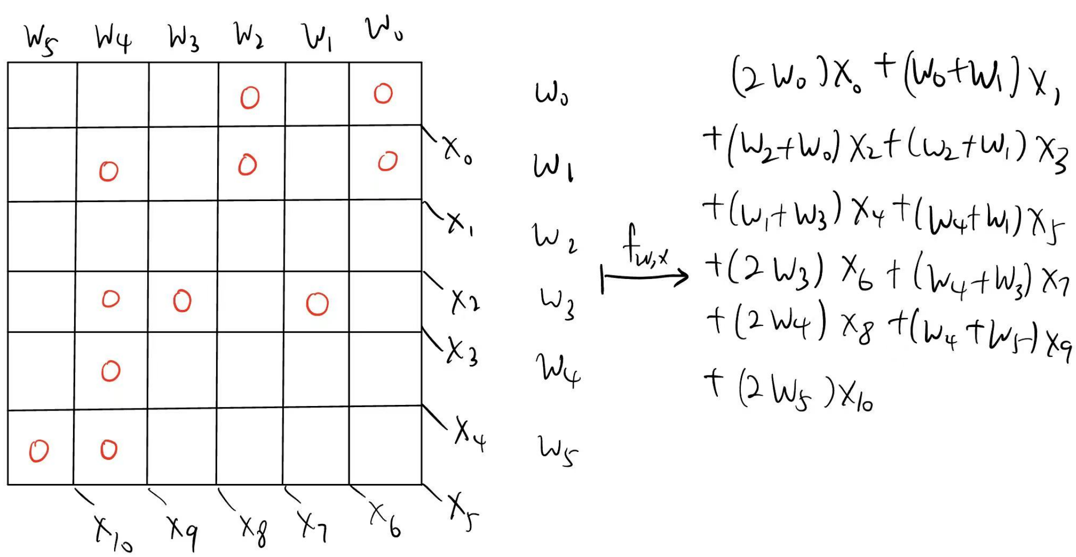
$$\text{Inequality } I(w) : \sum_{k=0}^{2n} \min_{\substack{0 \leq i,j \leq n \\ i+j=k}} (w_i + w_j) \leq 2\sum_{i=0}^{n} w_i.$$

Now let us see some specific $I(w)$'s:

$$I(k, k-1, \dots, 1, 0, 0, \dots, 0) : 2kx_0 + (2k-1)x_1 + \cdots + x_{2k-1} \leq k(k+1), 0 \leq k \leq n+1$$
$$I(0, \dots, 0, 1, 2, \dots, k) : 2kx_{2n} + (2k-1)x_{2n-1} + \cdots + x_{2n-2k+1}, 0 \leq k \leq n+1$$
$$I(1, 1, \dots, 1) : x_0 + x_1 + \cdots + x_{2n} \leq n+1.$$

The first two inequalities corresponds to exactly the inequality in the alternative x constraints, while the third can be implied by the alternative x constraints. In this section, we will show that for $x \in [0,1]^{2n+1}$, if $I(w)$ holds for all above w's, then $I(w)$ holds for all $w \in \mathbb{R}_{\geq 0}^{n+1}$. In order to do this, let us set up some notations.

For each $w = (w_0, \dots, w_n) \in \mathbb{R}_{\geq 0}^{n+1}$ and $x = (x_0, \dots, x_{2n}) \in \mathbb{R}_{\geq 0}^{2n+1}$, we obtain a function $f_{w,x} : \mathcal{L}_n \to \mathbb{R}_{\geq 0}$, where $\mathcal{L}_n$ is the collection of layouts of stones on a $(n+1) \times (n+1)$ chessboard, such that each diagonal has exactly 1 stone. The function $f_{w,x}$ is given by the following diagram:



Let $\mathcal{W}_n = \{(1, 0, \dots, 0), (2, 1, 0, \dots, 0), \dots, (n, n-1, \dots, 1, 0)\} \subset \mathbb{R}_{\geq 0}^{n+1}$. There exists a map $\Sigma : \mathbb{R}_{\geq 0}^{n+1} \to \mathbb{R}_{\geq 0}, w \mapsto \sum_{i=0}^{n} w_i$. Then the inequality $I(w)$ becomes $\min_{\mathcal{L}_n} f_{w,x} \leq 2\sum w$.

