PERELMAN'S $\lambda-$FUNCTIONAL ON ALG MANIFOLDS

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ABSTRACT. We study Perelman's λ-functional on ALG manifolds. For ALE and ALF manifolds, the modified functional incorporating mass terms provide a way of detecting the (in)stability. We show that the naive analogue fails in the ALG setting. Hardy-type inequality fails to hold, and the $\lambda-$functional might be $-\infty$ under base perturbations. Moreover, we suggest that the alternative approach of finding a solution to $-4\Delta_g w + R_g w = 0$ would also fail. Our results show that new tools are required to characterize stability in the ALG setting.

1. INTRODUCTION

Ricci flow, first introduced by Hamilton [Ham82], is a geometric flow of metrics on a manifold obeying

$$\partial_t g = -2\text{Ric}_g.$$

It played a central role in Perelman's resolution of the Poincaré conjecture. More recently, attention has shifted to understanding its behavior on 4-manifolds. A key aspect of this study is the (in)stability of singularities, since (in)stability determines whether a singularity can be avoided under the folow.

In the long-time behavior of Ricci flow on 4-manifolds, one expects to see singularity models given by ALE, ALF, and ALG spaces. These are complete noncompact 4-manifolds with specific asymptotics and curvature decay properties see Section 2 for the definition of ALG).

The stability and instability of ALE and ALF manifolds have been investigated in the works of Deruelle–Ozuch [DO23] and Kim–Ozuch [KO], following ideas introduced by Haslhofer [Has12]. Haslhofer proposed an adaptation of Perelman's λ-functional to the noncompact setting by defining

$$\lambda^{\circ}_{\text{ALE}}(g) = \inf_{\varphi - 1 \in C^\infty_c(M)} (4|\nabla^g \varphi|^2_g + R_g \varphi^2),$$

in contrast to Perelman's original formulation, which restricted to test functions whose L^2-norm is 1. Deruelle–Ozuch refined this construction by subtracting the ADM mass, and introduced

$$\lambda_{\text{ALE}}(g) = \lambda^{\circ}_{\text{ALE}}(g) - m_{\text{ADM}}(g).$$

Date: Aug 2025.

A key observation is that Ricci flow can be regarded as the gradient flow of λ_{ALE}, even in cases where neither Haslhofer's $\lambda^{\circ}_{\mathrm{ALE}}$ nor the ADM mass is defined. This suggests that λ_{ALE} may detect instability of Ricci flat ALE manifolds for broad class of perturbations. However, all known Ricci flat ALE spaces are hyperkähler and therefore linearly stable.

This approach extends to the ALF case in the work of Kim–Ozuch, where the ADM mass is replaced by a suitable "relative mass," leading to a functional λ_{ALF}. Using this framework, several classical non-hyperkähler Ricci flat ALF manifolds, Kerr, Taub-Bolt, and Chen-Teo metrics, are shown to be dynamically unstable under Ricci flow. This confirms that λ_{ALF} provides an indicator of (in)stability.

The next natural question is whether an analogous construction applies to Ricci-flat ALG manifolds. In this work we argue that substantial modifications, or possibly entirely new methods, are required even in the simplest case of $\mathbb{R}^2 \times \mathbb{T}^2$.

Suppose one attempts to define a functional λ_{ALG}, mirroring those of λ_{ALE} and λ_{ALF}:

$$\lambda^{\circ}_{\mathrm{ALG}}(g) := \inf_{\varphi \in \mathcal{C}} (4|\nabla^g \varphi|_g^2 + R_g \varphi^2),$$

for a suitable class of test functions $\mathcal{C}$.

In the ALE and ALF settings, the definition is nontrivial by a Hardy-type inequality which guarantees that the integrand admits a uniform lower bound, and hence the infimum is finite. However, in the ALG case, the Hardy-type inequality does not hold. Moreover, we prove that for some base direction perturbation, $\lambda_{\mathrm{ALG}}(g)$ is indeed $-\infty$.

Another key ingredient in [DO23] and [KO] is the existence of a minimizer w in a suitable Hölder space that attains the infimum. Such a minimizer plays the role of a weight function, enabling weighted analysis. If such a function w exists in the ALG setting, then one computes

$$\lambda_{\mathrm{ALG}}(w+\psi) - \lambda_{\mathrm{ALG}}(w) = \int_M 2(-4\Delta_g w + R_g w)\psi + (4|\nabla^g \psi|_g^2 + R_g \psi^2),$$

which formally forces the condition

$$-4\Delta_g w + R_g w = 0,$$

since otherwise one could change the scale or sign of the test function ψ to make the variation negative. We however provide a counterexample of g which is only compactly perturbed from the standard metric $\mathbb{R}^2 \times \mathbb{T}^2$ which does not admit such w approaching to 1 at infinity.

