Representations of semidirect products of finite groups with lattices

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1 Introduction

Semidirect products of a group acting on a lattice are of importance in representation theory. Examples include the extended affine Weyl group, the semidirect product of the affine Weyl group of a root system and the lattice it acts upon. Deformations of the group algebra of the extended affine Weyl group produce affine Hecke algebras which are of significance to studying the representation theory of the Lie algebra behind the root system.

In this paper we investigate a problem posed by Prof. R. Bezrukavnikov, namely the question of when do algebras of such semidirect products possess the density of characters property – the property of the virtual characters of the algebra being dense in the dual of the cocenter of the algebra – or analogous properties. Criteria for this property are well-known for finite-dimensional algebras; one set of criteria is that of being Noetherian and of finite homological dimension. However, not many criteria are known for infinite-dimensional group algebras. This paper will prove similar properties to the density of characters for this kind of group algebra; namely, the injectivity of a trace-induced map and the density of its image in a subset of the dual of the cocenter, which we call the dual of the reduced cocenter.

2 Notation

Elements of an abelian lattice $\mathbb{Z}^n$ are composed multiplicatively like tt', or as an expression inside square brackets like $[t + t']$ or $[g(t) - t']$ for clarity. If a group G acts on a group H, the semidirect product $H \rtimes G$ consists of elements hg, where $h \in H$ and $g \in G$, with group law

$$(hg)(h'g') = [h + g(h')]gg'.$$

For an algebra A over $\mathbb{C}$, let $\mathrm{Gr}(A)$ be the complexification (tensor with $\mathbb{C}$ as a $\mathbb{Z}$-module) of the Grothendieck group of A. Let $\overline{A} = A/[A, A]$ be the cocenter of A. For a finite-dimensional A-module M, let $\mathrm{tr}_M(a)$ denote the trace by which $a \in A$ or $a \in \overline{A}$ acts on M. Let the trace-induced map

$$\tilde{\text{tr}} : \text{Gr}(A) \to \overline{A}^*$$

be defined by

$$\tilde{\text{tr}}(M)(a) = \text{tr}_M(a).$$

Density of characters in the strict sense refers to $\tilde{\text{tr}}$ being an isomorphism.

The notation e_j denotes the jth elementary basis vector; the vector space depends on the context. The notation e_j^* denotes the dual of e_j under the dot product.

If S is a set, we may use the notation $\mathbb{C}[S]$ to denote the vector space with formal basis $[s]$ for $s \in S$.

The power $\mathcal{O}^n$ of a conjugacy class $\mathcal{O}$ of a group G is the set of elements $\{x^n \mid x \in \mathcal{O}\}$. It is itself a conjugacy class. The order of a conjugacy class $\mathcal{O}$ is the minimum positive n such that $\mathcal{O}^n = 1$, where 1 is the conjugacy class containing only the identity element. It is the order of any element of the conjugacy class.

3 Semidirect products

Let G be a finite group acting linearly on an abelian lattice $\mathbb{Z}^n$. Each element of G must then have determinant ± 1 on $\mathbb{Z}^n$. The group G acts linearly on the set of additive characters $\chi \in \text{Hom}(\mathbb{Z}^n, \mathbb{C}) \cong \mathbb{C}^n$ of $\mathbb{Z}^n$ as follows:

$$(g\chi)(t) = \chi(g^{-1}(t)),$$

for $t \in \mathbb{Z}^n$.

Let $\mathbb{C}|_\chi$ denote the 1-dimensional $\mathbb{C}[\mathbb{Z}^n]$-module with $\mathbb{Z}^n$-action

$$ta = (\exp \chi(t))a, \quad t \in \mathbb{Z}^n, a \in \mathbb{C}|_\chi.$$

The isomorphism relations between these modules induce equivalence classes $\tilde{\chi}$ between additive characters χ which respects addition of characters; in particular, $\mathbb{C}|_\chi \cong \mathbb{C}|_\xi$ iff $\chi - \xi \in 2\pi i\mathbb{Z}^n$. Let G_χ denote the stabilizer of the equivalence class $\tilde{\chi}$ under G, i.e. those g for which $g(\chi) - \chi \in 2\pi i\mathbb{Z}^n$. For $g \in G$, let the fixed subspace $(\mathbb{C}^n)^g$ of g be the set of characters $\chi \in \mathbb{C}^n$ such that g stabilizes $\tilde{\chi}$.

Claim 1. *The fixed set $(\mathbb{C}^n)^g$ is the direct sum, as $\mathbb{Z}$-modules, of a $\mathbb{C}$-vector space and a $\mathbb{Z}$-lattice, the latter spanned by finitely many elements of $2\pi i\mathbb{Z}^n$.*

Proof. Let U be the kernel of $g - 1$ over $\mathbb{C}^n$. Then U is a $\mathbb{C}$-vector space and $U \subset (\mathbb{C}^n)^g$.

Note that $(\mathbb{C}^n)^g$ is a $\mathbb{Z}$-module, because if

$$g(\chi) = \chi + 2\pi i\alpha, g(\xi) = \xi + 2\pi i\beta,$$

with $\alpha, \beta \in \mathbb{Z}^n$, then

$$g(\chi + \xi) = \chi + \xi + 2\pi i(\alpha + \beta).$$

Let $T = (\mathbb{C}^n)^g/U$, the quotient as $\mathbb{Z}$-modules. Because for any $\alpha \in \mathbb{Z}^n$, we can say

$$g(\alpha) - \alpha \in \mathbb{Z}^n$$

because g has integer matrix coefficients over the standard basis of $\mathbb{C}^n$ (g^{-1} has integer matrix coefficients over the standard basis of $\mathbb{Z}^n$), so

$$g(2\pi i\alpha) - 2\pi i\alpha \in 2\pi i\mathbb{Z}^n,$$

and thus $2\pi i\mathbb{Z}^n \subset (\mathbb{C}^n)^g$. Furthermore, if $k \in \mathbb{Z}^n$, then the set of $\chi \in \mathbb{C}^n$ such that $(g-1)\chi = 2\pi ik$ is the coset of U by $2\pi ij$ with $j \in \mathbb{Z}^n$ such that $(g-1)j = k$. Any such j equals $gj - k$ which is in $\mathbb{Z}^n$. Thus, as $T \subset 2\pi i\mathbb{Z}^n$, and $2\pi i\mathbb{Z}^n \subset T \oplus U$, T is spanned by finitely many elements of $2\pi i\mathbb{Z}^n$. □

As every $g \in G$ takes $2\pi i\mathbb{Z}^n$ to itself, we can make an action of G on equivalence classes $\tilde{\chi}$ of $\chi \in \mathbb{C}^n$.