4.2. **One-sided case.** In this section we are going to show that for any $x \in [0,1]^{2n+1}$, if $\min_{\mathcal{L}_n} f_{v,x} \leq 2\Sigma v$ for all $v \in \mathcal{W}_n$, then $\min_{\mathcal{L}_n} f_{w,x} \leq 2\Sigma w$ for all $w \in \mathbb{R}^n_{\geq 0} \times \{0\}$.

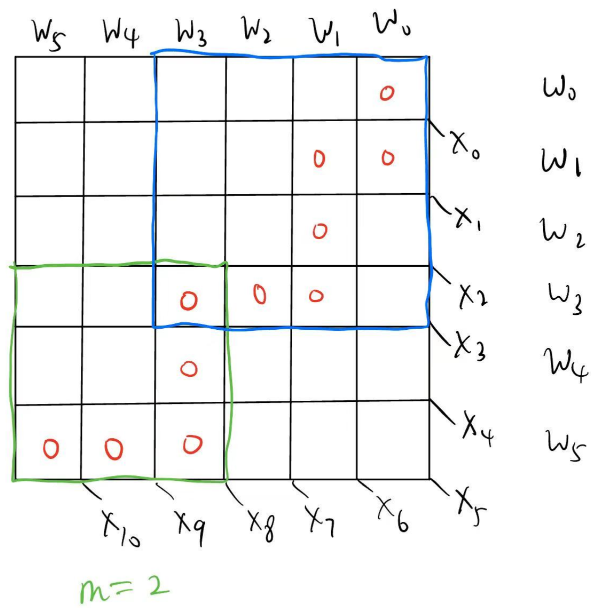
We use induction on n. For $n = 1$ this is trivial, since every element in $\mathbb{R}_{\geq 0} \times \{0\}$ is a scalar multiple of $(1,0)$, and the condition $\min f_{w,x} \leq 2\Sigma w$ is invariant under scalar multiplication. Thus, we assume $n \geq 2$.

Note that when $v = (n, n-1, \dots, 0) \in \mathcal{W}_n$, the condition $\min_{\mathcal{L}_n} f_{v,x} \leq 2\Sigma v$ becomes $2nx_0 + (2n-1)x_1 + \cdots + x_{2n-1} \leq n(n+1)$.

Pick $1 \leq m \leq n$ such that w_{n-m}/m is minimal. We now consider the new weight

$$w' = \left(w_0 - \frac{n}{m}w_{n-m}, w_1 - \frac{n-1}{m}w_{n-m}, \dots, w_{n-m-1} - \frac{m+1}{m}w_{n-m}, 0\right) \in \mathbb{R}^{n-m+1}_{\geq 0}.$$

Then from induction hypothesis there exists a layout $L_1 \in \mathcal{L}_{n-m}$ such that $f_{w',x'}(L_1) \leq 2\Sigma w'$, where $x' = (x_0, \dots, x_{2(n-m)})$. Now we pick $L_2 \in \mathcal{L}_m$ consisting of a horizontal line of stones in the bottom line and a vertical line of stones in the rightmost column. Then let $L = L_1 \cdot L_2$ be the "concatenating of layouts" illustrated as below (the blue one is L_1 and the green one is L_2).



Finally, we have

$$\begin{aligned}
f_{w,x}(L) &= f_{w',x'}(L_1) + \frac{w_{n-m}}{m}(2nx_0 + (2n-1)x_1 + \cdots + x_{2n-1}) \\
&\quad + \left(w_{n-m} + w_{n-m+1} - \frac{2m-1}{m}w_{n-m}\right)x_{2n-2m+1} \\
&\quad + \left(w_{n-m} + w_{n-m+2} - \frac{2m-2}{m}w_{n-m}\right)x_{2n-2m+2} \\
&\quad + \cdots \\
&\quad + \left(w_{n-m} + w_{n-1} - \frac{m+1}{m}w_{n-m}\right)x_{2n-m-1} \\
&\quad + \left(w_{n-m+1} - \frac{m-1}{m}w_{n-m}\right)x_{2n-m+1} \\
&\quad + \left(w_{n-m+2} - \frac{m-2}{m}w_{n-m}\right)x_{2n-m+2} \\
&\quad + \cdots \\
&\quad + \left(w_{n-1} - \frac{1}{m}w_{n-m}\right)x_{2n-1} \\
&\le 2\Sigma w' + n(n+1)\frac{w_{n-m}}{m} + 2\left(w_{n-m+1} + w_{n-m+2} + \cdots + w_{n-1} - \frac{1}{2}(m-1)m\frac{w_{n-m}}{m}\right) \\
&= 2(w_0 + w_1 + \cdots + w_{n-m}) - 2\frac{(n+m)(n-m+1)}{2}\frac{w_{n-m}}{m} + n(n+1)\frac{w_{n-m}}{m} \\
&\quad + 2(w_{n-m+1} + w_{n-m+2} + \cdots + w_{n-1}) - (m-1)w_{n-m} \\
&= 2\Sigma w + \frac{w_{n-m}}{m}(n(n+1) - (n+m)(n-m+1) - m(m-1)) \\
&= 2\Sigma w,
\end{aligned}$$

as desired.