These observations indicate that the naïve extension of the λ_{ALE} and λ_{ALF} to ALG manifolds encounters significant obstructions. Nevertheless, they do not rule out the possibility of a well-defined λ_{ALG}. For instance, one might restricting the class of test functions $\mathcal{C}$ so that λ_{ALG} is finite and the minimizer is obtainable, or by subtracting a mass-like term so that a similar argument goes through.

In Section 2, we define ALG manifolds which are analogously defined to ALE, ALF manifolds. In Section 3, we construct a counterexample for the Hardy-type inequality and show that the λ-functional cannot be defined in base direction perturbations. In Section 4, we construct an explicit example in which $-4\Delta_g v + R_g v = -R_g$ does not admit a solution in $C^{2,\alpha}_\tau(M)$.

The main difference between this case and the previous cases, ALE and ALF, is that $\mathbb{R}^2$ admits a green function $\log r$, which is not of the form of a polynomial (growing in radial direction; technically not a polynomial). In the previous cases, this was not a problem, since the base has a $\mathbb{R}^4$ or $\mathbb{R}^3$ structure. This paper show that we need a different tool to establish the (un)stability of Ricci-flat manifolds in the ALG setting. The presence of such function will be the main point in constructing counterexamples throughout Section 3 and Section 4.

2. ALG manifolds

In this section, we define ALG manifolds and corresponding weighted Hölder norms.

Definition 2.1 (ALG manifolds)**.** We call a smooth Riemannian manifold (M^{n+2}, g) *ALG* of order $\tau \in \mathbb{R}$ if it satisfies the following conditions:

(1) There exists a compact set $K \subset M$ and a diffeomorphism $\Phi : M \setminus K \to (1,\infty) \times E$ where E is the total space of a principal $\mathbb{T}^2$-bundle over $\mathbb{S}^{n-1}$
(2) For the standard round metric σ on $\mathbb{S}^{n-1}$ and connection 1-form θ with fiber area A, if we let r be the projection of Φ onto the radial direction and $g_0 = \Phi^*(dr^2 + r^2\sigma + \theta^2)$,

$$r^k |\nabla^{g_0,k}(g - g_0)| = O(r^{-\tau}) \text{ on } M \setminus K$$

holds for all $k \geq 0$.

Remark. Given the definition, we have the projection map $\pi : M \setminus K \to \mathbb{R}^n \setminus \overline{B_1(0)} \simeq (1,\infty) \times \mathbb{S}^{n-1}$. From now on, we will fix a smooth positive function $\rho = \rho_{g_o} > \frac{1}{2}$ that agrees to $\pi^* r_{\mathbb{R}^n \setminus \overline{B_1(0)}}$.

One of the main examples of ALG manifolds is $\mathbb{R}^2 \times \mathbb{T}^2$, which can be seen directly from the definition.

Remark. As in the case of ALE and ALF manifolds, one can see that ALG structure is preserved under the Ricci Flow. The proof is basically the same as [KO, Theorem 3.1]

Analagously to ALE and ALF manifolds, we can define weighted Hölder spaces as the following.

Definition 2.2 (Weighted Hölder norms)**.** For $\eta \in \mathbb{R}$ and a tensor s on M, define its $C^k_\eta(M)$ norm as

$$||s||^g_{C^k_\eta(M)} = \sup_M \rho^\eta (\sum_{i=0}^k \rho^i |\nabla^{g,i} s|_g).$$

For $\alpha \in (0,1)$, define $C^{k,\alpha}_\eta(M)$ norm as

$$||s||^g_{C^{k,\alpha}_\eta(M)} = \sup_M \rho^\eta (\sum_{i=0}^k \rho^i |\nabla^{g,i} s|_g + \rho^{k+\alpha}[\nabla^{g,k} s]_{C^{0,\alpha}}).$$

3. Perelman's λ-functional λ_{ALG}

Perelman's λ-functional was originally discussed in [Has12] to show the stability of Ricci Flat manifolds. This discussion was extended to the noncompact setting in [DO23] and [KO]. In this section, we show that for $\mathbb{R}^2 \times \mathbb{T}^2$, in order for the $\lambda^\circ_{\mathrm{ALG}}$ functional to be well defined, the base direction perturbation should not be allowed. This is made precise in Theorem 3.2. This shows that unlike the ALE and ALF setting, λ_{ALG} functional needs a modification.