Let $O(\chi)$ be the orbit of χ.

Let $\mathbb{C}'|_\chi$ be the representation of $\mathbb{Z}^n \rtimes G_\chi$ such that

$$(tg)(a) = t(a) = (\exp\chi(t))a, \quad t \in \mathbb{Z}^n, g \in G_\chi, a \in \mathbb{C}'|_\chi.$$

A standard result says that all irreducible representations of $\mathbb{Z}^n \rtimes G$ arise from irreducible representations of $\mathbb{Z}^n$ as follows:

$$V_\chi = \operatorname{Ind}_{\mathbb{Z}^n \rtimes G_\chi}^{\mathbb{Z}^n \rtimes G} \mathbb{C}'|_\chi.$$

In module notation, this is

$$V_\chi = \mathbb{C}[\mathbb{Z}^n \rtimes G] \otimes_{\mathbb{C}[\mathbb{Z}^n \rtimes G_\chi]} \mathbb{C}'|_\chi.$$

What is a basis for this representation? The action of any element of $\mathbb{Z}^n \rtimes G_\chi$ can be pulled out of the right-hand side of the tensor product exchanging for a scalar factor, and then applied to the left-hand side of the tensor product. Thus, the basis can be identified with right cosets

$$(\mathbb{Z}^n \rtimes G_\chi)\backslash(\mathbb{Z}^n \rtimes G).$$

However, because for any $g \in G$, $h \in G_\chi$, $t, t' \in \mathbb{Z}^n$, we can say

$$\begin{aligned} t'htg &= t'[h(t)]hg \\ &= [t' + h(t)]hg, \end{aligned}$$

right cosets of $\mathbb{Z}^n \rtimes G_\chi$ in $\mathbb{Z}^n \rtimes G$ can be identified with right cosets of G_χ in G. Thus, the basis of V_χ can be identified with

$G_\chi \backslash G.$

Here, $g \in G$ acts on the above by taking $G_\chi h$ to $G_\chi gh$, and $t \in \mathbb{Z}^n$ acts on the above by taking $G_\chi h$ to $(\exp \chi(t)) G_\chi h$.

3.1 Traces of irreducible representations

With what trace does $tg \in \mathbb{Z}^n \rtimes G$ act on V_χ?

We use the Fröbenius formula for traces of induced representations:

$$\begin{aligned}
\mathrm{tr}_{V_\chi}(tg) &= \sum_{\substack{G_\chi h \in G_\chi \backslash G, \\ htgh^{-1} \in \mathbb{Z}^n \rtimes G_\chi}} \exp \chi(htgh^{-1}) \\
&= \sum_{\substack{G_\chi h \in G_\chi \backslash G, \\ hgh^{-1} \in G_\chi}} \exp \chi(h(t)) \\
&= \sum_{\substack{G_\chi h \in G_\chi \backslash G, \\ g \in G_{h^{-1}(\chi)}}} \exp \chi(h(t)) \\
&= \sum_{\substack{G_\chi h \in G_\chi \backslash G, \\ h^{-1}(\chi) \in (\mathbb{C}^n)^g}} \exp h^{-1}(\chi)(t) \\
&= \sum_{\xi \in (\mathbb{C}^n)^g \cap O(\chi)} \exp \xi(t).
\end{aligned}$$

As we can see, every stabilized character ξ of g that is in the orbit of χ contributes $\exp \xi(t)$ to the trace. Also note that χ which are in the same orbit produce isomorphic V_χ.

There are indeed a set of discrete categories that we can put the representations in, according to which elements of G stabilize χ. If the subgroup H stabilizes χ, then for any $k \in G$, the subgroup kHk^{-1} stabilizes $k\chi$; as representations are the same if χ is replaced with something in its orbit, we consider only equivalence classes of subgroups of G under conjugation.

Definition. *The representation class $C(\chi)$ of χ is the equivalence class of the stabilizer of χ under conjugation.*

3.2 Conjugacy classes

Let $tg, t'h \in \mathbb{Z}^n \rtimes G$. Then

$$\begin{aligned}
(t'h)tg(t'h)^{-1} &= t'htg[-h^{-1}(t')]h^{-1} \\
&= t'h[t - gh^{-1}(t')]gh^{-1} \\
&= [h(t) + t' - hgh^{-1}(t')]hgh^{-1}.
\end{aligned}$$

Letting $t'h$ be arbitrary, we see that any conjugacy class is generated from a single element by the two operations

$$tg \mapsto [t+a]g, \quad a \in \operatorname{im}(g-1),$$
$$tg \mapsto [h(t)]hgh^{-1}, \quad h \in G.$$

Claim 2. *Assuming* ***every element of*** G ***has finite order****, the space* $\mathbb{Z}^n$ *is the inner direct sum of the lattice fixed by* g *and* $\operatorname{im}(g-1)$ *as* $\mathbb{Z}$*-modules.*

Proof. Let the order of g be N. If $gv = v$ and $(g-1)u = v$, then $v = (g^2 - g)u$. Indeed, $v = (g^{k+1} - g^k)u$ for any integer k. By adding all these relations over all $k = 0, \dots, n-1$, and recalling that $g^N = 1$, we get $Nv = 0$, so $v = 0$. Thus, the intersection of the lattice fixed by g and $\operatorname{im}(g-1)$ is 0.