4.3. **Two sided case.** Now let

$$\widetilde{\mathcal{W}}_n = \mathcal{W}_n \cup \{(0,\ldots,0,1),(0,\ldots,0,1,2),\ldots,(0,1,2,\ldots,n)\} \cup \{(1,1,\ldots,1)\} \subset \mathbb{R}^{n+1}_{\ge 0}.$$

We will show that for any $x \in [0,1]^{2n+1}$, if $\min_{\mathcal{L}_n} f_{v,x} \le 2\Sigma v$ for all $v \in \widetilde{\mathcal{W}}_n$, then $\min_{\mathcal{L}_n} f_{w,x} \le 2\Sigma w$ for all $w \in \mathbb{R}^{n+1}_{\ge 0}$.

We first show this when some $w_m = 0$. Then by the one-sided case there exists $L_1 \in \mathcal{L}_m$ and $L_2 \in \mathcal{L}_{n-m}$ such that $f_{w_{0\to m},x_{0\to m}}(L_1) \le 2\Sigma w_{0\to m}$ and $f_{w_{n\to(n-m)},x_{n\to(n-m)}}(L_2) \le 2\Sigma w_{n\to(n-m)}$. Now let $L = L_1 \cdot L_2$, then

$$\begin{aligned}
f_{w,x}(L) &= f_{w_{0\to m},x_{0\to m}}(L_1) + f_{w_{n\to(n-m)},x_{n\to(n-m)}}(L_2) \\
&\le 2(w_0 + w_1 + \cdots + w_m) + 2(w_m + w_{m+1} + \cdots + w_n) \\
&= 2\Sigma w,
\end{aligned}$$

as desired.

Note that when $v = (1,1,\ldots,1)$, the condition $\min_{\mathcal{L}_n} f_{v,x} \le 2\Sigma v$ becomes $x_0 + x_1 + \cdots + x_{2n} \le n+1$. Now for the general case, we pick the minimal w_m and consider $w'_i = w_i - w_m$, then by previous discussion there exists $L \in \mathcal{L}_n$ such that $f_{w',x}(L) \le 2\Sigma w'$. Finally,

$$\begin{aligned} f_{w,x}(L) &= f_{w',x}(L) + 2w_m(x_0 + x_1 + \cdots + x_{2n}) \\ &\le 2\Sigma w' + 2(n+1)w_m \\ &= 2\Sigma w, \end{aligned}$$

as desired.

5. Existence of Non-negative Real Solution

Now we are going to show that when all $I(w)$'s are satisfied, then there exists a non-negative real solution. Let $V = \mathbb{R}^{n\times n}$ be the collection of possible ways to fill in real numbers into a n by n board. We can view each $A \in V$ as a real matrix. For convenience we will write A_{ij} where i, j runs from 0 to $n-1$.

Theorem 5.1. *Let $S \subset [\![0, 2n-2]\!]$ be the collection of diagonal indices with cardinality n and $I(w)$ be the corresponding weight inequalities. If $I(w)$ holds for all $w \in \mathbb{R}^n_{\ge 0}$, then there exists $A \in V$ such that*

1. $A_{ij} \ge 0$ *for all* $i, j \in [\![0, n-1]\!]$.
2. $\sum_j A_{ij} = 1$ *for each* $i \in [\![0, n-1]\!]$.
3. $\sum_i A_{ij} = 1$ *for each* $j \in [\![0, n-1]\!]$.
4. $d_k(A) = 1$ *for each* $k \in S$, *and* 0 *otherwise, where* $d_k : V \to \mathbb{R}$ *represents the summation through the* k*'s diagonal.*

Proof. Let $P \subset V$ be the polytope consists of $A \in V$ such that:

1. $A_{ij} \ge 0$ for all $i, j \in [\![0, n-1]\!]$.
2. $\sum_j A_{ij} \le 1$ for each $i \in [\![0, n-1]\!]$.
3. $\sum_i A_{ij} \le 1$ for each $j \in [\![0, n-1]\!]$.