Notation. Given an ALG Ricci Flat metric g_b, let $\mathcal{M}^{2,\alpha}_\tau(g_b)$ be the space of metrics g such that $g - g_b \in C^{2,\alpha}_\tau(S^2T^*M)$ and $R_g = O(\rho^{\tau'}_{g_b})$ for some $\tau' > n$.

Definition 3.1. Let (M,g) be an ALG manifold with order $\tau > 0$. For $w - 1 \in C^\infty_c(M)$, define $\tilde{\mathcal{F}}_{\mathrm{ALG}}(g,w)$ by:

$$\tilde{\mathcal{F}}_{\mathrm{ALG}}(g,w) = \int_M (4|\nabla^g w|^2 + R_g w^2) d\mu_g.$$

The $\lambda^\circ_{\mathrm{ALG}}$ functional is defined as:

$$\lambda^\circ_{\mathrm{ALG}}(g) = \inf_{w-1\in C^\infty_c(M)} \tilde{\mathcal{F}}_{\mathrm{ALG}}(g,w).$$

Theorem 3.2. Given the standard metric g_b on $\mathbb{R}^2 \times \mathbb{T}^2$, for any $\epsilon > 0$, there exists base direction perturbation $g \in B_{C^{2,\alpha}_\tau}(g_b, \epsilon) \cap \mathcal{M}^{2,\alpha}_\tau(g_b)$, such that

$$\lambda^\circ_{\mathrm{ALG}}(g) = -\infty.$$

We state a theorem of Rosenberg as an lemma for the theorem.

Lemma 3.3. [Ros82, Theorem A] Let M be a finitely connected complete noncompact Riemannian surface with Gaussian curvature K and area form dA. If $\int_M KdA$ is absolutely integrable, then $\chi(M) \geq \int_M KdA$.

The next lemma gives a counterexample for a Hardy-type inequality which will be used in the construction for the proof of Theorem 3.2.

Lemma 3.4. For every $C > 0$, there exists a compactly supported radial function f on $\mathbb{R}^2$ such that

$$\int_{\mathbb{R}^2} f^2 dA \geq C \int_{\mathbb{R}^2} |\nabla f|^2 dA$$

Proof. For each $C = l$ with l a positive integer, consider

$$f(r) = \begin{cases} \log k & r < 1, \\ \log k - \log r & 1 \leq r < k, \\ 0 & r > k. \end{cases}$$

We can take a converging sequence of C^1 functions that converges to this function such that $\int_{\mathbb{R}^2} |\nabla f|^2 dA$ also converges.

With this function, $\int_{\mathbb{R}^2} f^2 dA$ has a k^2 scale and $\int_{\mathbb{R}^2} |\nabla f|^2 dA$ has a $\log k$ scale. □

Proof of Theorem 3.2. For a given ϵ_b, the base $\mathbb{R}^2$ satisfies the conditions of Lemma 3.3. Applying the lemma, we have that $\int_M K dA \leq 0$.

This shows that either $K = 0$ for all points or $K < 0$ for some point on $\mathbb{R}^2$. If $K = 0$ for all points, since $\mathbb{R}^2$ is two dimensional, it gives that the curvature is identically zero. Then, the metric on the base should be a multiple of the standard metric on $\mathbb{R}^2$. Since we are considering small base direction perturbations, this isn't the case.

If $K < 0$ for some point $a \in \mathbb{R}^2$, there exists some constant $c > 0$ and a neighborhood $B_{g_b}(a, r)$ of a such that $K < -c$ on every point of $B_{g_b}(a, r)$. Therefore using Lemma 3.4, one obtains the theorem. □

4. Possibility of the alternative approach

As in the Introduction, the minimizer of the $\lambda^{\circ}_{\text{ALG}}$ functional satisfies the equation $-4\Delta_g w + R_g w = 0$, where $w - 1 \in C^{2,\alpha}_{\tau}(M)$. Setting $v = w - 1$, the equation becomes $-4\Delta_g v + R_g v = -R_g$. Now, one alternative approach is to directly find a function $v \in C^{2,\alpha}_{\tau}(M)$ that satisfies $-4\Delta_g v + R_g v = -R_g$. Although we showed that the $\lambda^{\circ}_{\text{ALG}}$ functional is not well defined in most cases, if this function v is given, one can hope to define a functional as $\tilde{\mathcal{F}}_{\text{ALG}}(g, v + 1)$ and apply the same analysis as [DO23]. However, in this section, we suggest that it should also be impossible. We aim to construct an arbitrarily small compact perturbation of the standard metric of $\mathbb{R}^2$ and show that there is no such function.