But, over $\mathbb{Q}^n$, the dimensions of $\ker_{\mathbb{Q}^n}(g-1)$ and $\operatorname{im}_{\mathbb{Q}^n}(g-1)$ sum to n. Because they have a trivial intersection, we thus can say $\mathbb{Q}^n = \ker_{\mathbb{Q}^n}(g-1) \oplus \operatorname{im}_{\mathbb{Q}^n}(g-1)$.

Let $u \in \mathbb{Z}^n$. Therefore, there are $f \in \ker_{\mathbb{Q}^n}(g-1)$ and $v \in \operatorname{im}_{\mathbb{Q}^n}(g-1)$ such that $u = f + v$. Then, because $(g-1)u = v$, we know that $v \in \mathbb{Z}^n$ and so $f \in \mathbb{Z}^n$. □

Let $\pi_g(t)$ be the projection of t onto the fixed sublattice of g as according to the direct sum given above. The first operation means every conjugacy class is a union of sets of the form

$$\widetilde{tg} = \{t'g \in \mathbb{Z}^n \rtimes G \mid \pi_g(t') = t\}, \quad g(t) = t$$

because $\mathbb{Z}^n$ is the direct sum of the lattice fixed by g and $\operatorname{im}(g-1)$.

The first and second operations together mean we can choose a system of representatives for the conjugacy classes

$$tg, \quad g(t) = t, \tag{1}$$

for every representative g of the conjugacy classes of G and every fixed point t of g.

Furthermore, the above is a basis for the cocenter of $\mathbb{C}[\mathbb{Z}^n \rtimes G]$.

3.3 Injectivity of trace map

We show that the trace map $\tilde{\operatorname{tr}}$ is injective.

Definition. *Let* $\alpha = (\alpha_j)_{j=1}^n \in \mathbb{C}^n$ *be a vector. Then let the set of powers* $P(\alpha)$ *of* α *be the set* $\{(\alpha_j^\lambda)_{j=1}^n \mid \lambda \in \mathbb{Z}\}$ *of pointwise integer powers of* α.

Definition. *For a system of continuous equations* E *over* $\mathbb{C}^n$*, let* $Z(E) \subset \mathbb{C}^n$ *be the set of common zeros of* E.

Note that $Z(E)$ is always closed under the standard topology of $\mathbb{C}^n$.

Definition. *A system of equations E has the set-of-powers property iff $\alpha \in Z(E)$ implies $P(\alpha) \subset Z(E)$.*

Lemma 1. *A system of continuous equations E over $\mathbb{C}^n$ with the set-of-powers property either has a solution where one of the components is zero, or all of its solutions consist of unit-norm components.*

Proof. Suppose there is a solution $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ where α_j has non-unit norm, and no components are zero. Then by taking either positive or negative integer powers of α, we obtain new elements of $Z(E)$ with the norm of the jth component arbitrarily close to zero. As $Z(E)$ is closed, there must then be a solution with a component equaling zero. □

Let $\mathbb{S}$ be the topological abelian group given by the quotient $\mathbb{Z}$-module $\mathbb{R}/\mathbb{Z}$, with a quotient topology induced by the topology of $\mathbb{R}$ and additive notation. This is the unit circle group of complex numbers S^1 with some superficial differences.

Definition. *For $v \in \mathbb{S}^k$, let $R(v)$ be the sublattice of $\mathbb{Z}^k$ consisting of integer linear relations on the components of v, i.e.*

$$R(v) = \left\{ \alpha \in \mathbb{Z}^k \mid \sum_{j=1}^{k} \alpha_j v_j \in \mathbb{Z}^k . \right\}$$

Definition. *For $v \in \mathbb{S}^k$, let $F(v)$ be the shortest-length lift of v to $\mathbb{R}^k$.*

Definition. *For $v \in \mathbb{S}^k$, let the norm $N(v)$ of v be the length of $F(v)$.*

Definition. *Let $\pi^k : \mathbb{R}^k \to \mathbb{S}^k$ be the projection. Given $B = \{b_1, b_2, \ldots, b_d\} \subset \mathbb{R}^k$, let $\operatorname{lspan} B \subset \mathbb{S}^k$ be $\pi^k(\operatorname{span} B)$, i.e.*

$$\left\{ \pi^k \left(\sum_{j=1}^{d} \alpha_j b_j \right) \mid \alpha_j \in \mathbb{R} \right\}.$$

Lemma 2. *Let $\beta \in \mathbb{S}^k$. The closure of the set of integer multiples of β is $Z(R(\beta))$, i.e. the subset of $\mathbb{S}^k$ which satisfies all the relations in $R(\beta)$.*

Proof. **Showing it suffices to consider the case $R(\beta) = 0$.** All integer multiples of β must satisfy $R(\beta)$, so the set $\mathbb{Z}\beta$ of integer multiples of β is a subset of $Z(R(\beta))$. As $Z(R(\beta))$ is a closed set, the closure $\overline{\mathbb{Z}\beta}$ of the aforementioned set is also a subset of $Z(R(\beta))$.

The space of common zeros of $R(\beta)$ in $\mathbb{R}^n$ is $\mathbb{R}$-spanned by irreducible integer vectors, so there is an integer vector basis of $Z(R(\beta)) \subset \mathbb{S}^k$, say $b_1, b_2, \ldots, b_d \in \mathbb{Z}^k$ – i.e., $Z(R(\beta)) = \operatorname{lspan}\{b_1, b_2, \ldots, b_d\}$. There is thus an $\mathbb{R}$-linear automorphism φ of $\mathbb{R}^k$ that sends b_j to e_j for $j = 1, 2, \ldots, d$ with integer matrix coefficients, with the undefined degrees of freedom fixed arbitrarily. (Integer matrix coefficients and invertibility is guaranteed because of the irreducibility of $b_1, b_2, \ldots, b_d$.) Because of these integer matrix coefficients, this automorphism of $\mathbb{R}^k$ restricts to an endomorphism $\tilde{\varphi}$ of $\mathbb{S}^k$:

$$\begin{array}{ccc} \mathbb{R}^k & \xrightarrow{\pi^k} & \mathbb{S}^k \\ {\scriptstyle\varphi}\uparrow & & {\scriptstyle\tilde{\varphi}}\uparrow \\ \mathbb{R}^k & \xrightarrow{\pi^k} & \mathbb{S}^k \end{array}$$

Because β is the projection of a real linear combination

$$\sum_{j=1}^{d} \eta_j b_j$$

of the b_j, then

$$\tilde{\varphi}(\beta) = \sum_{j=1}^{d} \eta_j e_j.$$

Let $\eta = (\eta_j)_{j=1}^d$. Then $R(\eta) = 0$, because if $r \in R(\eta)$, then $\varphi^{-1}(r)$ would be an integer linear dependence relation on $b_1, b_2, \ldots, b_d$.