Let $H \subset V$ be the plane consists of A such that $d_k(A) = 1$ for each $k \in S$ and 0 otherwise. We claim that $P \cap H \ne \varnothing$. Assume the contrary, $P \cap H = \varnothing$. Then by the hyperplane separation theorem, there exists a hyperplane separating P and H, i.e., there exists a linear functional $f : V \to \mathbb{R}$ such that $f(P)$ and $f(H)$ are disjoint.

Note that since H is a plane, in order for $f(H) \ne \mathbb{R}$, we need $f(H)$ consists of one point. After possibly scaling f we can assume $f(H) = \{1\}$. Then by the definition of H we see $f = \sum_{k=0}^{2n-2} \alpha_k d_k$ for some $\alpha_k \in \mathbb{R}$ such that $\sum_{k\in S} \alpha_k = 1$. Now $0 \in P$ and $f(0) = 0$, so since P is compact there exists $\epsilon > 0$ such that $f(p) \le 1 - \epsilon$ for all $p \in P$.

We now have the linear programming problem:

$$\begin{aligned} \text{Inequality } R_i\text{:} \quad & \sum_j A_{ij} \le 1, 0 \le i \le n-1 \\ \text{Inequality } C_j\text{:} \quad & \sum_i A_{ij} \le 1, 0 \le j \le n-1 \\ \text{Inequality } E_{ij}\text{:} \quad & -A_{ij} \le 0, 0 \le i, j \le n-1 \\ \text{They imply inequality } D\text{:} \quad & \sum_{i,j=0}^{n-1} \alpha_{i-j+n-1} A_{ij} \le 1 - \epsilon. \end{aligned}$$

Then from LP duality, there exist $r_i \in \mathbb{R}_{\ge 0}, c_i \in \mathbb{R}_{\ge 0}, e_{ij} \in \mathbb{R}_{\ge 0}$ such that if we add r_i multiple of inequality R_i, c_i multiple of inequality C_i, and e_{ij} multiple of inequality E_{ij} together, the resulting inequality would have left hand side being exactly that of inequality D, and right hand side being $\leq 1-\epsilon$.

Thus, if we let $c'_j = c_{n-1-j}$, then $r_i + c'_j \geq \alpha_{i+j}$ for all $i, j \in [\![0, n-1]\!]$, so

$$\sum_i r_i + \sum_j c'_j \leq 1-\epsilon$$

$$\sum_{k\in S} \min_{i+j=k}(r_i + c'_j) \geq \sum_{k\in S} \alpha_k = 1.$$

Let $w_i = (r_i + c'_i)/2$, then we claim that for each k, we have $\min_{i+j=k}(w_i+w_j) \geq \min_{i+j=k}(r_i+c'_j)$. To see this, just note that if w_i+w_j achieve minimal value, then from $(r_i+c'_j)+(r_j+c'_i) = 2(w_i+w_j)$ we see one of $r_i + c'_j$ or $r_j + c'_i$ must be at most $w_i + w_j$, as desired. Therefore we get

$$2\sum_i w_i = \sum_i r_i + \sum_j c'_j \leq 1-\epsilon$$

$$\sum_{k\in S} \min_{i+j=k}(w_i + w_j) \geq \sum_{k\in S} \min_{i+j=k}(r_i + c'_j) \geq 1,$$

which implies $I(w)$ does not hold, a contradiction.

As a result, there exists $A_0 \in P \cap H$. Now sum A_0 through all diagonals we see the summation of all entries of A_0 is n, so the summation of each row and each column must be 1 in order for the summation of all entries to be n. This shows that A_0 is the desired element. □

As a result, we get

Corollary 5.2. *Suppose* $(x_0, x_1, \ldots, x_{2n-2}) \in \mathbb{Z}_{\geq 0}^{2n-1}$ *satisfies:*

$$\begin{aligned} x_i \in \{0,1\}, 0 \leq i \leq 2n-2 \\ x_0 + x_1 + \cdots + x_{2n-2} = n \\ 2kx_{2n-2} + (2k-1)x_{2n-3} + \cdots + x_{2n-1-2k} \leq k(k+1), 1 \leq k \leq n \\ 2kx_0 + (2k-1)x_1 + \cdots + x_{2k-1} \leq k(k+1), 1 \leq k \leq n, \end{aligned}$$

where we agree $x_{-1} = x_{2n-1} = 0$. *Then* $(x_0, x_1, \ldots, x_{2n-2})$ *can be realized as the X-ray of a doubly stochastic matrix.*

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