The following conjecture is of the main discussion in this section. The conjecture states that even in the one of the smallest classes of perturbations, namely compact perturbations, there exists a metric that does not admit a solution to the equation $-4\Delta_g v + R_g v = -R_g$. In our discussion, we give an equivalent criterion of the statement.

Conjecture 4.1. For any $\epsilon > 0$, there exists a compact perturbation $g \in B_{C^{2,\alpha}_{\tau}}(g_b, \epsilon)$ of the standard metric of $\mathbb{R}^2 \times \mathbb{T}^2$ such that there is no solution to the equation $-4\Delta_g v + R_g v = -R_g$ in any $C^{2,\alpha}_{\tau}(M)$ for all $\tau > 0$.

Theorem 4.1. Given the following differential equation with the initial condition $u(1) = 1, u'(1) = 0$, if there is a compactly supported f with $\max|f|$ arbitrary small such that u has a $\log r$ scale infinity behaviour, Conjecture 4.1 is true.

$$2u\left(\ddot{f} + \frac{1}{r}\dot{f}\right) + \left(\ddot{u} + \frac{1}{r}\dot{u}\right) = 0.$$

We state four simple lemmata before starting the proof of Theorem 4.1. The first lemma is a well known fact in functional analysis.

Lemma 4.2. [Rud73, Theorem 4.7] Given a Banach space X and its closed subspace Y, $Y = {}^{\perp}(Y^{\perp})$.

We use this well known result of harmonic functions on $\mathbb{R}^2$.

Lemma 4.3. [ABR01, Exercise 10.1] For all $r > 0$, given a harmonic function u on $\mathbb{R}^2 \setminus B_r(0)$, u can be written as the following series:

$$u(x) = \sum_{n=0}^{\infty} P_n(x) + a\log|x| + \sum_{n=1}^{\infty} \frac{Q_n(x)}{|x|^{2n}},$$

where each P_n, Q_n are degree n harmonic polynomials and the series converge on every compact set.

Lemma 4.4. $0 < \tau < 1$ is given. For a compact perturbation g of the standard metric of $\mathbb{R}^2 \times \mathbb{T}^2$, if $u \in C^{2,\alpha}_{-\tau}(M)$ satisfies $-4\Delta_g u + R_g u = 0$, $u \in C^{2,\alpha}_{-\tau'}(M)$ for all $\tau' < \tau$.

Proof. Since the metric is a compact perturbation, u satisfies $\Delta_{g_{euc}} u = 0$ outside a compact set. Invoking Lemma 4.3, we have that

$$u(x) = \sum_{n=0}^{\infty} P_n(x) + a\log|x| + \sum_{n=1}^{\infty} \frac{Q_n(x)}{|x|^{2n}}.$$

Since $u \in C^{2,\alpha}_{-\tau}(M)$, $P_n(x) = 0$ for all $n \geq 1$. This shows that u must be in $u \in C^{2,\alpha}_{-\tau'}(M)$ for all $\tau' < \tau$. □

Proof of Theorem 4.1. It suffices to show that $R_g \notin \operatorname{im}(-4\Delta_g + R_g : C^{2,\alpha}_{\tau}(M) \to C^{0,\alpha}_{\tau+2}(M))$. We first show that the following spaces are equal, considered as subspaces of $C^{2,\alpha}_{\tau+2}(M)^*$.

$$\ker(-4\Delta_g+R_g : C^{2,\alpha}_{-\tau}(M) \to C^{0,\alpha}_{-\tau+2}(M)) = \operatorname{im}(-4\Delta_g+R_g : C^{2,\alpha}_{\tau}(M) \to C^{0,\alpha}_{\tau+2}(M))^{\perp}$$

We start with $\ker \subseteq \operatorname{im}^{\perp}$. Assume $(-4\Delta_g + R_g)u = 0$ for $u \in C^{2,\alpha}_{-\tau}(M)$. Then, invoking Lemma 4.4, we have that $u \in C^{2,\alpha}_{-\tau'}(M)$ for some constant $\tau' < \tau$. It suffices to show that $\int_M u(-4\Delta_g + R_g)v = 0$ for all $v \in C^{2,\alpha}_{\tau}(M)$.