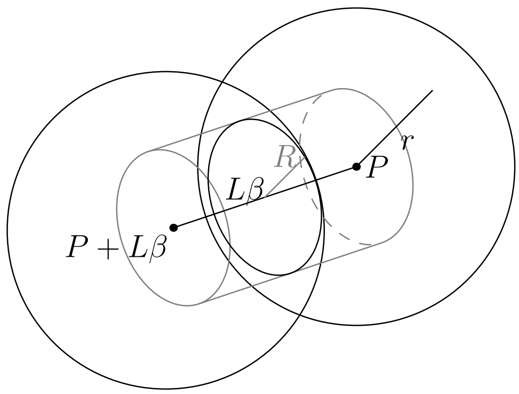
The claim is then equivalent to the closure $\overline{\mathbb{Z}\eta}$ being $\operatorname{lspan}\{e_1, e_2, \ldots, e_d\}$.

Therefore, it suffices to consider the claim in the case when $R(\beta) = \{0\}$, being that $\mathbb{Z}\beta$ is dense in $\mathbb{S}^k$.

Showing the density of $\mathbb{Z}\beta$. Suppose, for the purpose of contradiction, there were some open ball $B(P, r)$ in $\mathbb{S}^k$ which does not intersect $\mathbb{Z}\beta$. Then, all cosets of this open ball $n\beta + B(P, r) = B(P + n\beta, r)$, where n is an integer, do not intersect $\mathbb{Z}\beta$ either. As these cosets are of positive Lebesgue measure and $\mathbb{S}^k$ has finite Lebesgue measure, two of these cosets, say $n\beta + B(P, r)$ and $l\beta + B(P, r)$, must intersect. Therefore, $B(P, r)$ intersects with

$$L\beta + B(P, r),$$

where $L = l - n$. If $L\beta = 0$, this is a nontrivial integer relation on the components of β; therefore, $L\beta \neq 0$ and $B(P, r) \neq B(P + L\beta, r)$. Then, letting $R = \sqrt{r^2 - \frac{N(L\beta)^2}{4}}$, the set C of points strictly within a distance R of the line segment from P to $P + L\beta$ does not intersect $\mathbb{Z}\beta$ because it sits within the union of the two spheres around P and $P + L\beta$. See the diagram below, where $C = C(P, P + L\beta, R)$ is the gray-outlined cylinder.



But also $C(P + L\beta, P + 2L\beta, R)$ does not intersect $\mathbb{Z}\beta$, and so on, so the union of these, which is an open hypercylinder $X = P + C'(\operatorname{lspan} F(L\beta), R)$ of points strictly within R of the line $P + \operatorname{lspan} F(L\beta)$, does not intersect $\mathbb{Z}\beta$.

Quotient $\mathbb{S}^k$ by the subgroup $\operatorname{lspan}\{F(L\beta)\}$ to get

$$\mathbb{S}^k / \operatorname{lspan}\{F(L\beta)\} \cong (\mathbb{R}^k / \operatorname{span}\{F(L\beta)\})/\mathbb{Z}^k \cong \mathbb{R}^{k-1}/A(\mathbb{Z}^k),$$

where A is an $\mathbb{R}$-linear matrix $\mathbb{R}^k \to \mathbb{R}^{k-1}$ with orthonormal rows which takes $F(L\beta)$ to 0. The last isomorphism in the sequence above is given by A. We can extend A with the row $F(L\beta)$ to get an orthogonal matrix A', and we can say that $A'(\mathbb{Z}^k) \cong \mathbb{Z}^k \cong A'^T(\mathbb{Z}^k)$ because $F(L\beta)$ has no nontrivial integer relations and thus $\ker A' \cap \mathbb{Z}^k = 0$. Therefore, $A(\mathbb{Z}^k) \cong A(A'^T(\mathbb{Z}^k)) \cong \mathbb{Z}^{k-1}$, so $\mathbb{S}^k / \operatorname{lspan}\{F(L\beta)\} \cong \mathbb{S}^{k-1}$ through a projection $p : \mathbb{S}^k \to \mathbb{S}^{k-1}$. Then the p-preimage of $B(P', R)$, where $P' = p(P)$, is X. We then have the integer multiples of $p(\beta)$ are not dense in $\mathbb{S}^{k-1}$. Thus, we can do reductio ad absurdum in this case, until we get to the situation $k = 1$.

When $k = 1$, then the hypercylinder X will cover all of $\mathbb{S}$. But this is impossible, because X does not intersect $\mathbb{Z}\beta$, and $\beta \in \mathbb{Z}\beta \subset \mathbb{S}$, which is a contradiction. □

Lemma 3. *For an integer N, there is a linear isomorphism Q of $\mathbb{Z}^N$ such that Q has order N, $Q^x(e_1 + e_2)$ is linearly independent from $e_1 + e_2$ if $N \nmid x$, and*

$$(e_1^* Q^x e_1)_{x=1}^N$$

is a permutation of the sequence $(1, 2, \ldots, N)$.