$$\int_D u(-4\Delta_g + R_g)v = \int_D v(-4\Delta_g + R_g)u + \int_{\partial D} (-4u\langle\nabla_g v, n\rangle + 4v\langle\nabla_g u, n\rangle)$$

As the radius of D tends to infinity, each term in the last integral goes to 0. Therefore, $\int_M u(-4\Delta_g + R_g)v = 0$

Now we show that $\mathrm{im}^\perp \subseteq \ker$. Assume $u \in C^{0,\alpha}_{\tau+2}(M)^*$ satisfies $\int_M u(-4\Delta_g + R_g)v = 0$ for all $v \in C^{2,\alpha}_\tau(M)$.

By elliptic regularity, $u \in C^{2,\alpha}(M)$. Therefore,

$$\int_M u(-4\Delta_g + R_g)v = \int_M v(-4\Delta_g + R_g)u$$

holds for all $v \in C^\infty_c(M)$, which shows that $(-4\Delta_g + R_g)u = 0$.

Since $\Delta_{euc} : C^{2,\alpha}_{-\tau}(M) \to C^{0,\alpha}_{-\tau+2}(M)$ is a Fredholm operator and g is the compact perturbation of the standard metric and R_g is a compact operator, $-4\Delta_g + R_g$ should also be a Fredholm operator. In particular, the image is closed, and by Lemma 4.2, ${}^\perp\mathrm{im}^\perp = \mathrm{im}$. This yields the following equivalence of spaces:

$${}^\perp \ker(-4\Delta_g+R_g : C^{2,\alpha}_{-\tau}(M) \to C^{0,\alpha}_{-\tau+2}(M)) = \mathrm{im}(-4\Delta_g+R_g : C^{2,\alpha}_{\tau}(M) \to C^{0,\alpha}_{\tau+2}(M))$$

Now, we obtain the following: $R_g \in \mathrm{im}$ is equivalent to $R_g \in {}^\perp\ker$. Then, in order for $R_g \in \mathrm{im}$, $\int_M R_g u$ should be 0 for all $u \in C^{2,\alpha}_{-\tau}(M)$ that satisfies $(-4\Delta_g + R_g)u = 0$.

However, for a sufficiently large ball that contains the perturbation,

$$\int_M R_g u = \int_B 4\Delta_g u = \int_{\partial B} 4\langle \nabla_g u, n\rangle.$$

Given Lemma 4.3, near ∂B, u is of the form $\sum_{n=0}^\infty P_n(x) + a\log|x| + \sum_{n=1}^\infty \frac{Q_n(x)}{|x|^{2n}}$. In particular, if u contains a $\log|x|$ term, i.e. if $a \neq 0$, the term in the righthandside would be equal to $8\pi a$, which is not zero. The problem that is left is to construct a $g \in B_{C^{2,\alpha}_\tau}(g_b, \epsilon)$ and a function $u \in C^{2,\alpha}_{-\tau}(M)$ that satisfies $-4\Delta_g u + R_g u = 0$, and that u contains a $\log|x|$ term in its infinity behaviour. This can be done by considering the conformal peturbtaion of g as in the following lemma.

Lemma 4.5. For a conformal metric $g = e^{2f} g_{\mathrm{euc}}$ of the standard metric on $\mathbb{R}^2$, the laplacian and the scalar curvature is changed by the following:

$$R_g = e^{-2f}(-2\Delta_{g_{\mathrm{euc}}} f + R_{g_{\mathrm{euc}}})$$

$$\Delta_g = e^{-2f}\Delta_{g_{\mathrm{euc}}}$$

Applying the lemma above, $-4\Delta_g u + R_g u = 0$ is equivalent to $2u\Delta f + \Delta u = 0$. Considering u, f as radial functions $u(r), f(r)$, the equation is equivalent to

$$2u\left(\ddot{f} + \frac{1}{r}\dot{f}\right) + \left(\ddot{u} + \frac{1}{r}\dot{u}\right) = 0.$$

This ends the proof of the theorem. □

Considering f as $0.1e^{-\frac{1}{(x-1)(2-x)}}$ and the initial condition for u as $u(1) = 1, u'(1) = 0$, using numerical methods, we obtain that this u contains a $\log r$ term. This gives a strong numerical evidence for the conjecture.

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