Proof. Choose recursively vectors $a_t \in \mathbb{Z}^n$ for $t = 1, \ldots, N$ as follows. Let $a_1 = (1, 1, 0, \ldots, 0) = e_1 + e_2$. Then, for $t = 2, \ldots, N$, let $M = \mathbb{Z}^n / \operatorname{span}\{a_1, \ldots, a_{t-1}\}$ as a $\mathbb{Z}$-module. We assume M is a free $\mathbb{Z}$-module with a basis $(b_1, \ldots, b_{n-t+1})$ where b_1 is the projection of $e_1 \in \mathbb{Z}^N$, and inductively show this remains so. Thus the subspace $\operatorname{span}\{e_2, \ldots, e_N\}$ of $\mathbb{Z}^N$ has a projection into M which is spanned by $\{b_2, \ldots, b_{n-t+1}\}$. Therefore, b_2 has a lift to $\mathbb{Z}^N$ with first component equal to 0. Let $a'_t = tb_1 + b_2$. Then $M / \operatorname{span}\{a'_t\}$ is still a free $\mathbb{Z}$-module which admits a basis with an element that is a projection of e_1. Afterwards, let a_t be a lift of a'_t to $\mathbb{Z}^N$ such that the first component is t.

Then, the vectors $a_1, \ldots, a_N$ form a basis. Let Q cycle these, mapping a_t to a_{t+1} for $t \in [1, N-1]$ and a_N to a_1.

□

Theorem 1 (Trace Independence)**.** *The irreducible traces* $\tilde{\mathrm{tr}}(V_\chi)$ *of* $\mathbb{Z}^n \rtimes G$ *are linearly independent (under finite sums).*

Proof. Suppose

$$\sum_{j=1}^{k} c_j \tilde{\mathrm{tr}}(V_{\chi_j}) = 0,$$

for additive characters $\chi_j \in \mathrm{Hom}(\mathbb{Z}^n, \mathbb{C})$ and coefficients $c_j \in \mathbb{C}$ that are nonzero. Assume WLOG that the stabilizer of χ_j is H_j. Then, for any $g \in G$, $t \in \mathbb{Z}^n$, the linear combination of traces is

$$\sum_{j=1}^{k} c_j \sum_{\substack{h \in G \\ hgh^{-1} \in H_j}} \exp \chi_j(h(t)) = \sum_{\substack{h \in G \\ hgh^{-1} \in H_j}} \sum_{j=1}^{k} c_j \exp \chi_j(h(t)).$$

Let the lth component of χ_j be α_{jl}, and the lth component of $h(t)$ be $h_l(t)$. Then, letting $\beta_{jl} = \exp \alpha_{jl}$, this sum is

$$\sum_{j=1}^{k} \sum_{\substack{h \in G \\ hgh^{-1} \in H_j}} c_j \prod_{l=1}^{n} \beta_{jl}^{h_l(t)} = 0.$$

Showing that it suffices to consider only β_{jl} with unit norm. This is a Laurent polynomial equation in the β_{jl}. Every choice of g and t produces another Laurent polynomial equation. Because scaling t by λ scales all the exponents in the Laurent polynomial similarly, if a set of values $(b_{jl})_{jl}$ for $(\beta_{jl})_{jl}$ is a solution, so are all sets of values $(b_{jl}^\lambda)_{jl}$ for integer powers λ. Thus, this system of equations has the set-of-powers property, so by Lemma 1, either there is a solution with $\beta_{jl} = 0$, or all solutions have unit-norm components. But if there is a solution with $\beta_{jl} = 0$, we can change the assumption to say $c_j = 0$ and get a same or greater set of solutions, so doing reductio ad absurdum we can assume all solutions consist of β_{jl} on the unit circle.

Showing that it suffices to consider only β_{jl} that are roots of unity. Let $\eta_{jl} = \frac{1}{2\pi} \arg(\beta_{jl}) \in \mathbb{S}$, and let $\eta = (\eta_{jl})_{jl} \in \mathbb{S}^{nk}$. We have the relations

$$\sum_{j=1}^{k} \sum_{\substack{h \in G \\ hgh^{-1} \in H_j}} c_j \exp\left(2\pi i \cdot \sum_{l=1}^{n} \eta_{jl} h_l(t)\right) \tag{2}$$

By the set-of-powers property and Lemma 2, we can add to η any set of phases which satisfy the relations $R(\eta)$ and get another solution:

$$\eta \to \eta + \kappa, \quad \kappa \in Z(R(\eta)).$$

(We also use here the fact that the set of solutions is closed.)

If any $\eta_{j'l'}$ is irrational, then we can find an irreducible integer vector $\theta \in \mathbb{Z}^{nk}$ such that the projection of any real multiple $y\theta$ of θ into $\mathbb{S}^{nk}$ satisfies $R(\eta)$. (This is because $Z(R(\eta))$ is infinite in this ase.) Take some t, g. Let the so-called jth integer offset to tg arising from θ be

$$\psi_j(tg) = \sum_{\substack{h \in G \\ hgh^{-1} \in H_j}} \sum_{l=1}^{n} \theta_{jl} h_l(t).$$

The amount by which the argument to the exponential in (2) changes when we take $\eta \to \eta + y\theta$ is $2\pi i \cdot y\psi_j$, hence why we call these the "offsets" corresponding to θ. Let $\psi = (\psi_1, \psi_2, \ldots, \psi_k)$. Choose a U that divides some of the components of ψ and not some other components (we will deal with the case that all components of ψ are equal later). Then $\frac{1}{U}\psi \in \mathbb{S}^k$ will have component 0 corresponding to those components of ψ which U divides and something nonzero corresponding to those which U does not divide. This offset $\frac{1}{U}\psi$ comes from a phase change

$$\eta \to \eta + \frac{1}{U}\theta$$

which satisfies $R(\eta)$, so it corresponds to a new solution, which when we plug into the relation (2) for tg, gives

$$\sum_{j=1}^{k} \sum_{\substack{h \in G \\ hgh^{-1} \in H_j}} c_j \exp\left(2\pi i \cdot \frac{1}{U}\psi_j\right) \exp\left(2\pi i \cdot \sum_{l=1}^{n} \eta_{jl} h_l(t)\right) = 0.$$

Note that the new phase factor $\exp\left(2\pi i \cdot \frac{1}{U}\psi_j\right)$ is 1 for some j and some other Uth root of unity for other j. These are the j for which U divides ψ_j and those j for which U does not divide ψ_j, respectively. By adding these relations for all offsets $\frac{u}{U}\psi$ with $u = 0, \ldots, U-1$, we obtain a resulting solution that has fewer nonzero c_j for the tg relation.

We did not yet consider the case if all components of ψ are equal. We deal with this by just choosing a tg where not all the components of ψ are equal. We show this is possible. Assume WLOG that all χ_i are distinct. Then there is a potential relation $e_{il} - e_{jl}$ which $R(\eta)$ does not contain; therefore, we can choose θ so $\theta_{il} \neq \theta_{jl}$ for some i, j, l. Thus, the equation $\psi_i(tg) = \psi_j(tg)$ is a nontrivial integer relation on t, so some t does not satisfy it.

Thus, we prove the stronger statement that no $(c_j)_j, (\beta_{jl})_{jl}$ where some $\beta_{j'l'}$ is not a root of unity satisfies the condition that all its integer powers satisfy the relation corresponding to the tg chosen above.

If there were, we could come up with another solution with fewer nonzero c_j and the same choice of θ, tg. Thus, we could do reductio ad absurdum.

Thus, every solution $(\beta_{jl})_{jl}$ consists of root-of-unity-components.

Showing the contradiction. Let $(\beta_{jl})_{jl}$ be a solution with N the product of all orders of the roots of unity β_{jl} and let $\zeta = \exp(2\pi i/N)$. Let $\beta_{jl} = \zeta^{\gamma_{jl}}$, and let $\gamma_j = (\gamma_{j1}, \gamma_{j2}, \cdots, \gamma_{jn})$. We use $\langle \cdot, \cdot \rangle$ to denote the standard inner product on $\mathbb{Z}^n$. The equations become

$$\sum_{j=1}^{k} \sum_{\substack{h \in G \\ hgh^{-1} \in H_j}} c_j \zeta^{\langle \gamma_j, h(t) \rangle} = 0.$$

We will show the stronger statement that the subset of the above equations where $t \neq 0$ and the last $N \left\lfloor \frac{n}{N} \right\rfloor$ components of t are a repetition of $e_1 \in \mathbb{Z}^N$, i.e. $t = (v, e_1, e_1, \ldots, e_1)$ where $v \in \mathbb{Z}^w$ and $w < N$, cannot simultaneously hold for all g. We call these the limited relations.

Suppose γ_f is an arbitrary one of the γ_j. Then let $\tilde{G} = G \times C_N$, where C_N is the cyclic group with order N. Let $(g, x) \in \tilde{G}$ act on $\mathbb{Z}^n \times \mathbb{Z}^N$ by

$$(g, x)(t, s) = (gt, Q^x s),$$

where Q is the linear isomorphism from Lemma 3.

Then the sequence $(Q^0 s, Q^1 s, \ldots, Q^{q-1} s)$, for $s = e_1 \in \mathbb{Z}^N$, has a permutation of the sequence $(1, 2, \ldots, N)$ as its first components.

Now let $\tilde{\gamma_f}$ be $(\gamma_f, 1, 0, 0, \ldots, 0) \in \mathbb{Z}^{n+N}$, and for γ_j with $j \neq f$ let $\tilde{\gamma_j} = (\gamma_j, N, N, 0, 0, \ldots, 0) \in \mathbb{Z}^{n+N}$. What are the linear dependence relations on $(\mathbb{Z}^n \times \mathbb{Z}^N) \rtimes \tilde{G}$ using characters $\tilde{\gamma_j}$, and are the limited ones satisfied?

The new character corresponding to $\tilde{\gamma_f}$ is stabilized by $H \times 1$, and every $\tilde{\gamma_j}$ is also stabilized by $H \times 1$, because the image under Q^x of $(N, 0, 0, \ldots, 0)$ is not that vector for $x \neq 0$. Consider $\tilde{g} = (g, x) \in G \times C_N$. Let U be the set of $h \in G$ such that $hgh^{-1} \in H$. Then the set of $\tilde{h} \in \tilde{G}$ such that $\tilde{h}\tilde{g}\tilde{h}^{-1} \in H \times 1$ is $U \times C_N$ if $x = 1$, and the empty set otherwise. Thus the relations are

$$\sum_{j=1}^{k} c_j \sum_{\substack{h \in G \\ hgh^{-1} \in H_j}} \sum_{x \in C_N} \zeta^{\langle \tilde{\gamma_j}, (h(t), Q^x s) \rangle} = 0,$$

over $(t, s) \in \mathbb{Z}^n \times \mathbb{Z}^N$.

The above rightmost sum for $\tilde{\gamma_j}$ with $j \neq f$ is N times $\zeta^{\langle \gamma_j, h(t) \rangle}$. Thus it is N times what it would be in the original equations.

For s equal to e_1 as it would be in the limited relations for $\tilde{G}$, the sum for $\tilde{\gamma_f}$ is zero, because it is $\zeta^{\langle \gamma_f, h(t) \rangle}$ times a $0 - c_j$ disappears in such a relation.

Note that in the new relations there is one fewer term in the linear combination. Thus, we can do reductio ad absurdum again.

□

3.4 Density of characters analog

Having characterized the kernel of the trace-induced map, we work towards characterizing its image. We show that the image of the trace-induced map is dense in a certain subspace of the cocenter, that we call the reduced cocenter, which produces a property analogous to density of characters, though strictly weaker.

Firstly we consider a direct failure of density of characters.

Theorem 2 (Trace-Always-Zero Group Elements)**.** *The set of group elements of $\mathbb{Z}^n \rtimes G$ which act by trace zero in all representations is the set of elements tg where g has a trivial fixed subspace.*

Proof. The relation for an element tg to have trace zero in irreducible representation V_χ, with $H = G_\chi$, is

$$\sum_{\substack{h \in G \\ hgh^{-1} \in H}} \exp \chi(h(t)) = 0.$$

This is equivalent to

$$\sum_{\substack{h \in G \\ hgh^{-1} \in H}} \prod_{j=1}^{n} (\exp \chi_j)^{h_j(t)},$$

where $h_j(t)$ is the jth component of $h(t)$ and χ_j is the jth component of χ.

In order for tg to have trace zero for all V_χ with stabilizer H, the above must hold for all χ in $(\mathbb{C}^n)^g$. As $(\mathbb{C}^n)^g$ is a subspace, say with dimension d, this set of χ is characterized as a the image of an integer linear transformation A over free indeterminates $u_1, u_2, \ldots, u_d$. Thus the above relation is a polynomial relation in $\exp u_j$, which must hold for all possible values of u_j, and thus for all positive values of $\exp u_j$. For a polynomial relation to hold for all positive values of its indeterminates, it must be identically zero. As all the coefficients in the sum above are 1, this means that there must be no terms, i.e., there are no $h \in G$ such that $hgh^{-1} \in H$; thus, g must not be conjugate to any element of H. This must be true for all possible stabilizer subgroups H. Thus g must have a trivial fixed subspace, as otherwise the subgroup generated by g is a possible stabilizer subgroup. Actually, if g has a trivial fixed subspace, then g is not conjugate to any element of a stabilizer subgroup H, as conjugating by k transforms the fixed subspace through an isomorphism, namely the action of k. □

We call elements of G with a trivial fixed subspace *transcendent*. Similarly we call $tg_0 \in \mathbb{Z}^n \rtimes G$ transcendent iff g_0 is transcendent.

Definition. *The reduced cocenter of the group algebra $A = \mathbb{C}[\mathbb{Z}^n \rtimes G]$ is its cocenter $\overline{A}$ quotiented by the (projected image of) the span of all transcendent elements tg_0 and all expressions of the form*

$$tg^l - tg$$

where l is coprime with the order of g, as $\mathbb{C}$-vector spaces, denoted $C_r(\mathbb{Z}^n \rtimes G)$.

Remark. *As this is a vector space quotient which must therefore split, the reduced cocenter can also be regarded as a subspace of the cocenter.*

We can define an equivalence relation $\mathcal{O} \sim \mathcal{V}$ on conjugacy classes of G which is so when $\mathcal{V} = \mathcal{O}^l$ and l is coprime with the order of $\mathcal{O}$. Thus, we can define a set P of equivalence classes of conjugacy classes, where $\overline{\mathcal{O}}$ is the equivalence class of $\mathcal{O}$. Note that taking of powers factors through this equivalence relation: if $\mathcal{O} \sim \mathcal{V}$, then $\overline{\mathcal{O}^k} = \overline{\mathcal{V}^k}$ for any k. If the order of $\mathcal{O}$ is m, the unique powers of $\overline{\mathcal{O}}$ correspond to positive divisors of m: $\overline{\mathcal{O}}^d$, where $d \mid m$.

Definition. *We can choose a subset, called the reduced representatives, of the representatives (1) whose images under the quotient map $\mathbb{C}[\mathbb{Z}^n \rtimes G] \to C_r(\mathbb{Z}^n \rtimes G)$ span the reduced cocenter. Picking out this set of representatives consists of choosing a $g_{\overline{\mathcal{O}}}$ in some $\mathcal{V} \in \overline{\mathcal{O}}$ for every equivalence class $\overline{\mathcal{O}}$, and then choosing a reduced representative $tg_{\overline{\mathcal{O}}}$ for every fixed point t of $g_{\overline{\mathcal{O}}}$.*

We shall show that no nonzero element of the reduced cocenter $C_r(\mathbb{Z}^n \rtimes G)$ has trace zero in all representations, which we call the reduced density of characters property.

Lemma 4. *Let $[H;t]$ for a subgroup H of G, $t \in \mathbb{Z}^n$ denote the element in $\mathbb{C}[E \times \mathbb{Z}^n]$, where E denotes the set of subgroups of G. Let $A_H \in \operatorname{End}_{\mathbb{Z}} \mathbb{Z}^n$ be a generating matrix for the fixed subspace of H: $\operatorname{im}(A) = (\mathbb{Z}^n)^H$. Then, for the reduced density of characters property, it suffices to show the condition that the $\mathbb{C}[E \times \mathbb{Z}^n]$ elements*

$$\sum_{H \in E} \sum_{\substack{h \in G \\ hgh^{-1} \in H}} [H; (A_H^T h)(t)] \tag{3}$$

are linearly independent over all reduced representatives tg.

Proof. Fix a subgroup H and let $A = A_H$. We use $[t]$, where $t \in \mathbb{Z}^n$, to denote the element of $\mathbb{C}[\mathbb{Z}^n]$. Let B be a basis for the reduced cocenter as in (1). The linear combination

$$\sum_{tg \in B} c_{tg} tg$$

having trace zero for all χ fixed by H is concretely that

$$\sum_{tg \in B} c_{tg} \sum_{\substack{h \in G \\ hgh^{-1} \in H}} \exp \langle \chi, h(t) \rangle = 0$$

for all $\chi \in (\mathbb{C}^n)^H$. However, this is saying that

$$\sum_{tg \in B} c_{tg} \sum_{\substack{h \in G \\ hgh^{-1} \in H}} \exp \langle Au, h(t) \rangle = 0$$

for all $u \in \mathbb{C}^d$, where d is the number of columns of A. But using that $\langle Au, h(t) \rangle = \langle u, A^T h(t) \rangle$ and substituting $u = iu'$, this is a relation on Fourier series, so

$$\sum_{tg \in B} c_{tg} \sum_{\substack{h \in G \\ hgh^{-1} \in H}} [(A^T h)(t)] = 0 \in \mathbb{C}[\mathbb{Z}^n].$$

Let

$$k_H(tg) = \sum_{\substack{h \in G \\ hgh^{-1} \in H}} [(A^T h)(t)].$$

Then, if $k_H(tg)$ for tg going over B have no nontrivial integer relations, then all $c_{tg} = 0$. The integer relations satisfied by the expressions (3) are the relations satisfied by the above for all H. We are done. □

We introduce a lemma regarding general linear algebra.

Lemma 5. *If $v_i, i \in I$ are vectors in a vector space V, and there is a pair of dual bases $\{\alpha_j\}, \{\alpha_j^*\}, j \in J$ of V, V^*, then if there is a cover $U_k, k \in K$ of I and corresponding sets $V_k \subset J$, such that*

$$\alpha_j^*(v_i) = 0 \tag{4}$$

if $j \in V_k, i \notin U_k$, and

$$\left\{ \sum_{j \in V_k} \alpha_j \alpha_j^*(v_i) \mid i \in U_k \right\} \tag{5}$$

forms a linearly independent set for all $k \in K$, then the v_i are linearly independent.

Proof. Let

$$\sum_{i \in I} c_i v_i = 0.$$

Choose $k \in K$ arbitrarily. Then, by applying

$$\Pi_k = \sum_{j \in V_k} \alpha_j \alpha_j^*$$

to the left side, we obtain

$$\sum_{i \in I} c_i \sum_{j \in V_k} \alpha_j \alpha_j^*(v_i) = 0.$$

All $i \notin U_k$ drop out by (4), so we obtain

$$\sum_{i \in U_k} c_i \sum_{j \in V_k} \alpha_j \alpha_j^*(v_i) = 0.$$

By the given linear independence (5), $c_i = 0$ for all $i \in U_k$. Because the U_k cover I, $c_i = 0$ for all i. □

Theorem 3. *If v is an element of the reduced cocenter of $\mathbb{Z}^n \rtimes G$ that has trace zero in all representations, then $v = 0$.*

Proof. By Lemma 4, we can simply show the condition given there. This is equivalent to

$$\sum_{H \in E} \sum_{\substack{h \in G \\ hg_{\mathcal{O}}h^{-1} \in H}} [H; (A_H^T h)(t)] \tag{6}$$

being linearly independent over all reduced representatives $tg_{\overline{\mathcal{O}}}$.

We call the above sum $\kappa(\overline{\mathcal{O}}, t)$.

The proof works by showing that the images $\kappa(\overline{\mathcal{O}})$ of $\kappa(\overline{\mathcal{O}}, t)$ under the canonical map $\mathbb{C}[E \times \mathbb{Z}^n] \to \mathbb{C}[E]$ are linearly independent.

Let $\langle k \rangle$ be the subgroup generated by k. We use the linear algebra lemma (4). We wish to show $\kappa(\overline{\mathcal{O}})$ are linearly independent. We construct the basis α_H of $\mathbb{C}[E]$ to be the standard basis $\{[H] \mid H \in E\}$ and the corresponding dual basis α_H^* is $[H]^*$. The cover of the set $I = P$ of conjugacy class equivalence classes is given by repeatedly choosing any remaining conjugacy class $\mathcal{O}$ with highest order, then placing all of the powers of $\overline{\mathcal{O}}$ into a cover set $U_{\overline{\mathcal{O}}}$. The corresponding sets $V_{\overline{\mathcal{O}}}$ are each the set of $\left\langle g_{\overline{\mathcal{O}}}^l \right\rangle$ for all l dividing the order of $\mathcal{O}$.

We first must show the condition (4) of the lemma. Consider some $v_i = \kappa(\overline{\mathcal{V}})$ and $\alpha_j^* = \left[\left\langle g_{\overline{\mathcal{O}}}^l \right\rangle\right]^*$, where $\overline{\mathcal{V}}$ is not a power of $\overline{\mathcal{O}}$. If $hg_{\overline{\mathcal{V}}}h^{-1} = g_{\overline{\mathcal{O}}}^{nl}$ for some n, then $g_{\overline{\mathcal{V}}}$ is in the same conjugacy class as $g_{\overline{\mathcal{O}}}^{nl}$, which means $\overline{\mathcal{V}} = \overline{\mathcal{O}}^{nl}$, which is a contradiction. Thus, in this case, $\alpha_j^*(v_i) = 0$, as there are no h to produce terms in (6).

Next, we show the condition (5) of the lemma. Fix a $\overline{\mathcal{O}}$. Let the order of $\mathcal{O}$ be m. Then, the elements of $U_{\mathcal{O}}$ are indexed by divisors l of m: $U_{\mathcal{O}} = \{\overline{\mathcal{O}}^l \mid l|m\}$. Let $A(\overline{\mathcal{O}})_{kl}$ be the number of $h \in G$ such that $hg_{\mathcal{O}}^l h^{-1} = g_{\mathcal{O}}^{nk}$ for some n. We wish $A(\overline{\mathcal{O}})$ as a matrix to be invertible. However, note that $A(\overline{\mathcal{O}})_{jk}$ may be nonzero only if $k \mid l$, so the matrix is triangular, while its diagonal element corresponding to k is simply the size of the centralizer of $g_{\mathcal{O}}^k$, hence it is so.

□

Theorem 4. *The image of the trace-induced map $\tilde{\mathrm{tr}}$ of $\mathbb{C}[\mathbb{Z}^n \rtimes G]$ is dense in $C_r(\mathbb{Z}^n \rtimes G)^*$.*

Proof. The image of $\tilde{\mathrm{tr}}$ is a vector space. If it were not dense in $C_r(\mathbb{Z}^n \rtimes G)^*$, it would have an orthogonal subspace containing a nonzero vector $v \in C_r(\mathbb{Z}^n \rtimes G)$;

then $\mathrm{tr}_M(v) = 0$ for all representations v, which is not possible by the previous lemma. Therefore, the image is dense in $C_r(\mathbb{Z}^n \rtimes G)$. $\square$